Differential Operators and Invariant Theory

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Outline

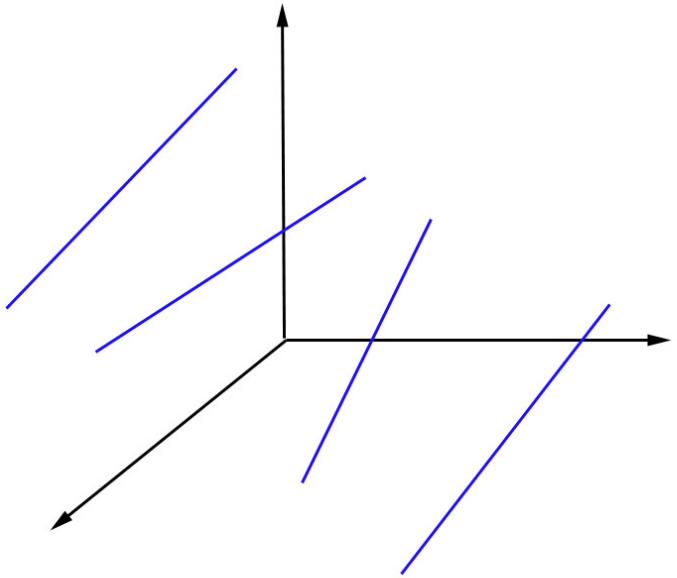
- Ring of invariants R^G
- Geometric problem
- Grassmann varieties
- Fundamental theorems of invariant theory
- Invariant operators
- Hilbert series computations
- Differential ops on Grassmann varieties

Counting Lines in 3-space

Perhaps surprisingly, the study of certain rings of invariants helps us answer geometric questions.

Question

Given four lines in general position in $\mathbb{P}^3$, how many lines meet all four?



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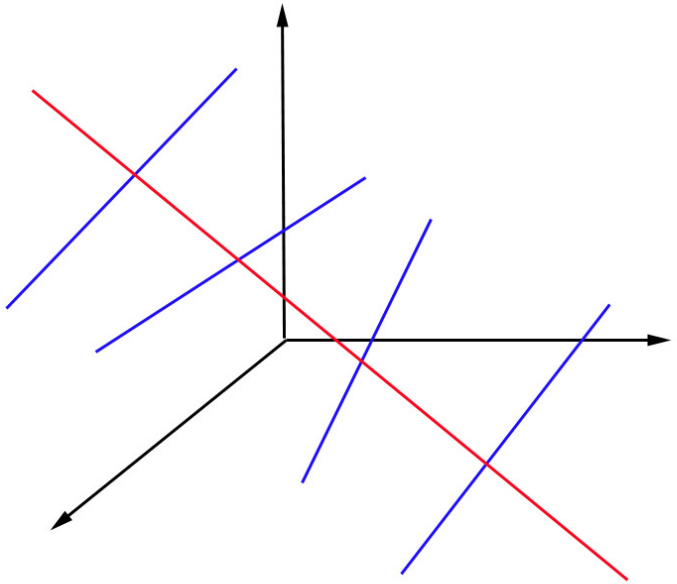


A ring of invariants

- Given a $k \times n$ matrix M filled with variables,

$$M = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix},$$

  we let SL_k act on the left by matrix multiplication.

- If $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\sigma \bullet x_{11} = ax_{11} + bx_{21}$.

- $\mathbb{C}[M_{k,n}] = \mathbb{C}[x_{11}, \ldots, x_{kn}]$

- $\mathbb{C}[M_{k,n}]^{SL_k}$ = polynomials that are invariant on SL_k orbits

Some obvious invariants

- SL_k acts by multiplication on the left so it acts on each column of

$$M = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

  simultaneously.

- The $k \times k$ subdeterminants are invariant because $\sigma \bullet \det(N) = \det(\sigma N) = \det(\sigma)\det(N) = 1\det(N)$.

- Denote the $k \times k$ minor involving rows $\mathrm{I}_1, \ldots, \mathrm{I}_k$ by

$$[\mathrm{I}] = [\mathrm{I}_1 \ldots \mathrm{I}_k].$$

  This is a polynomial of degree k in the variables x_{ij}.
  $\det(N) = [12]$

First Fundamental Theorem of Invariant Theory

The First Fundamental Theorem of Invariant Theory

The $k \times k$ minors generate the ring of invariants $\mathbb{C}[M_{k,n}]^{SL_k}$.

