From Pascal's Mystic Hexagon Theorem to Secant Varieties

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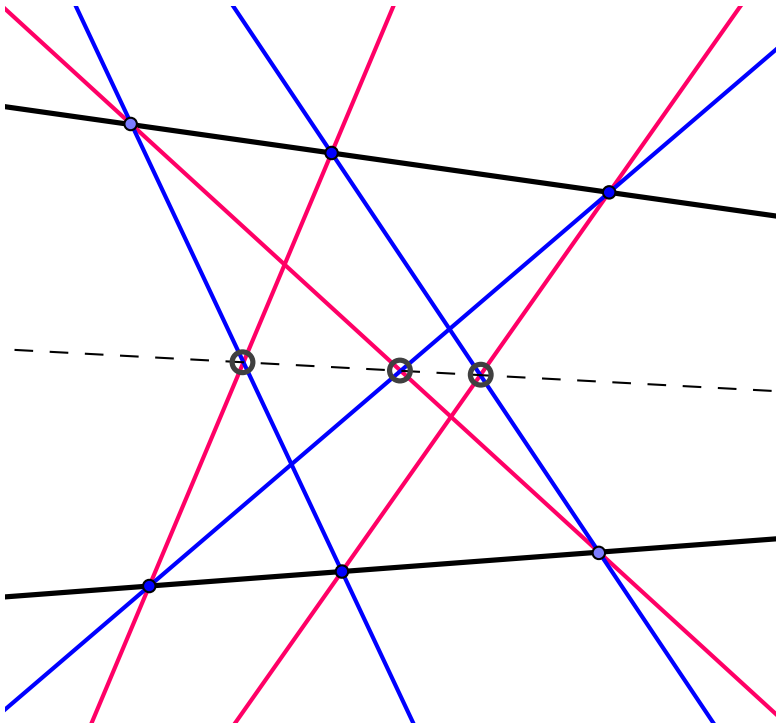
Start at the Beginning

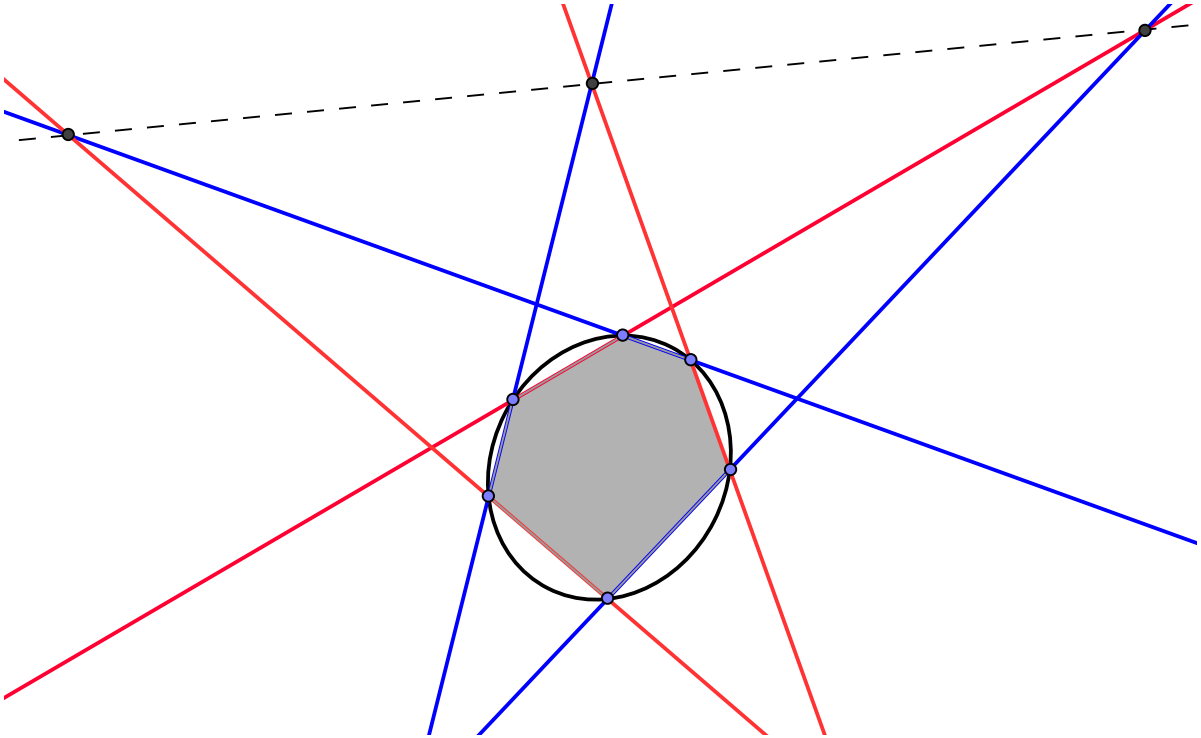
Line Arrangement due to Pappus of Alexandria (4th century C.E.)



