Solving Systems of Polynomial Equations

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Systems of Polynomial Equations

Systems of equations like

$$\begin{cases} 3x^2 + 2y^2 = 5 \\ 4x^2 - 3y^3 = 1 \end{cases}$$

arise in all sorts of applications, such as:

- finding critical points of a function
- finding max/mins using Lagrange multipliers
- finding good statistical estimators - e.g. method of moments
- geometric problems in engineering and science

Solution Methods

Today we'll concentrate on three types of solution methods

- Linearization
- Grobner basis techniques
- Eigenspace methods

The common theme in these methods is the use of **linear algebra** to solve **nonlinear equations**.

Linearization

Solve the **nonlinear** system,

$$\begin{cases} x^2 + y^2 & = 2 \\ x^2 - y^2 & = 0 \end{cases}$$

Setting $u = x^2$ and $v = y^2$ we get the **linear** system,

$$\begin{cases} u + v & = 2 \\ u - v & = 0 \end{cases}$$

with solutions $u = v = 1$. Then $x^2 = y^2 = 1$ so $x = \pm 1$ and $y = \pm 1$. We get 4 solutions $(x, y) = (\pm 1, \pm 1)$.

Problems: Need enough equations to uniquely determine monomials.

Elimination

Strategy: Use one equation to solve for one of the variables. Then **eliminate** this variable from the other equations.

Solve

$$\begin{cases} x^2 + y^2 & = 2 \\ x^2 - 4x + y^2 - 4y & = -6 \end{cases}$$

[1] - [2] gives $4x + 4y = 8$ so $y = 2 - x$ and thus

$$x^2 + (2 - x)^2 = 2.$$

This solves to give $x = y = 1$.

Elimination Ideals

Solve

$$\begin{cases} x^2 + y^2 + z^2 &= 4 \\ x^2 + 2y^2 &= 5 \\ xz &= 1 \end{cases}$$

This is equivalent to the system

$$\begin{cases} \mathbf{f} = & x^2 + y^2 + z^2 - 4 &= 0 \\ \mathbf{g} = & x^2 + 2y^2 - 5 &= 0 \\ \mathbf{h} = & xz - 1 &= 0 \end{cases}$$

The solutions also satisfy $x^2\mathbf{f} + 3y^3\mathbf{g} - (z+1)\mathbf{h} = 0$.
In fact, the solutions are killed by all the polynomials in the ideal $(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

Can we find polynomials $F(z), G(y,z), H(x,y,z)$ that generate the same ideal?

Grobner Bases

A Grobner basis is a redundant set of generating polynomials with some extra combinatorial structure.

Ordering on monomials $\rightarrow$ **leading term** of each polynomial
e.g. lex: $x^2 > xy > x > yz$ $\qquad$ $\mathrm{LT}(2x^2 - xy + 3yz) = x^2$.

$\mathrm{LT}(\text{Ideal } I) = \{\mathrm{LT}(f)\colon f \in I\}$ is a monomial ideal in $S = k[x, y, z]$

The **Grobner basis** of I is a generating set $\{F_1, \ldots, F_k\}$ of I so that $\mathrm{LT}(I) = (\mathrm{LT}(F_1), \ldots, \mathrm{LT}(F_k))$.

The monomials not in $LT(I)$ are a basis for the vector space S/I.

Can we find polynomials $F(z), G(y,z), H(x,y,z)$ that generate the ideal $I = (x^2 + y^2 + z^2 - 4, x^2 + 2y^2 - 5, xz - 1)$?

ANS: YES. Compute a gb in lex order:
$I = (\mathbf{x} + 2z^3 - 3z, \mathbf{y^2} - z^2 - 1, 2\mathbf{z^4} - 3z^2 + 1)$.

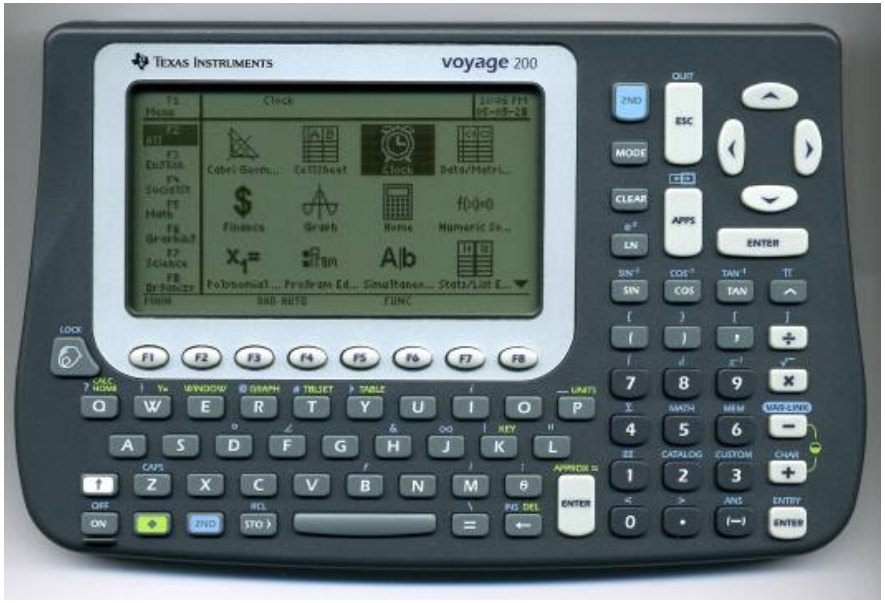
Last poly factors as $(z-1)(z+1)(2z^2-1)$ so $z = \pm 1, \pm 1/\sqrt{2}$.
Plug into second equation to get corresponding y-values (8).
Plug into first equation to get the x-values.