- The ring $\mathbb{C}[M_{k,n}]^{SL_k}$ is called the bracket algebra.

Geometric interpretation

- $G(k,n)$ = k-dimensional subspaces of $\mathbb{C}^n$

$$\begin{array}{lcl} G(k,n) & \longrightarrow & \mathbb{P}(\wedge^k(\mathbb{C}^n)) = \mathbb{P}^{\binom{n}{k}-1} \\ V_k \subset \mathbb{C}^n & \longmapsto & \wedge^k V \end{array}$$

- Plücker coordinates on $\mathbb{P}(\wedge^k(\mathbb{C}^n))$
  = coeff of $dv_{I_1} \wedge \cdots \wedge dv_{I_k}$
  correspond to the brackets $[\mathrm{I}]$

- By 1st FTIT,
  $\mathbb{C}[G(k,n)] = \mathbb{C}\langle[\mathrm{I}]\rangle = \mathbb{C}[M_{k,n}]^{SL_k}$

H. Grassmann

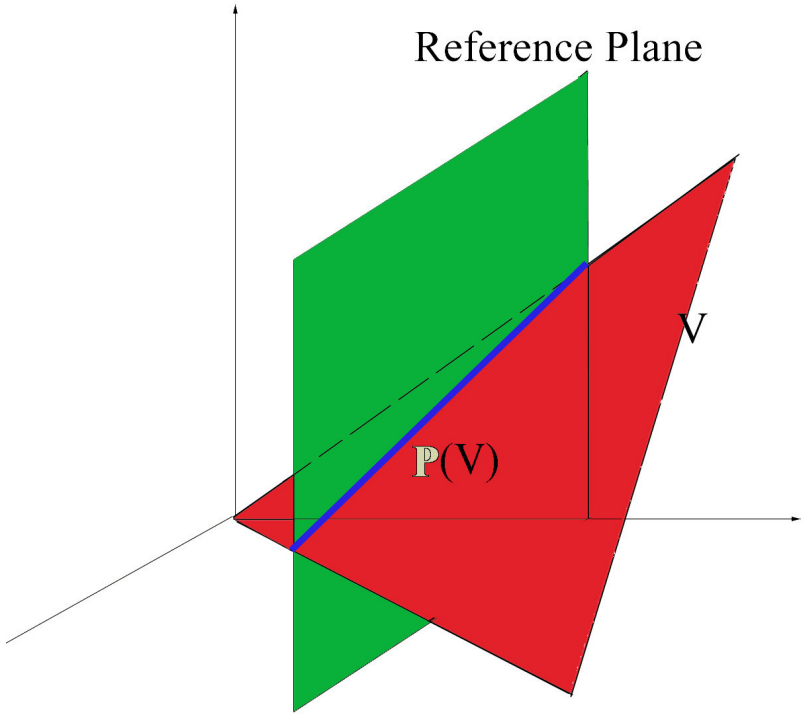
Projectivization

- $k = 2$, $n = 4 \Rightarrow \mathbb{C}(G(2,4)) = \mathbb{C}[M_{2,4}]^{SL_2}$ generated by $\binom{4}{2} = 6$ brackets so $G(2,4) \subset \mathbb{P}^5$.

- Correspondence: $G(2,4) \cong \mathbb{G}(1,3) =$ lines in $\mathbb{P}^3$

- Notation $G(2,4) =$ affine cone over $\mathbb{G}(1,3) \subset \mathbb{P}^5$.

- $\mathbb{G}(1,3)$ is the proj variety with proj coordinate ring $\mathbb{C}[M_{2,4}]^{SL_2}$



Second Fundamental Theorem of Invariant Theory

The 2nd Fundamental Theorem of Invariant Theory

The generators of $\mathbb{C}[M_{k,n}]^{SL_k}$ satisfy the relation

$$\sum_{\ell=1}^{k+1}(-1)^{\ell}[\mathrm{I}_1\ldots\mathrm{I}_{k-1}\mathrm{J}_\ell][\mathrm{J}_1\ldots\widehat{\mathrm{J}_\ell}\ldots\mathrm{J}_{k+1}]=0.$$

All the relations come from these quadratic relations.