Richter-Gebert: Perspectives on Projective Geometry 2011

Pascal's Mystic Hexagon Theorem

Pascal: placed the 6 intersection points on a conic



Converse: Braikenridge-Maclaurin Theorem

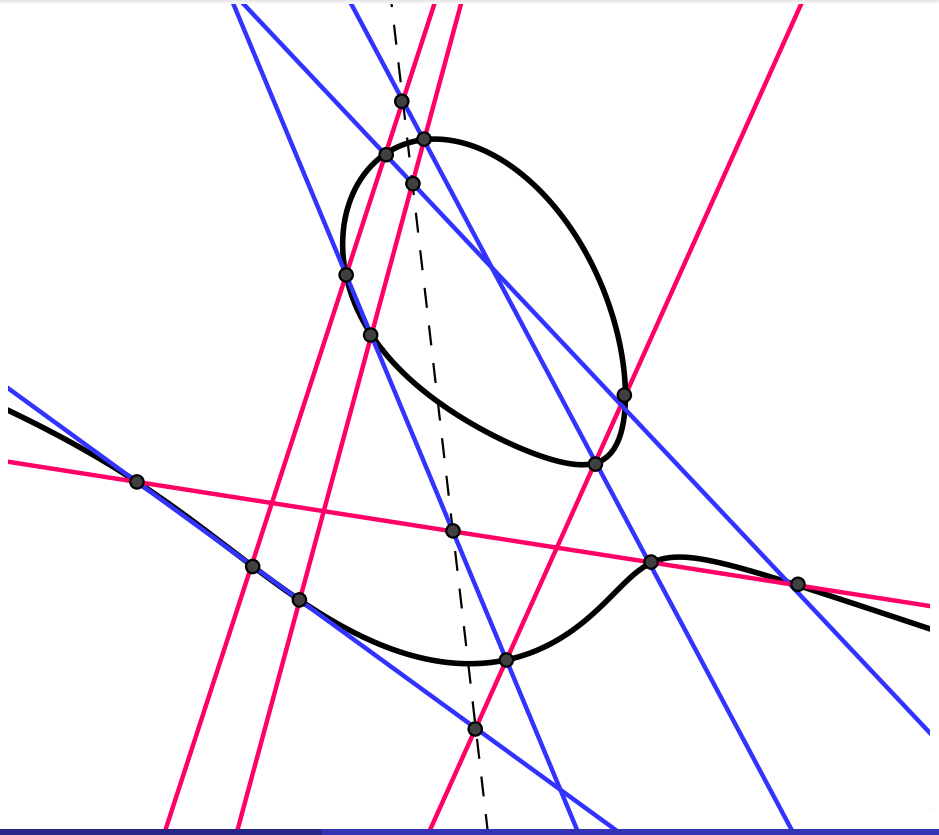
Folklore

Theorem

*Suppose that k **red** lines meet k **blue** lines in a set Γ of k^2 distinct points. If $S = 0$ is an irreducible curve of degree d through kd points of Γ then the remaining points lie on a unique curve C of degree $t = k - d$.*

Proof (Existence): Choose $F_{a,b} = aR + bB$ to vanish at $P \in S \setminus \Gamma$. Then $S = 0$ is a component of $F_{a,b} = 0$ and $F_{a,b}/S$ defines C. $\square$

S degree-d $\rightsquigarrow$ C degree-t
S and C are directly linked



d-Constructible Curves

Definition

A curve C of degree t is **d-constructible** if there exist $k = d + t$ red lines and k blue lines so that:
(1) red lines meet blue lines in set Γ of k^2 distinct points
(2) dk points in Γ lie on a degree-d curve S
(3) the remaining tk points of Γ lie on C.

