Subspace Arrangements Yearning for Freedom

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AMS Section Meeting, Lexington, KY, 27 MAR 2010
Special Session on Combinatorial Algebra

Outline

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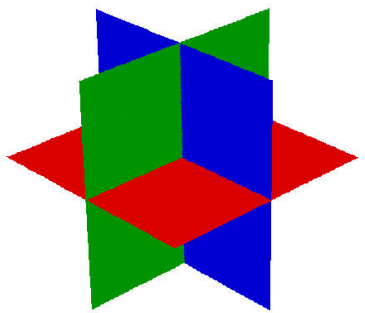
- **Free** hyperplane arrangements
- From hyperplane arrangements to **subspace** arrangements
- **Derivation radical** arrangements
- **Equations** for subspace arrangements
- Questions for further investigation

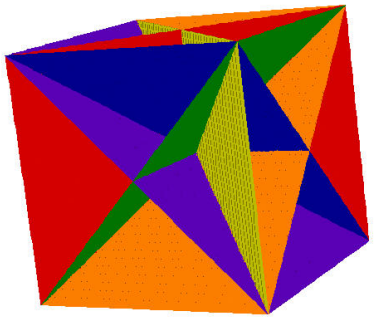
Hyperplane Arrangements

An **arrangement** $\mathcal{A}$ is a finite collection of hyperplanes in $V \cong \mathbb{C}^\ell$

Boolean Arrangement
$\mathcal{A} = \{\mathbb{V}(x_1), \ldots, \mathbb{V}(x_\ell)\}$
Defining polynomial $Q = x_1 \ldots x_\ell$.



Braid Arrangement
$\mathcal{A}_\ell = \{\mathbb{V}(x_i - x_j) : 1 \leq i < j \leq \ell\}$
Defining polynomial $Q = \prod_{i<j}(x_i - x_j)$.



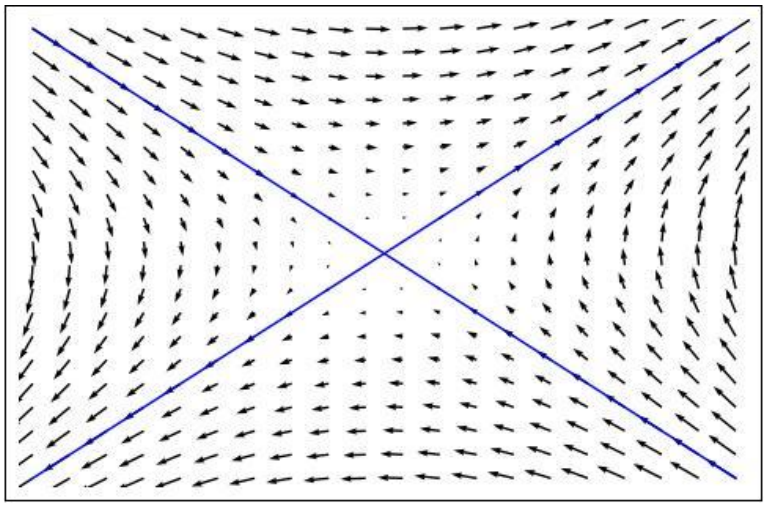
Both arrangements are highly symmetric and highly singular.
They are **free** arrangements.

Free Arrangements

$\mathcal{A}$: a hyperplane arrangement with defining polynomial $Q \in S = \mathbb{C}[x_1, \ldots, x_\ell]$

A **derivation on $\mathcal{A}$** is a derivation θ on S so that $\theta(Q) \in (Q)$ (polynomial vector fields on V **tangent** to all the hyperplanes in $\mathcal{A}$)

The derivation $y\partial_x + x\partial_y$ on $\mathcal{A}$ with $Q = (x - y)(x + y)$ corresponds to the polynomial vector field $\langle y, x \rangle$



The collection $Der(\mathcal{A})$ of derivations on $\mathcal{A}$ is a module over S.

The arrangement $\mathcal{A}$ is **free** if $Der(\mathcal{A})$ is a free S-module.

Topology of Free Arrangements

The complement of a free hyperplane arrangement has a nice topology.

Theorem (Terao's factorization theorem)

If $\mathcal{A}$ is a free hyperplane arrangement then the Hilbert series of $H^*(V \setminus \bigcup \mathcal{A})$ factors as $\prod(1 + d_i t)$ where the d_i are the degrees of a minimal system of generators of $Der(\mathcal{A})$.

