

Game Theory



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Game Theory

- Study of conflict and cooperation
- Two or more players, each having several possible strategies
- The outcome of the game depends on the strategies
- Outcome leads to numerical payoffs to the players
- Example: Rock-Paper-Scissors



A two person zero sum game

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	20
	C	3	2	4	3
	D	-16	0	0	16

These payoffs are to Rose: Rose tries to increase payoffs while Colin plays to hold them down.

Domination

- Is Colin C a reasonable strategy?

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	20
	C	3	2	4	3
	D	-16	0	0	16

Best Play

- Rose C & Colin B with payoff 2
- saddle point (equilibrium)

		Colin			
		A	B	C	D
Rose	A	12	-1	1	0
	B	5	1	7	20
	C	3	2	4	3
	D	-16	0	0	16

Domination & Pruning

- Games with players alternating (checkers, chess, etc)
 - Game trees list positions
 - Can prune the game tree to determine the value of the game
- How chess computers play (search & evaluation)
- Checkers is a draw.



Is there always a best strategy?

Colin

		A	B	C
Rose	A	2	-3	2
	B	-3	4	-3
	C	2	-3	6

Domination

Colin

		Colin		
		A	B	C
Rose	A	2	-3	2
	B	-3	4	-3
	C	2	-3	6

Who has the advantage?

Is there a best single strategy?

Mixed Strategies

Colin

		Colin	
		A	B
Rose	B	-3	4
	C	2	-3



Since opponent can take advantage of any predictable strategy, the only way forward is to use a random strategy!

Expected Payoffs

		Colin	
		A	B
Rose	B	-3	4
	C	2	-3

$$E(\text{Rose B}) = 0.5(-3) + 0.5(4) = 0.5$$

$$E(\text{Rose C}) = 0.5(2) + 0.5(-3) = -0.5$$

If Colin plays 0.5A, 0.5B then Rose expects to win 0.5 by playing B and lose 0.5 with C.

Rose profits by exploiting the difference in the expected payoffs.

No profit!

Rose profits by exploiting the difference in the expected payoffs. So Colin sets a random strategy at x A, $(1-x)$ B so that Rose cannot profit.

		Colin	
		A	B
Rose	B	-3	4
	C	2	-3

$$E(\text{Rose B}) = x(-3) + (1-x)(4) = 4 - 7x$$

$$E(\text{Rose C}) = x(2) + (1-x)(-3) = -3 + 5x$$

$$4 - 7x = -3 + 5x$$

$$7 = 12x$$

$$x = 7/12 \quad \text{so } E(\text{Rose B or C}) = -1/12$$

Colin can ensure Rose wins no more than $-1/12$ on average by playing $7/12$ A, $5/12$ B.

Two can play at that game

Rose can also randomize to eliminate the difference in Colin's expected payoffs.

Rose: $yB, (1-y)C$

		Colin	
		A	B
Rose	B	-3	4
	C	2	-3

$$E(\text{Colin A}) = y(-3) + (1-y)(2) = -5y + 2$$

$$E(\text{Colin B}) = y(4) + (1-y)(-3) = 7y - 3$$

$$-5y + 2 = 7y - 3$$

$$5 = 12y$$

$$y = 5/12 \quad \text{so } E(\text{Colin A or B}) = -1/12$$

Rose can ensure Colin wins no more than $1/12$ on average by playing $5/12B, 7/12C$.

Summary

- Rose has a mixed strategy so that Colin wins no more than $1/12$
- Colin has a mixed strategy so that Colin wins at least $1/12$
- If Rose lost more than $1/12$ then she could have played better; if Colin wins less than $1/12$ then he could have played better
- Value of the game is $1/12$ to Colin (i.e. to make the game fair, he should pay Rose $1/12$ to play)



Von Neumann's Minimax Theorem

- Every two person zero sum game admits optimal (mixed) strategies for each player.

The corresponding outcome is the value of the game.

There are efficient ways to find the best strategies (linear programming).



The Prisoner's Dilemma

- This is an important **non-zero sum game**

	Squeal	Silent
Squeal	$(-1, -1)$	$(1, -2)$
Silent	$(-2, 1)$	$(0, 0)$



Both have incentives to squeal, but doing so causes a non-optimal result.

Other applications

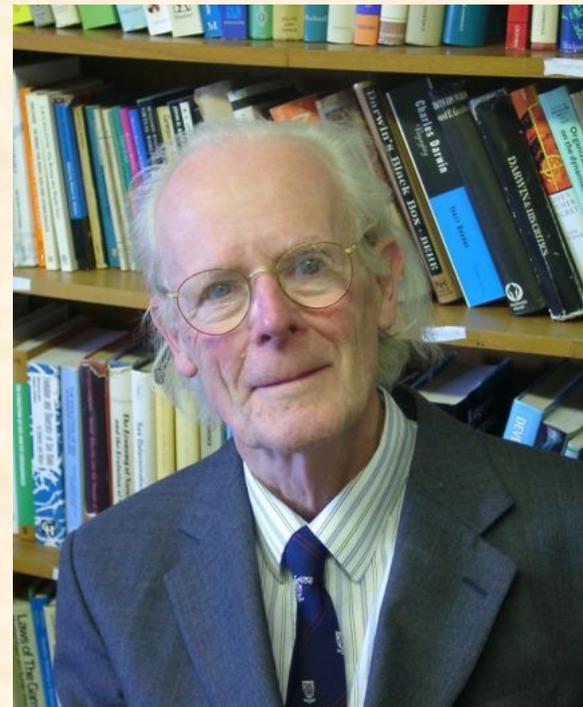
- The Prisoner's dilemma arises in many situations
 - Arms control: both sides have an incentive to cheat, but doing so puts them both in worse positions (RAND Corp.)
 - Business competition: Two nearby gas stations might both have incentive to lower prices (to attract customers); if they both do so then they both lose profits (Nobel prizes)



Applications to Evolutionary Biology

John Maynard Smith studied applications of game theory to evolutionary biology.

The importance of irrelevant factors in animal behavior can be explained as a form of randomization.



Other important games

- Games against nature
 - Is nature a rational decision maker?
 - Statistical decision theory (business school)
- Multiplayer games
 - No satisfactory theory about best play
 - General approach is to study formation of coalitions (obvious applications to politics and elections)