Definition

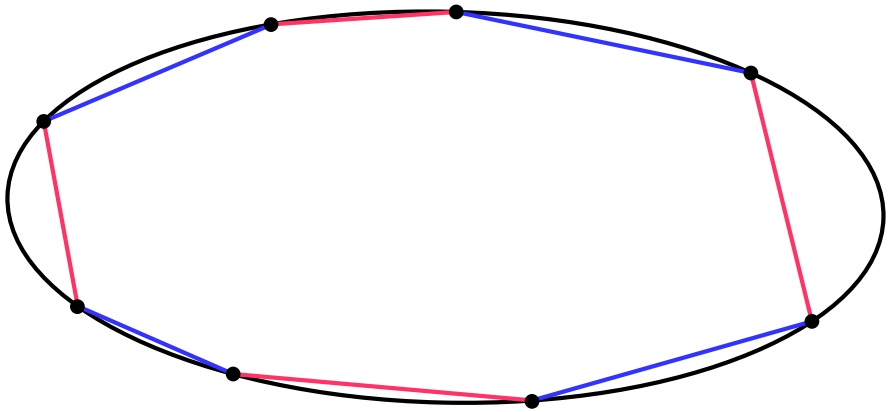
The d-construction is **dense in degree t** if there is a nonempty Zariski-open set $U \subset \mathbb{P}(S_t)$ so that every degree-t curve in U is d-constructible (here $S = \mathbb{C}[x, y, z]$).

Two Examples

Example

For any d, all lines and all irreducible conics are d-constructible.

For lines: arrange $d+1$ points on the line and red and blue lines through these points (with distinct intersection points).

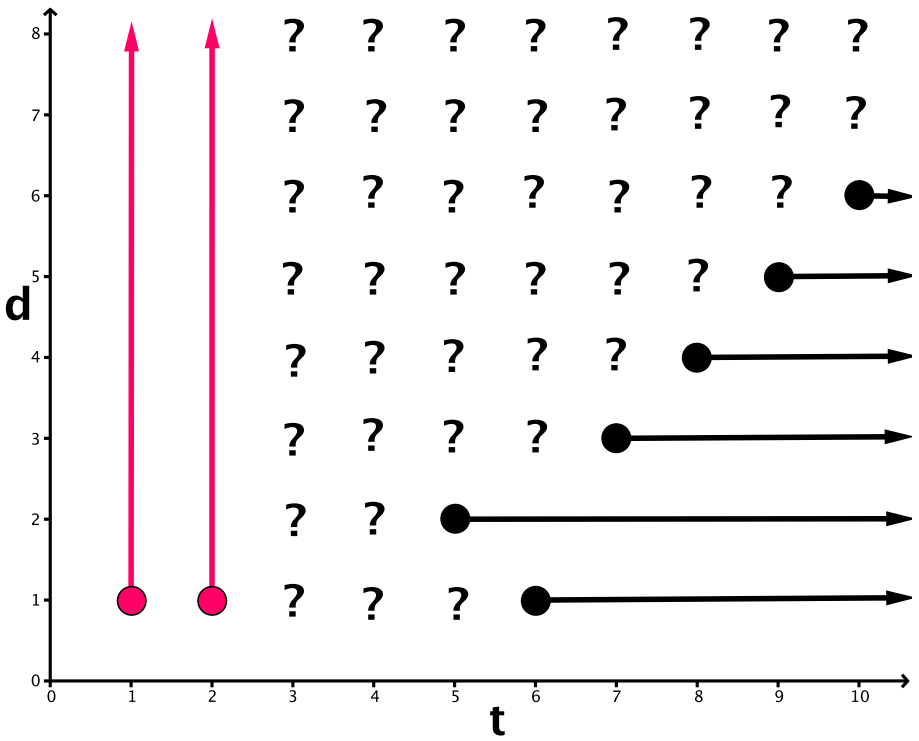


For conics: arrange $2(d+2)$ points, joined cyclicly by alternating red and blue lines.

Counting Dimensions

Theorem

If $d \geq 3$, then the d-construction is not dense in degrees $d + 4$ or higher. The 2-construction is not dense in degrees five or higher. The 1-construction is not dense in degrees six or higher.