Example

($k=2$, $n=4$) If $\mathrm{I}=1$ and $\mathrm{J}=234$ then the relation is

$$-[12][34]+[13][24]-[14][23]=0.$$

All other relations are multiples of this one.

Lines in 3 space

Question

Given four lines in general position in $\mathbb{P}^3$, how many lines meet all four?

Answer

- $\mathbb{G}(1,3)$ is a quadratic hypersurface in $\mathbb{P}^5$ so has dimension 4.
- To meet a fixed line is a linear condition on $\mathbb{G}(1,3)$.
- If 4 independent linear conditions are imposed, we expect $\deg(\mathbb{G}(1,3)) = 2$ points.

Another invariant ring

- $\mathbb{C}[M_{k,n}] = \mathbb{C}[V^n] = \mathbb{C}[V_k \oplus V_k \oplus \cdots \oplus V_k]$
- $\mathbb{C}[V^s \oplus (V^*)^t] = \mathbb{C}\begin{bmatrix} x_{11} & \cdots & x_{1s} & \xi_{11} & \cdots & \xi_{1s} \\ & \vdots & & & \vdots & \\ x_{k1} & \cdots & x_{ks} & \xi_{k1} & \cdots & \xi_{ks} \end{bmatrix}$
- Coordinates on (V^*) are $\xi_{1\cdot}, \ldots, \xi_{k\cdot}$ with $\langle \xi_{i\cdot}, x_{j\cdot} \rangle = \delta_{ij}$.
- SL_k acts on the ξ_{ij} by the contragredient representation
  - If $\sigma \in SL_k$ acts on $\mathbb{C}[V]$ by A then σ acts on $\mathbb{C}[V^*]$ by $(A^{-1})^T$

Some invariants

- $\begin{bmatrix} x_{11} & \cdots & x_{1s} & \xi_{11} & \cdots & \xi_{1s} \\ & \vdots & & & \vdots & \\ x_{k1} & \cdots & x_{ks} & \xi_{k1} & \cdots & \xi_{ks} \end{bmatrix}$
- Define: $|\mathrm{J}| = |\mathrm{J}_1 \ldots \mathrm{J}_k| = k \times k$ subdeterminant of the ξ_{ij}
- Inner product: $v \in V, w \in V^* \Rightarrow$

$$\langle Av, (A^{-1})^T w \rangle = (Av)^T (A^{-1})^T w = v^T A^T (A^T)^{-1} w = v^T w = \langle v, w \rangle$$

- New invariants:

$$\begin{array}{rcl} \langle ij \rangle : V^s \oplus (V^*)^t & \longrightarrow & \mathbb{C} \\ (v_1, \ldots, v_s, w_1, \ldots, w_t) & \longmapsto & \langle v_i, w_j \rangle \end{array}.$$

- In coordinates: $\langle ij \rangle = \sum_{\ell=1}^{k} x_{\ell i} \xi_{\ell j}$.

First Fundamental Theorem of Invariant Theory

First Fundamental Theorem of Invariant Theory

The invariants $[\mathrm{I}]$, $|\mathrm{J}|$ and $\langle ij \rangle$ generate the ring $\mathbb{C}[V^s \oplus (V^*)^t]^{SL_k}$.

H. Weyl

Second Fundamental Theorem of Invariant Theory

Second Fundamental Theorem of Invariant Theory

The relations are:

