Enumerative Geometry of Hyperplane Arrangements

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Given an arrangement $A \subset \mathbb{P}^n$ with intersection lattice L there is a moduli space M_L of arrangements with intersection lattice isomorphic to L.

Main Question: Find D = dim M_L and **count the number of arrangements in M_L that pass through D points in $\mathbb{P}^n$ in general position**. Ideally, the answers should be expressed in terms of the combinatorics of the lattice L.

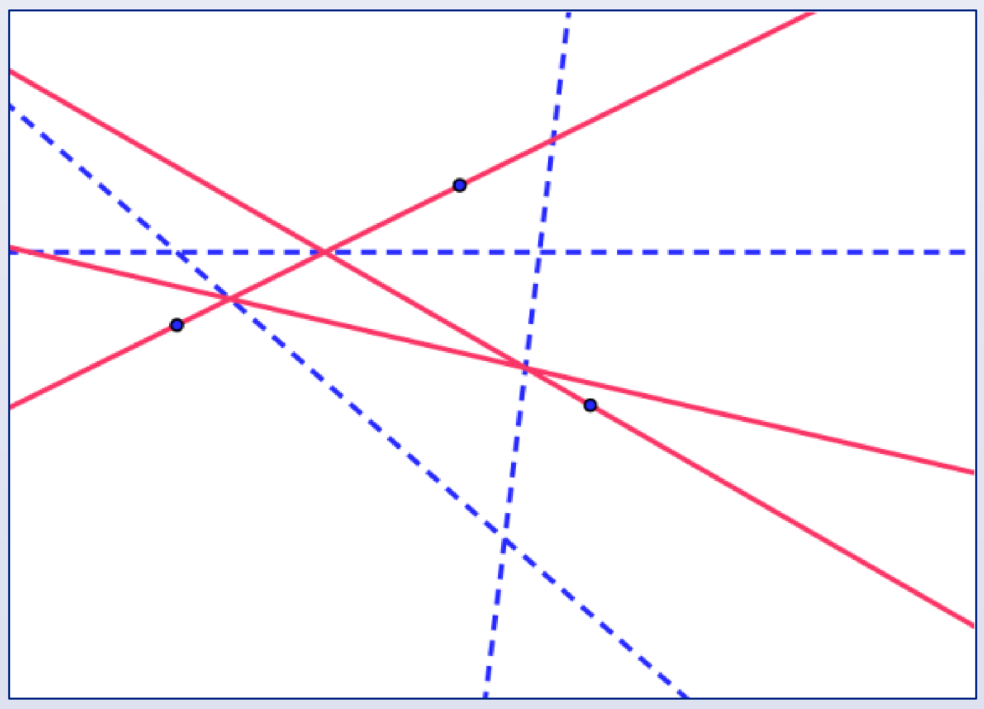
Enumerative questions were studied extensively for curves during the 19th and 20th centuries. Our problem seems related to Schubert Calculus but the questions are complicated by the presence of many hyperplanes. The only enumerative paper related to arrangements that we know of is by Fehér, Némethi and Rimányi but its emphasis is somewhat different than ours.

Example 1: If A consists of k generic hyperplanes in $\mathbb{P}^n$ then dim M_L is kn and there are $\frac{(kn)!}{(n!)^k(k!)}$ arrangements in M_L that pass through kn points in general position. In the projective plane (when n=2) the count equals $T(q=1; x_1=0, x_2=2, x_3=4,\dots,x_k=2k-2)$, where $T(q,x_1,\dots, x_k)$ is the **multivariate Tutte polynomial of L**, a polynomial that encodes the combinatorics of L.

Example 2: When A consists of k generic lines in $\mathbb{P}^2$ we can ask for the **characteristic numbers** – the numbers of such arrangements that pass through p points and are tangent to ℓ lines in general position (with $p+\ell = 2k$).

p	6	5	4	3	2	1	0
ℓ	0	1	2	3	4	5	6
count	15	30	48	57	48	30	15

Characteristic numbers of 3 generic lines through p points and tangent to ℓ lines ($p+\ell=6$).



The 57 arrangements solving the $p=\ell=3$ problem break into 4 combinatorial types, one of which is interesting because each line goes through only one fixed point and no vertices of the triangle (see the picture on the right).