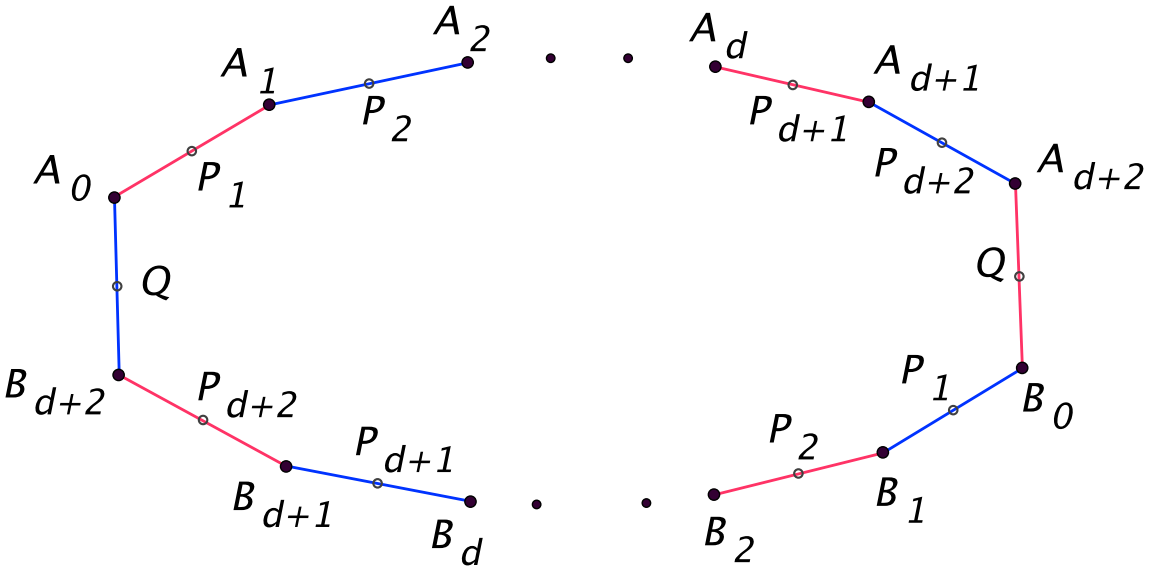


Elliptic Curves

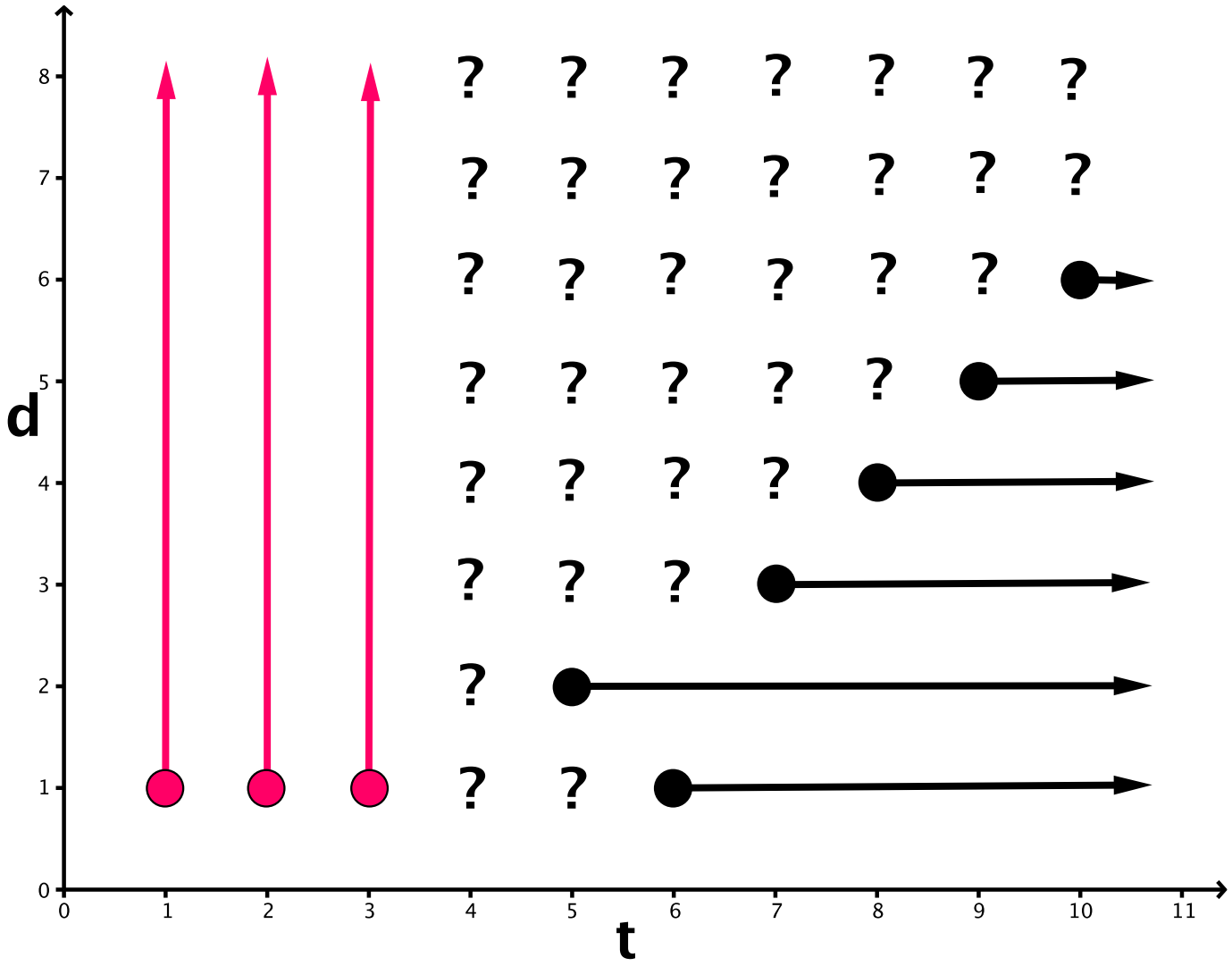
Theorem

All elliptic curves are d-constructible so the d-construction is dense in degree 3.

Need to arrange $3(d+3)$ points on E with a blue and red line through each of them.



Density



Secants and ideas related to tensors

Space of Completely Reducible Polynomials (Chow Variety):

$$\begin{aligned} \mathbb{X}_{1^n} &:= \{[F_1 \cdots F_n] : \text{ each } F_i \in S_1\} \\ &= \text{Image}(\text{Sym}^n(\mathbb{P}(S_1)) \hookrightarrow \mathbb{P}(S_n)) \end{aligned}$$

$$\text{Sec}(\mathbb{X}_{1^n}) = \sigma_2(\mathbb{X}_{1^n}) = \overline{\{aF_1 + bF_2 \in \mathbb{P}(S_n) : F_i \in \mathbb{X}_{1^n} \text{ and } [a:b] \in \mathbb{P}^1\}}$$

Reducible points in $\text{Sec}(\mathbb{X}_{1^n})$ give rise to constructible curves:
$F_1, F_2 \in \mathbb{X}_{1^n}$, and $GH = aF_1 + bF_2 \Rightarrow \mathbb{V}(F_1) \cap \mathbb{V}(F_2) \subset \mathbb{V}(G) \cup \mathbb{V}(H)$.

Expect $\dim \text{Sec}(\mathbb{X}_{1^5}) = \min(2\dim(\mathbb{X}_{1^5}) + 1, \dim \mathbb{P}(S_5)) = 20$.
[Correct: verify with Terracini's Lemma and explicit computation.]

1-Constructible Curves

Theorem

The 1-construction is dense in degrees 4 *and* 5.