$$\sum_{\ell=1}^{k+1}(-1)^\ell[\mathrm{I}_1\ldots\mathrm{I}_{k-1}\mathrm{J}_\ell][\mathrm{J}_1\ldots\widehat{\mathrm{J}_\ell}\ldots\mathrm{J}_{k+1}]=0$$
$$\sum_{\ell=1}^{k+1}(-1)^\ell|\mathrm{I}_1\ldots\mathrm{I}_{k-1}\mathrm{J}_\ell||\mathrm{J}_1\ldots\widehat{\mathrm{J}_\ell}\ldots\mathrm{J}_{k+1}|=0$$
$$\sum_{\ell=1}^{k+1}(-1)^\ell[\mathrm{I}_1\mathrm{I}_2\cdots\hat{\mathrm{I}}_\ell\cdots\mathrm{I}_{k+1}]\langle\mathrm{I}_\ell\mathrm{J}_1\rangle=0$$
$$\sum_{\ell=1}^{k+1}(-1)^\ell\langle\mathrm{I}_1\mathrm{J}_\ell\rangle|\mathrm{J}_1\mathrm{J}_2\cdots\hat{\mathrm{J}}_\ell\cdots\mathrm{J}_{k+1}|=0$$
$$\det(\langle\mathrm{IJ}\rangle)=\det(\langle\mathrm{I}_a\mathrm{J}_b\rangle)_{a,b=1}^k=[\mathrm{I}_1\mathrm{I}_2\cdots\mathrm{I}_k]|\mathrm{J}_1\mathrm{J}_2\ldots\mathrm{J}_k|$$

Example

If $[\mathrm{I}]=[134]$, $|\mathrm{J}|=[235]$ then

$$\det\begin{bmatrix}\langle 12\rangle & \langle 13\rangle & \langle 15\rangle\\ \langle 32\rangle & \langle 33\rangle & \langle 35\rangle\\ \langle 42\rangle & \langle 43\rangle & \langle 45\rangle\end{bmatrix}=\det\begin{bmatrix}x_{11} & x_{13} & x_{14}\\ x_{21} & x_{23} & x_{24}\\ x_{31} & x_{33} & x_{34}\end{bmatrix}\det\begin{bmatrix}\xi_{12} & \xi_{13} & \xi_{15}\\ \xi_{22} & \xi_{23} & \xi_{25}\\ \xi_{32} & \xi_{33} & \xi_{35}\end{bmatrix}$$

The Weyl algebra

- $R = \mathbb{C}[x_1, \ldots, x_n]$ is the coordinate ring for $\mathbb{C}^n$
- The ring of differential operators on $\mathbb{C}^n$ is the Weyl algebra

$$D(\mathbb{C}^n) = D(R) = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

- $\partial_i = \partial/\partial x_i \Rightarrow \partial_i x_j = x_j \partial_i + \delta_{ij}$ Product rule
  - $$\begin{aligned} (\partial_i x_j) \bullet f &= \partial_i \bullet (x_j f) \\ &= x_j \partial_i \bullet f + \delta_{ij} f \\ &= (x_j \partial_i + 1) \bullet f \end{aligned}$$

Filtration by order

- $R = \mathbb{C}[x_1, \ldots, x_n]$ is the coordinate ring for $\mathbb{C}^n$
  $D(R) = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$

- Filtration by order: $x_1\partial_1\partial_2 + \partial_1 - 3x_2$ has order 2

- Symbol map:

$$\begin{array}{rcl} D(R) & \longrightarrow & GrD(R) = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n] \\ \partial_i & \longmapsto & \xi_i \\ x_1\partial_1\partial_2 + \partial_1 - 3x_2 & \longmapsto & x_1\xi_1\xi_2 \end{array}$$

- If M is a $D(R)$ module then M is generated by lifts of the generators of $Gr(M)$ as a $GrD(R)$ module.

The group action on $D(R)$

- We are interested in the case where $R = \mathbb{C}[M_{k,n}]$
- SL_k acts on each variable ∂_{ij} in the same way it acts on ξ_{ij} $(A^{-1})^T$
- $Gr[D(R)^{SL_k}] = [GrD(R)]^{SL_k} = \mathbb{C}[V^n \oplus (V^*)^n]^{SL_k}$
- Lifts of the generators for $Gr[D(R)^{SL_k}]$ generate $D(R)^{SL_k}$

1st Fundamental Theorem for $D(R)^{SL_k}$

The generators have the form $[\mathrm{I}]$, $|\mathrm{J}|$ and $\langle ij \rangle$, where ξ_{ij} is replaced by ∂_{ij}.