Saito's criterion for Freeness

$\mathcal{A}$: hyperplane arrangement in $\mathbb{C}^\ell$ with $\theta_1, \ldots, \theta_t$ generating $Der(\mathcal{A})$
$\mathbf{M}(\mathcal{A}) = (\theta_i(x_j))$
Minors$(\mathcal{A})$ = ideal of $\ell \times \ell$-minors of $\mathbf{M}(\mathcal{A})$

Theorem (Saito)

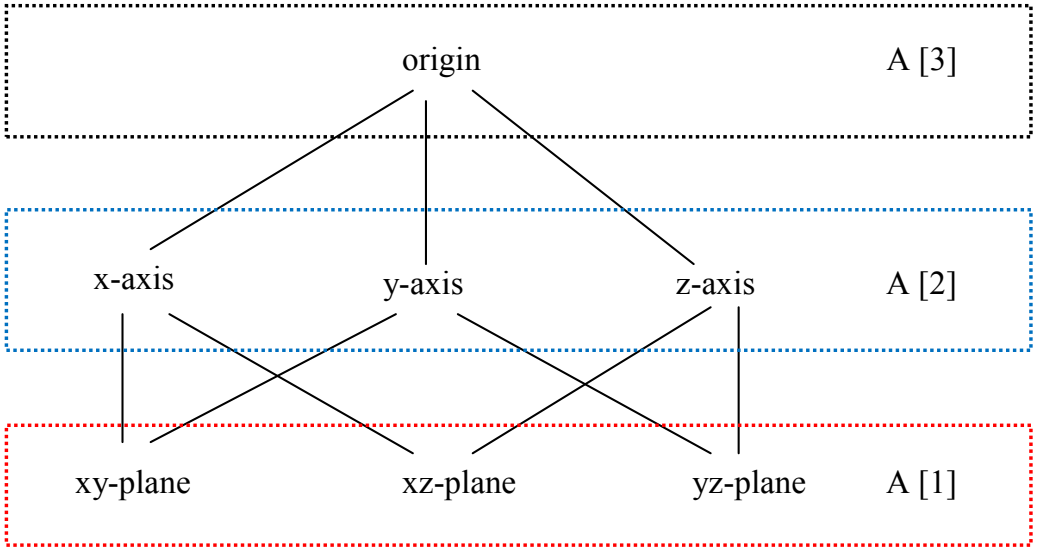
$\sqrt{Minors(\mathcal{A})} = (Q)$. Moreover, $\mathcal{A}$ is free $\iff$ $Minors(\mathcal{A}) = (Q)$.

Restatement: $\mathcal{A}$ is free $\iff$ Minors$(\mathcal{A})$ is radical.

Kyoji Saito

From Hyperplane Arrangements to Subspace Arrangements

To each hyperplane arrangement $\mathcal{A}$ we associate a **lattice of subspaces**.



$\mathcal{A}[c]$: All subspaces of codimension c formed by intersecting hyperplanes in $\mathcal{A}$.

Subspace arrangements are never free

A **subspace** arrangement $\mathcal{A}$ is **free** if
$Der(\mathcal{A}) = \{\theta \in S : \ \theta(\mathbb{I}(\mathcal{A})) \subseteq \mathbb{I}(\mathcal{A})\}$ is a **free** S-module.

Theorem (Wiens)

Free arrangements must have codimension 1 – so proper subspace arrangements are ***never*** *free.*

Need a different notion of freeness for subspace arrangements.

Derivation radical arrangements

$\mathcal{A}$: **pure** subspace arrangement of $\mathbb{C}^\ell$ of codimension c
$Der(\mathcal{A})$ = S-module generated by $\theta_1, \ldots, \theta_t$

$\mathbf{M}(\mathcal{A}) = (\theta_i(x_j))$
Minors($\mathcal{A}$) $=$ ideal of $\ell - c + 1$-minors of $\mathbf{M}(\mathcal{A})$

Inspired by Saito, M. Wakefield and I showed that for pure subspace arrangements, $\sqrt{Minors(\mathcal{A})} = \mathbb{I}(\mathcal{A})$.

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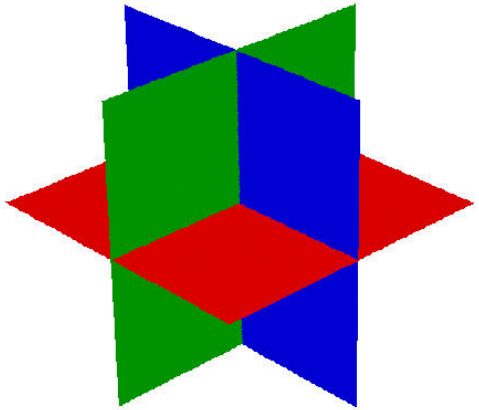
Definition

A subspace arrangement $\mathcal{A}$ is **derivation radical** if Minors($\mathcal{A}$) is radical.

Saito's Freeness Criterion: A hyperplane arrangement is **free** $\iff$ it is **derivation radical**.

Coordinate subspace arrangements

A **coordinate subspace arrangement** consists of subspaces cut out by setting some variables to zero.