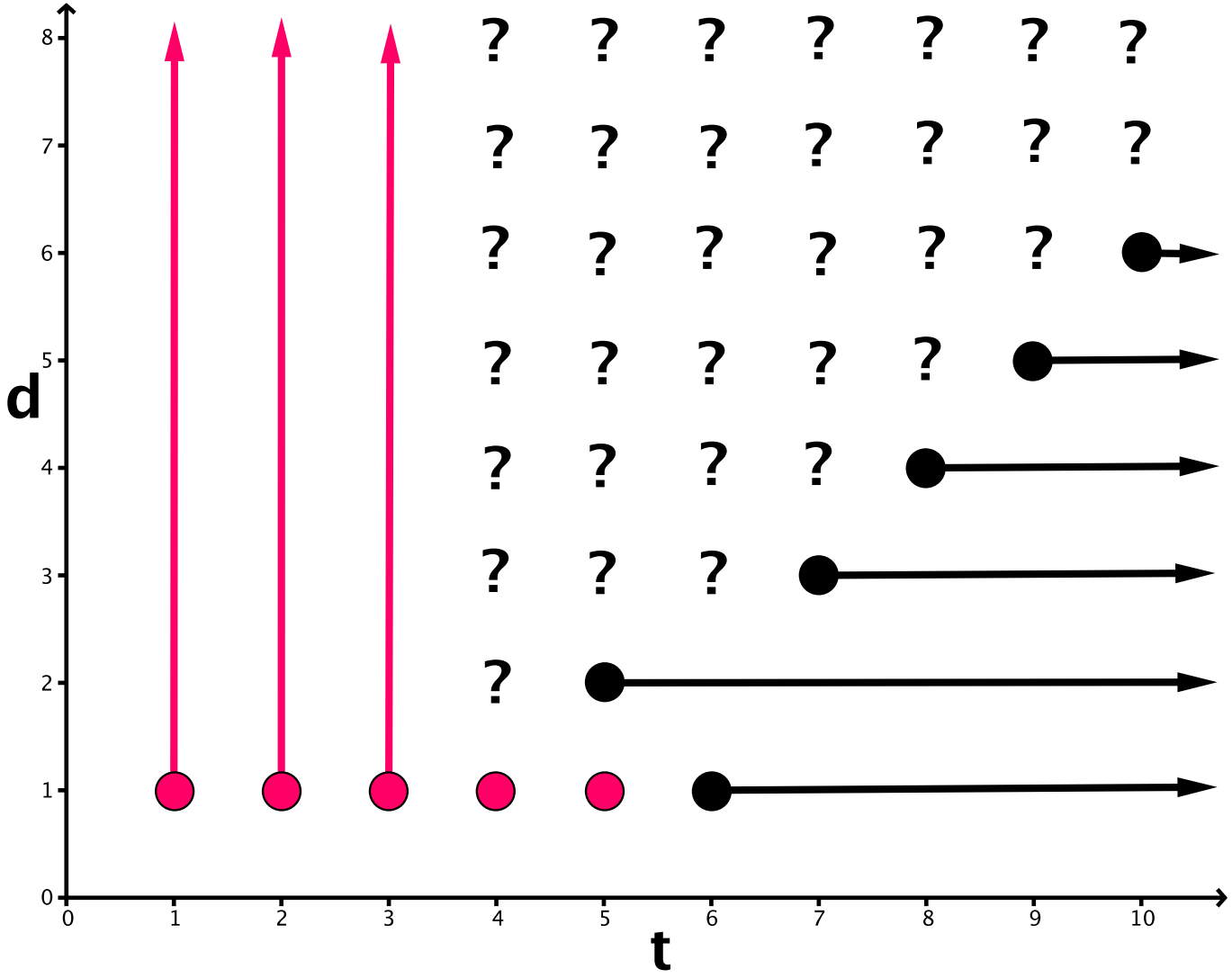
Sketch of proof (degree 4 case only):
$Y = \{(F_1, F_2) \in \mathbb{X}_{1^5} \times \mathbb{X}_{1^5} : \ |\mathbb{V}(F_1, F_2)| = 25\}$ is open.

Image of dominant map $\mathbb{P}^1 \times Y \to Sec(X_{1^5}) = \mathbb{P}(S_5)$ given by $([a : b], F_1, F_2) \mapsto F = aF_1 + bF_2$ is an open set U of $\mathbb{P}(S_5)$.

For $L \in \mathbb{P}(S_1)$, define $\phi_L : \mathbb{P}(S_4) \xrightarrow{\times L} \mathbb{P}(S_5)$. Then $V = \phi_L^{-1}(U) \cap \{\text{irreducible forms}\}$ is open in $\mathbb{P}(S_4)$.