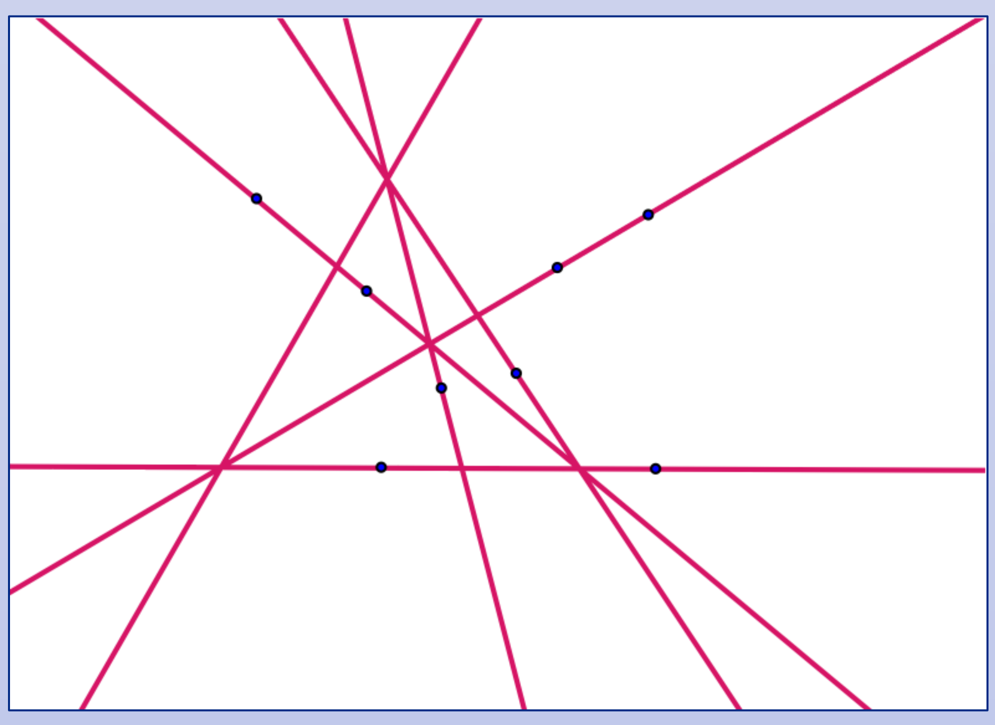
We computed the characteristic numbers using **intersection theory**. Parameterize the 3 generic lines and their 3 intersection points as a single point in $(\mathbb{P}^{2*})^3\times(\mathbb{P}^2)^3$. We need to ensure that each fixed point lies on the union of the 3 lines (a homogeneous condition of multi-degree (1,1,1,0,0,0)) and that each of the 3 given lines pass through one of the 3 intersection points (a multi-degree (0,0,0,1,1,1) condition). Also the intersection point p_{12} must lie on lines ℓ_1 and ℓ_2 (homogeneous conditions of degrees (1,0,0,1,0,0) and (0,1,0,1,0,0)) (similarly for p_{13} and p_{23}). Applying Poincaré duality, each condition corresponds to a **cohomology class** and their intersection (the solution space) has cohomology class

$$(\ell_1+\ell_2+\ell_3)^3 (p_{12}+p_{13}+p_{23})^3(\ell_1+p_{12})(\ell_2+p_{12}) (\ell_1+p_{13})(\ell_3+p_{13}) (\ell_2+p_{23})(\ell_3+p_{23}) \equiv 342(\ell_1\ell_2\ell_3p_{12}p_{13}p_{23})^2 \qquad (*)$$

in $H^*((\mathbb{P}^{2*})^3\times(\mathbb{P}^2)^3, \mathbb{Z}) = \mathbb{Z}[\ell_1,\ell_2,\ell_3,p_{12},p_{13},p_{23}]/(\ell_1^3,\ell_2^3,\ell_3,p_{12}^3,p_{13}^3,p_{23}^3)$. Using Poincaré duality again, the class (*) represents 342 points, each corresponding to a **labeled** arrangement solving our problem. Each arrangement can be labeled in 3! ways, giving 342/3!=57 arrangements of 3 generic lines through 3 points and tangent to 3 lines in general position. The involution on $(\mathbb{P}^{2*})^3\times(\mathbb{P}^2)^3$ gives rise to the symmetry in the table.

Example 3: We can use the characteristic numbers to answer more general enumerative problems using a method developed by **Fulton, Kleiman, and MacPherson**. To compute the number of arrangements with 3 generic lines through 2 points and tangent to 4 smooth curves of degree d, we expand the polynomial $\mu^2(d\mu+d(d-1)\nu)^4/3!$ and replace each term $\mu^p\nu^\ell$ by the corresponding characteristic number.

Example 4: We used the same method to compute the number of arrangements of 4 generic lines tangent to 8 general lines, giving 551880 labeled arrangements. However, there are 151200 **multi-arrangements** with one line of multiplicity 4 and six labeled intersection points $p_{12},\dots,p_{34}$ so that each of the 8 lines contains an intersection point. This leaves 400680 labeled generic arrangements, giving 400680/4! = 16695 arrangements of 4 generic lines tangent to 8 lines in general position. **Dualizing** gives 16695 arrangements of 6 lines with 4 triple points, a **braid arrangement**, through 8 points. We verified this result by counting each of the possible combinatorial types of braid arrangements through 8 points.



One of 16695 braid arrangements through 8 points.

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