2nd Fundamental Theorem for $D(R)^{SL_k}$

2nd Fundamental Theorem of Invariant Theory for $GrD(R)^{SL_k}$

The relations are:

$$\sum_{\ell=1}^{k+1}(-1)^\ell[\mathrm{I}_1\ldots\mathrm{I}_{k-1}\mathrm{J}_\ell][\mathrm{J}_1\ldots\widehat{\mathrm{J}_\ell}\ldots\mathrm{J}_{k+1}]=0$$
$$\sum_{\ell=1}^{k+1}(-1)^\ell|\mathrm{I}_1\ldots\mathrm{I}_{k-1}\mathrm{J}_\ell||\mathrm{J}_1\ldots\widehat{\mathrm{J}_\ell}\ldots\mathrm{J}_{k+1}|=0$$
$$\sum_{\ell=1}^{k+1}(-1)^\ell[\mathrm{I}_1\mathrm{I}_2\cdots\hat{\mathrm{I}}_\ell\cdots\mathrm{I}_{k+1}]\langle\mathrm{I}_\ell\mathrm{J}_1\rangle=0$$
$$\sum_{\ell=1}^{k+1}(-1)^\ell\langle\mathrm{I}_1\mathrm{J}_\ell\rangle|\mathrm{J}_1\mathrm{J}_2\cdots\hat{\mathrm{J}}_\ell\cdots\mathrm{J}_{k+1}|=0$$
$$\det(\langle\mathrm{IJ}\rangle)=\det(\langle\mathrm{I}_a\mathrm{J}_b\rangle)_{a,b=1}^k=[\mathrm{I}_1\mathrm{I}_2\cdots\mathrm{I}_k]|\mathrm{J}_1\mathrm{J}_2\ldots\mathrm{J}_k|$$

- The first four relations extend trivially to $D(R)^{SL_k}$
- The last relation needs to be modified:

$$[\mathrm{I}]|\mathrm{J}|=\det\langle\mathrm{IJ}\rangle-\sum_{\emptyset\neq S\subset\mathrm{I}\cap\mathrm{J}}\alpha_{S\mathrm{IJ}}\beta_{S\mathrm{IJ}}[\mathrm{I}\setminus S]|\mathrm{J}\setminus S|$$

- $[134]|234|=\det\langle 134,234\rangle+\det\langle 14,24\rangle+2\det\langle 13,23\rangle+2\langle 12\rangle$

The Hilbert Series of $GrD(M_{2,4})^{SL_2}$

- Set $k = 2$ and $n = 4$. Then $R = \mathbb{C}[M_{2,4}]$
- $GrD(R) = \mathbb{C}[x_{ij}, \xi_{ij}]$ is a graded ring using total degree
- $H(GrD(R)^{SL_2}, t) = \sum_{d=0}^{\infty} \dim \left(GrD(R)_d^{SL_2} \right) \; t^d$
  - tells us about the structure of the invariant ring
- Usually need a resolution to get the Hilbert Series but we can avoid using the relations entirely!

The Torus-Hilbert series

- SL_2 contains a diagonal torus: $\sigma_z = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \in T$
- $\sigma_z x_{1j} = z x_{1j}, \qquad \sigma_z x_{2j} = z^{-1} x_{2j}, \qquad \sigma_z \xi_{1j} = z^{-1} \xi_{1j}, \qquad \sigma_z \xi_{2j} = z \xi_{2j}$
- If T acts diagonally on a v.s. W with matrix $\rho_z = \begin{bmatrix} z^{a_1} & & \\ & \ddots & \\ & & z^{a_n} \end{bmatrix}$

  then dim W^T = coeff of z^0 in $Tr(\rho_z)$
- $W = \oplus W_d \Rightarrow Define : H_T(W, z, t) = \sum_d Tr(\rho_z \text{ on } W_d) t^d$
- $W = GrD(R) \Rightarrow H_T(W, z, t) = \dfrac{1}{(1 - zt)^8 (1 - z^{-1} t)^8}$

The Weyl character formula

- $W = \oplus_\lambda W_\lambda^{a_\lambda}$ as SL_2-modules
- $a_0 = \dim W^{SL_2}$

Theorem (Weyl character formula)

$Tr(\rho_z \text{ on } W_\lambda)(z - z^{-1}) = z^{1+\lambda} - z^{-(1+\lambda)}$

- Summing over the representations in W:
  $Tr(\rho_z \text{ on } W)(z - z^{-1}) = \sum_\lambda a_\lambda (z^{1+\lambda} - z^{-(1+\lambda)})$
- $\dim W^{SL_2} = a_0 \quad = \quad$ coeff of z^1 in $Tr(\rho_z \text{ on } W)(z - z^{-1})$