For $G \in V$, $LG = aF_1 + bF_2$ and $\mathbb{V}(LG) \supset \mathbb{V}(F_1, F_2)$. Bézout $\Rightarrow$ 5 points on $\mathbb{V}(L)$ and 20 points on $\mathbb{V}(G)$. So $\mathbb{V}(G)$ is 1-constructible. Open set $V \subset \mathbb{P}(S_4)$ of curves is 1-constructible. An example shows $V \neq \emptyset$. $\square$

Density



Some Questions

- Are the d-constructible curves in a given degree Zariski-closed?
- Is the d-construction dense in degree 4 for all d?
  (Owen Gwilliam, a new postdoc at U.C. Berkeley, suggested that I try to use the Jacobian variety on a degree 4 curve to prove this.)

A test ....

$\mathbb{X}_{1^3} \subset \mathbb{P}(S_3) \cong \mathbb{P}^9$ has dim 6 so most lines $L \in \mathbb{G}(1,9)$ miss $\mathbb{X}_{1^3}$.

Construct secant line $L \in \mathbb{G}(1,9)$ meeting $\mathbb{X}_{1^3}$ in two points (**red** and **blue**).

Question

Are there lines that meet $\mathbb{X}_{1^3}$ in just 3 points?
(Hint: think back 1600 years)

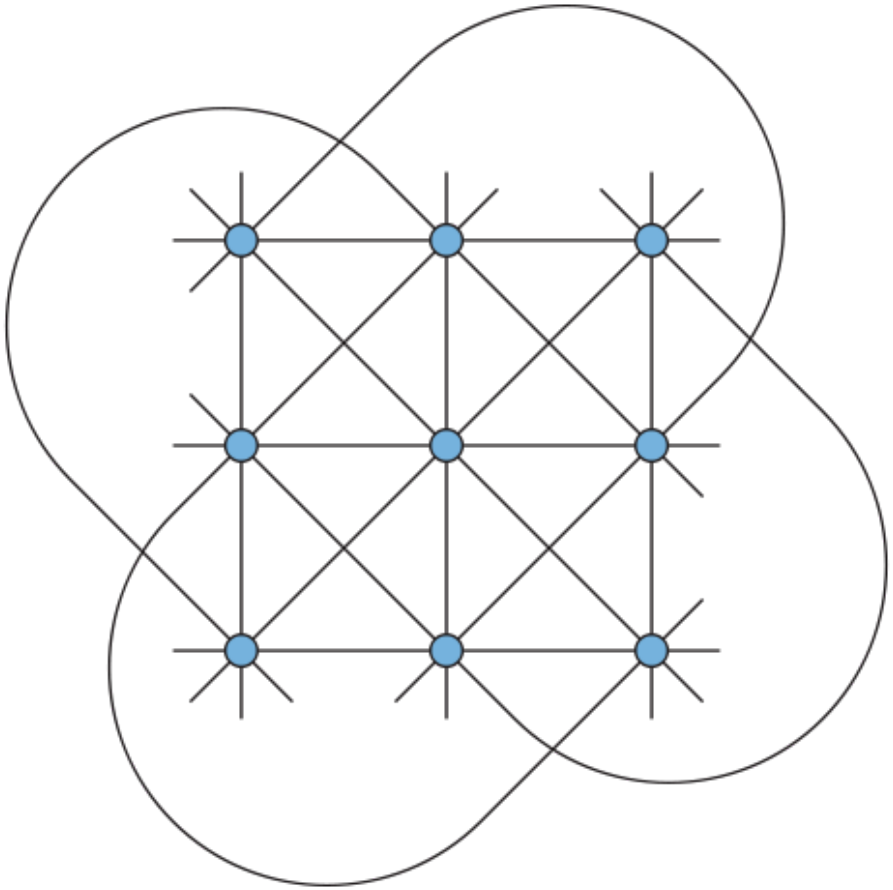
A test .... Part 2

Question

Are there lines that meet $\mathbb{X}_{13}$ in just 4 points?

Hesse Configuration

Answer: The nine points of inflection on an elliptic curve lie on four collections of 3 lines.



Parting Question

Yesterday Daniel Litt and I proved that there are no lines that meet $\mathbb{X}_{1^3}$ in just k points for $4 < k < \infty$.

Question

Let $\phi(d)$ = max finite k s.t. $\exists$ line L in $\mathbb{P}(S_d)$ that meets X_{1^d} in just k points. So $\phi(3) = 4$. What can you say about the behavior of $\phi(d)$ as $d \to \infty$?

(Daniel and I showed that $\phi(d) \leq d + 1$.)