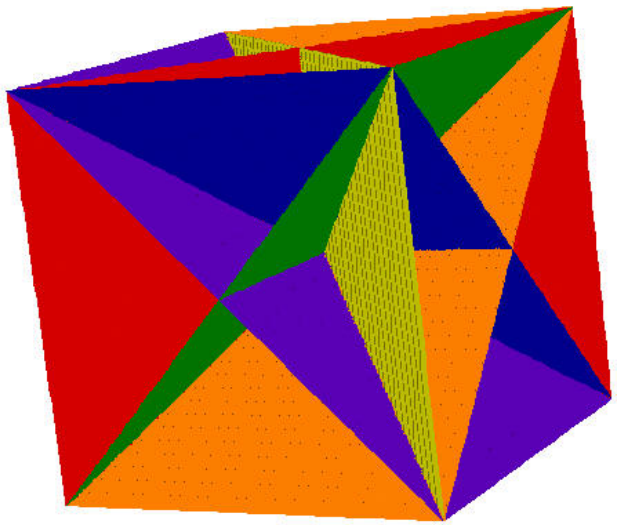


Theorem (___,Wakefield)

Truncations of Boolean arrangements are derivation radical. Taking the product with a linear space preserves the derivation radical property. All derivation radical coordinate subspace arrangements occur in this way.

Truncations of the braid arrangement

The braid arrangement $\mathcal{A}_\ell$ consists of all the hyperplanes $x_i = x_j$ in $V = \mathbb{C}^\ell$.



Theorem (___,Wakefield)

All truncations of the braid arrangement $\mathcal{A}_\ell$ are derivation radical.

The proof exploits the symmetry of the braid arrangement and uses a structural result due to Kleitman and Lovász.

Equations for Truncations

Problem: Find equations that cut out the truncation $\mathcal{A}[c]$ of a hyperplane arrangement $\mathcal{A}$.

In the examples, the $\ell - c + 1$-minors of $\mathbf{M}(\mathcal{A})$ generated $\mathbb{I}(\mathcal{A}[c])$.

This does not follow immediately from the fact that $\mathcal{A}[c]$ is derivation radical because the minors of $\mathbf{M}(\mathcal{A})$ need not generate Minors($\mathcal{A}$**[c]**).

Example

The Edelman-Reiner hyperplane arrangement $\mathcal{A}$ is an important counterexample to many natural questions. The 3-minors of $\mathbf{M}(\mathcal{A})$ do not generate Minors($\mathcal{A}[3]$).

Equations for Truncations

Theorem (___,Wakefield)

The minors of $\mathbf{M}(\mathcal{A})$ *generate* $\mathbb{I}(\mathcal{A}[c])$ *when the ideal of minors of* $\mathbf{M}(\mathcal{A})$ *has no* **embedded primes**. *As well, under this condition, the truncation* $\mathcal{A}[c]$ *is derivation radical.*

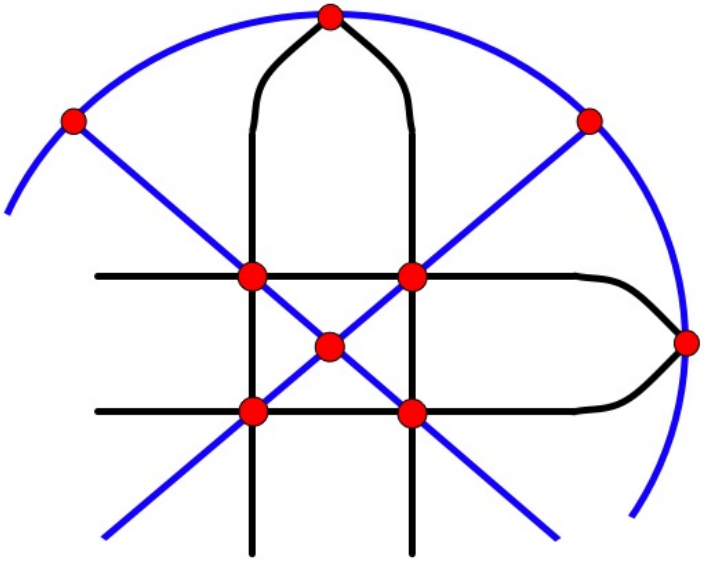
Question: Is there a condition that is easier to check that implies that all truncations are radical? Freeness?

Truncations of free arrangements

Perhaps all **free** hyperplane arrangements have derivation radical truncations? This is false.

Example

The non-Fano arrangement, the hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^3$ defined by $Q = xyz(x-y)(x-z)(y-z)(x+y-z)$, is free but $\mathcal{A}[2]$ is not derivation radical.



- $\mathrm{Der}(\mathcal{A})$ is free with generators of degrees $1, 3, 3$.
- $\mathcal{A}[2]$ is nine points, so $\exists$ cubic in $\mathbb{I}(\mathcal{A}[2])$
- 2-minors of $\mathbf{M}(\mathcal{A})$ have degree ≥ 4
- M2: 2-minors of $\mathbf{M}(\mathcal{A})$ generate $\mathrm{Minors}(\mathcal{A}[2])$.

Open questions

1. Which classes of hyperplane arrangements have the property that all truncations are derivation radical: B_ℓ, D_ℓ?
2. Does some other property (e.g. inductively-free or supersolvable) of hyperplane arrangements imply that all truncations are derivation radical? Does it imply that the minors of $\mathbf{M}(\mathcal{A})$ generate $\mathbb{I}(A[c])$?
3. If $\mathcal{A}$ is a generic subspace arrangement does it follow that $\mathcal{A}$ is not derivation radical?
4. What can be said about the **topology** of derivation radical subspace arrangements? (E.g. Betti numbers, etc)