The Roller Coaster Conjecture for Artin Level Algebras

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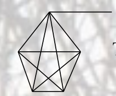
The **Roller Coaster Conjecture** was originally introduced to describe the possible clique sequences of well-covered graphs. After hearing Tony Geramita speak about **Artin level algebras** in January in San Diego, I realized that the conjecture naturally extends to a statement about the possible h-vectors of Artin level algebras.

Some Graph Theory: Let G be a graph with no loops or multiple edges. A subset of vertices V is said to be a **clique** if each vertex in V is joined to all other vertices in V.



The graph G contains a clique on 5 vertices.

The clique sequence $(c_0, c_1, \ldots, c_\alpha)$ of G is a vector of positive integers, where c_n denotes the number of cliques on n vertices in G. Erdös et. al. [AMSE] showed that there are no constraints on the relative sizes of the clique numbers: for any permutation σ of $\{0, 1, \ldots, \alpha\}$ there is a graph with $c_{\sigma(0)} \geq c_{\sigma(1)} \geq \ldots \geq c_{\sigma(\alpha)}$.

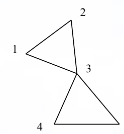
A graph is said to be **well-covered** if all its maximal cliques have the same size. The clique numbers for well-covered graphs were originally thought to be **unimodal** [BDN]; however, T.S. Michael and I showed that this is far from the case [MT]. In fact, the clique sequence must increase up to its half-way point. After that, our Roller Coaster Conjecture asserts that there are no constraints on the relative sizes of the last half of the clique numbers.

Artin Level Algebras: An Artin level algebra is a graded zero-dimensional algebra in which all the **socle** elements ($Soc(R) = Ann_R(m)$) are homogeneous of the same degree. Many researchers (Bigatti, Hulett, Pardue, Iarrobino, Froberg, Laksov, Geramita, etc.) have asked after the possible h-vectors (Poincaré series) of Artin level algebras. The h-vector of R is $(h_0, h_1, \ldots, h_\alpha)$, where h_k is the dimension of the k^{th} graded piece of R.

To each graph, we may associate an ideal of quadratic monomials. Assign a variable to each vertex of the graph to get $S = k[x_1,\ldots,x_n]$. The ideal

$$I = (x_i x_k : [x_i,x_k] \text{ is } \textbf{not} \text{ an edge of } G)$$

contains the square of each variable. Note that R=S/I is zero-dimensional. Moreover, a square-free monomial x^a is in Soc(R) if and only if $supp(x^a)$ is a clique in G but $supp(x^a x_i)$ is not a clique in G for all vertices i.



Well-covered Graph G ⟷ Artin level algebra with h-vector (1,5,6,2)

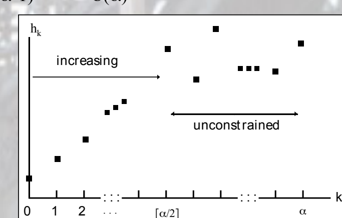
$$R = \frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_1x_4,x_1x_5,x_2x_4,x_2x_5,\; x_1^2,x_2^2,x_3^2,x_4^2,x_5^2)}$$

Thm: (1) homog. elts of Soc(R) correspond to the maximal cliques in G.
(2) G is well-covered if and only if R is an Artin level algebra.
(3) The clique sequence of G agrees with the h-vector of R.

Hibi [H] extended results of Stanley and Björner on simplicial complexes to Artin level algebras. Using combinatorial means he showed that the first half of the h_i's must increase and $h_i \leq h_{\alpha-i}$ for Artin level algebras.

Conjecture: There are no constraints on the **relative** sizes of the last half of the h-vectors of Artin level algebras: if σ is a permutation on $\{\lceil\alpha/2\rceil, \ldots, \alpha-1, \alpha\}$ then there is an Artin level algebra with socle concentrated in level α and $h_{\sigma(\lceil\alpha/2\rceil)} \geq h_{\sigma(\alpha-1)} \geq h_{\sigma(\alpha)}$.

The Gemini Roller Coaster at Cedar Point, Sandusky, OH, USA



Corollary: The Roller Coaster conjecture for well-covered graphs implies the conjecture for Artin level algebras.

References:
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