The Hilbert series of $GrD(R)^{SL_2}$

- $\dim W^G = a_0 =$ coeff of z^1 in $Tr(\rho_z \text{ on } W)(z - z^{-1})$
- $W = \oplus_d W_d \Rightarrow$

$$\begin{aligned} H(W^{SL_2}, t) &= \textstyle\sum_d \dim W_d^{SL_2} t^d \\ &= \text{coeff of } z \text{ in } \textstyle\sum_d Tr(\rho_z \text{ on } W) t^d (z - z^{-1}) \\ &= \text{coeff of } z \text{ in } H_T(W, z, t)(z - z^{-1}) \end{aligned}$$

- $H(GrD(R)^{SL_2}, t) =$ coeff of z^0 in $H_T(GrD(R), z, t)(1 - z^{-2})$

Coefficient extraction

- $H(GrD(R)^{SL_2}, t) =$ coeff of z^0 in

$$H_T(GrD(R), z, t)(1 - z^{-2}) = \frac{1 - z^{-2}}{(1 - zt)^8(1 - z^{-1}t)^8}$$

- Series converges if $|zt| < 1$ and $|z^{-1}t| < 1$: choose $t < 1$ & $|z| = 1$
- Coeff of z^0

$$\begin{aligned} &= \frac{1}{2\pi i}\int_C \frac{1-z^{-2}\,dz}{z(1-zt)^8(1-z^{-1}t)^8} \\ &= \mathrm{Res}_{z=t}\left(\frac{(1-z^{-2})z^7}{(1-zt)^8(z-t)^8}\right) \\ &= 7^{\text{th}} \text{ coeff in p.s. exp of } \frac{(1-z^{-2})z^7}{(1-zt)^8} \text{ at } z = t \\ &= \frac{1+15t^2+50t^4+50t^6+15t^8+t^{10}}{(1-t^2)^{13}}. \end{aligned}$$

Example

When $R = \mathbb{C}[M_{2,4}]$, $\quad H(GrD(R)^{SL_2}, t) = \frac{1+15t^2+50t^4+50t^6+15t^8+t^{10}}{(1-t^2)^{13}}$.

Differential operators on the Grassmann variety

- $Y = \mathbb{V}(\mathrm{I}) \subseteq \mathbb{C}^n$: $\mathbb{C}[Y] = R/\mathrm{I} = \mathbb{C}[x_1, \ldots, x_n]/\mathrm{I}$
  $Y = G(2,4) \subset \mathbb{C}^6$
- $D(Y) = D(R/\mathrm{I}) = \dfrac{\{\theta \in D(R) : \theta(\mathrm{I}) \subseteq \mathrm{I}\}}{\mathrm{I}D(R)}$
- Practical problems in determining $D(Y)$
- Restricting operators to R^{SL_k} gives $\pi_* : D(R)^{SL_k} \to D(R^{SL_k})$

From $D(R)^{SL_k}$ to $D(R^{SL_k})$

Theorem (Schwarz)

The map $\pi_* : D(R)^{SL_k} \to D(R^{SL_k})$ is surjective. So $D(R^{SL_k})$ is generated by the restriction of the operators $[\mathrm{I}], |\mathrm{J}|$, and $\langle ij \rangle$.

Theorem (Schwarz)

The kernel of π_* is just $(D(R)\mathfrak{sl}_k)^{SL_k}$

- $\mathfrak{sl}_2 =$ Trace zero matrices
- $\mathfrak{sl}_2$ is generated by $g_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $g_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and

$$g_{11} - g_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Each matrix acts as a derivation on $R = \mathbb{C}[M_{2,4}]$

The kernel of π_*

- $\ker \pi_* = (D(R)\mathfrak{sl}_2)^{SL_2}$
- At one time I believed that the kernel of π_* was generated by the Casimir operator
- This is incorrect! One way to show this is to compute the Hilbert series of $Gr(\ker \pi_*)$