Bruno Buchberger

Comments on Elimination

Elimination using Grobner bases in lex order can take a long time. The Grobner walk speeds up computations but is complicated.

The TI-92+ used Grobner basis methods to run its solve function in exact mode. I don't know about the Voyage 200.



Eigenspace Methods

Solve $x^3 - x^2 + x - 1 = 0$. [Newton's method; factoring]

Let $f = x^2 + 1$ and let $I = (x^3 - x^2 + x - 1)$.
Map $m_f \colon S/I \to S/I$ given by $m_f(g) = fg$.
Basis of S/I is $\{1, x, x^2\}$. Since $fx = x^2 + 1 \bmod I$ and $fx^2 = x^2 + 1 \bmod I$, matrix of m_f wrt basis is

$$M_f = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

THM: The eigenvalues of M_f are precisely the evaluation of f on the zeros of I.

Eigenvalues are 0 and 2. Solutions to $x^3 - x^2 + x - 1 = 0$ are $\{\pm i, 1\}$ and $f(\pm i) = 0$, $f(1) = 2$.

Down to the grid

THM: The eigenvalues of M_f are precisely the evaluation of f on the zeros of I.

COR: The eigenvalues of M_{x_i} are precisely the i-th coordinates of the zeros of I.

This gives a **grid** of points containing the solutions to polynomials in I. Checking each of them gives the actual solutions.

Problems: 1. There may be many more points than solutions.
2. Numerical stability – the eigenvalues may only be determined approximately.

Eigenvector methods

THM: Let $\{1, x_1, \ldots, x_n, \ldots\}$ be a basis of $k[x_1, \ldots, x_n]/I$. Suppose $(M_f)^{tr}$ has only **1-dimensional eigenspaces** with eigenvectors $V_i = (1, a_1, \ldots, a_n, \ldots)$. Then the solutions to I are the truncations $tr(V_i) = (a_1, \ldots, a_n)$.

$$\text{Solve} \begin{cases} 3x^2 + 4xy + y^2 - 7x - 5y + 4 & = 0 \\ 3y^3 + 10xy + 7y^2 - 4x - 20y + 4 & = 0 \\ 3xy^2 - 7xy - 7y^2 - 2x + 11y + 2 & = 0 \end{cases}$$

$$M_{3y} = \begin{bmatrix} 0 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 2 & 4 \\ 3 & 0 & 0 & -11 & 20 \\ 0 & 3 & 0 & 7 & -10 \\ 0 & 0 & 3 & 7 & -7 \end{bmatrix}$$

Basis of S/I: $\{1, x, y, xy, y^2\}$
Eigenvectors: (1,1,0,0,0), (1,1,1,1,1), (1,2,-1,-2,1),(1,-1,2,-2,4),(1,2,-2,-4,4)
Solutions:
(1,0),(1,1),(2,-1),(-1,2),(2,-2).

Approximate Eigenvalues

Use **SVD** to compute eigenvectors of the $n \times n$ matrix $M_f - \lambda\mathbb{I}$:

$$M_f - \lambda\mathbb{I} = USV^{tr}$$

where U, V are $n \times n$ orthonormal matrices and $S = \mathrm{Diag}(s_1, \ldots, s_n)$ with $s_1 \geq \cdots \geq s_n \geq 0$. If $r = \mathrm{rank} M_f$ then the last $n - r$ columns of V are a basis for $\mathrm{Ker}(M_f - \lambda\mathbb{I})$.

If $\epsilon > 0$ is our error tolerance, then choose k so that $s_k > \epsilon > s_{k+1}$. The last $n - k$ rows of V are **approximate eigenvectors** in the sense that $||M_f v - \lambda v||_2 < \epsilon$.

Scale each v to have first component 1 and then truncate to get **approximate** solutions.

Other Methods

There are many other methods. One is the **homotopy method**. Find a deformation P_t of your original polynomial system P_0 with P_1 easy to solve. Then the solutions to P_t satisfy an ODE, the Davidenko DE, and the solutions to P_0 can be computed by path tracing.