An equivariant resolution

- $Gr(D(R)\mathfrak{sl}_2)$ generated by symbols of g_{12}, g_{21} and $g_{11} - g_{22}$
- These form a regular sequence in the polynomial ring $GrD(R)$
- Koszul resolution

$$\begin{aligned} 0 \quad &\to GrD(\mathbb{C}[V^4])(-6) \quad \to GrD(\mathbb{C}[V^4])(-4)^3 \\ &\to GrD(\mathbb{C}[V^4])(-2)^3 \quad \to GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C} \quad \to 0. \end{aligned}$$

- These three operators are eigenvectors for the torus action
- Torus weights can be assigned to the resolution's generators so that it is SL_2 equivariant

Hilbert series of $Gr(\ker \pi_*)$

- Koszul resolution

$$\begin{aligned} 0 \quad &\to GrD(\mathbb{C}[V^4])(-6) \quad \to GrD(\mathbb{C}[V^4])(-4)^3 \\ &\to GrD(\mathbb{C}[V^4])(-2)^3 \quad \to GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C} \quad \to 0. \end{aligned}$$

- Find $H_T(\mathrm{Res}, z, t)$ as before and find $H(Res^{SL_2}, t)$ as the coeff of z^0 in $(1 - z^{-2})H_T(\mathrm{Res}, z, t)$
- Hilbert series for $Gr(\ker \pi_*) = Gr(D(R)\mathfrak{sl}_2)^{SL_2}$ is alternating sum of $H(Res^{SL_2}, t)$ so

$$H(Gr \ker \pi_*, t) = \frac{36t^4 + 127t^6 + 15t^8 - 76t^{10} + 14t^{12} + 15t^{14} + t^{16}}{(1 - t^2)^{13}}$$

Hilbert series of $GrD(R^{SL_2})$

- Get the Hilbert series for

$$Gr(D(G(2,4))) = GrD(R^{SL_2}) = Gr\left(\frac{D(R)^{SL_2}}{\ker \pi_*}\right) = \frac{GrD(R)^{SL_2}}{Gr \ker \pi_*}$$

by subtraction

- $$\begin{aligned} H(GrD(R^{SL_2}), t) &= H(GrD(R)^{SL_2}, t) - H(Gr \ker \pi_*, t) \\ &= \frac{1 + 18t^2 + 65t^4 + 65t^6 + 18t^8 + t^{10}}{(1-t^2)^{10}} \end{aligned}$$

Summary

- $R = \mathbb{C}[M_{k,n}]^{SL_k}$ coordinate ring of Grassmann variety used in intersection theory of linear spaces
- Fundamental Theorem of Invariant theory gives gens & relations
- Extension: presented $D(R)^{SL_k}$
- Computed Hilbert series $H(GrD(R)^{SL_2}, t)$
- Investigated $D(R^{SL_2})$ using the map π_* and found $H(GrD(R^{SL_2}), t)$

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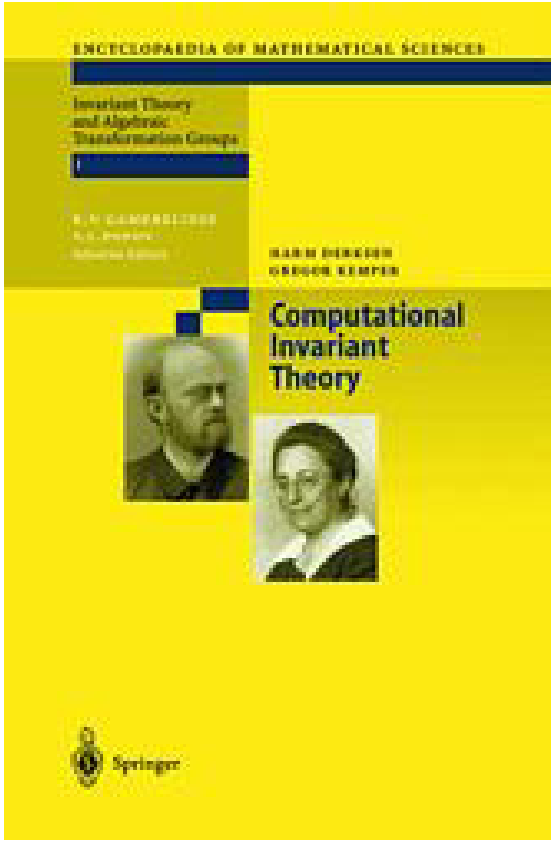
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