Families of Free Hyperplane Arrangements

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Outline

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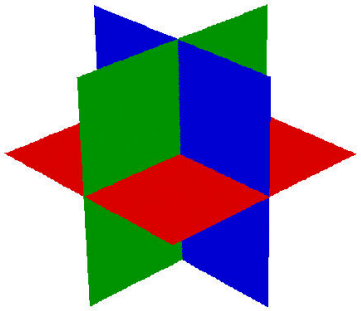
- **Free** hyperplane arrangements, the Jacobian ideal and Terao's conjecture
- Hilbert function of arrangements in flat families
- Symmetry and families of **multi**-arrangements

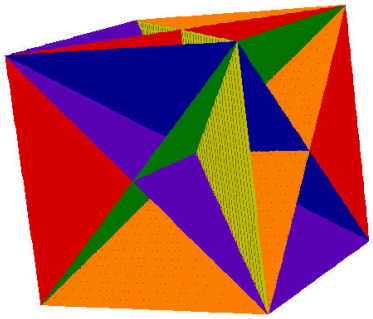
Hyperplane Arrangements

An **arrangement** $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{P}^n = \mathbb{P}(V)$

Boolean Arrangement
$\mathcal{A} = \{\mathbb{V}(x_0), \ldots, \mathbb{V}(x_n)\}$
Defining polynomial: $Q = x_0 \cdots x_n$.



Braid Arrangement
$\mathcal{A}_n = \{\mathbb{V}(x_i - x_j) : \ 0 \leq i < j \leq n\}$
Defining polynomial: $Q = \prod_{i<j} x_i - x_j$.



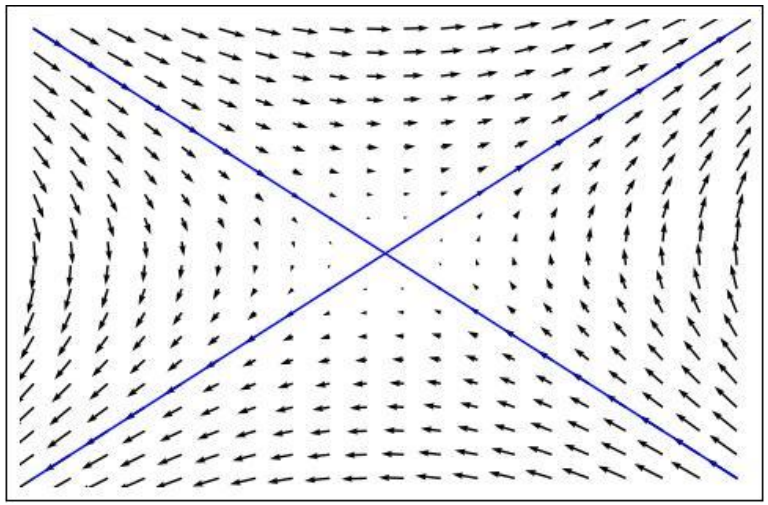
Both arrangements are highly symmetric and highly singular.
They are **free** arrangements.

Free Arrangements

$\mathcal{A}$: a hyperplane arrangement $\{\mathbb{V}(F_1), \ldots, \mathbb{V}(F_d)\}$ where each F_i is a linear form in $S = \mathbb{C}[x_0, \ldots, x_n]$

A **derivation on $\mathcal{A}$** is a derivation θ on S so that for all i, $\theta(F_i) \in (F_i)$ (polynomial vector fields on V **tangent** to all the hyperplanes in $\mathcal{A}$)

The derivation $y\partial_x + x\partial_y$ on $\mathcal{A}$ with $Q = (x - y)(x + y)$ corresponds to the polynomial vector field $\langle y, x \rangle$



The collection $Der(\mathcal{A})$ of derivations on $\mathcal{A}$ is a module over S.

The arrangement $\mathcal{A}$ is **free** if $Der(\mathcal{A})$ is a free S-module.

Topology of Free Arrangements

The complement of a free hyperplane arrangement has a nice topology.

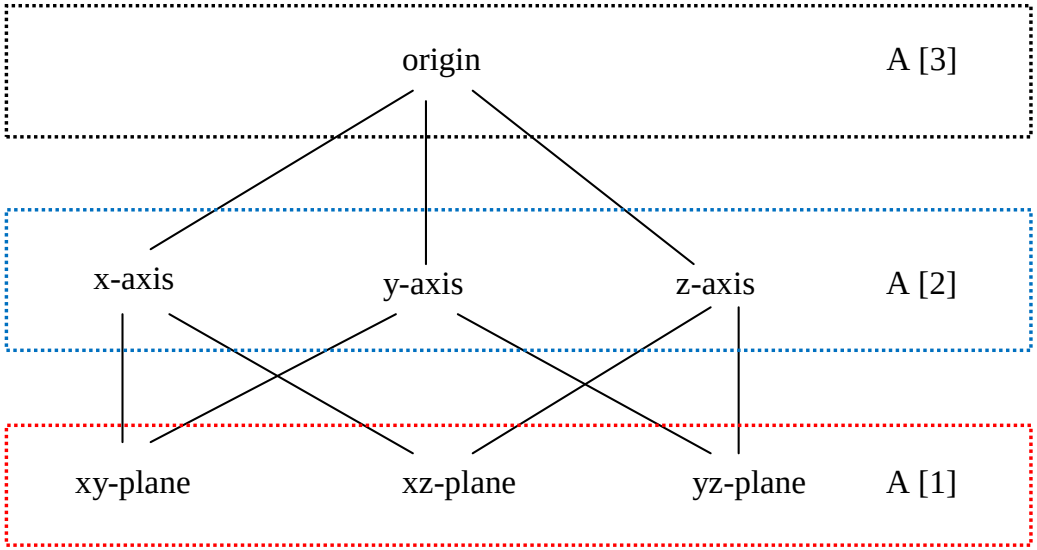
Theorem (Terao's factorization theorem)

If $\mathcal{A}$ is a free hyperplane arrangement then the Hilbert series of $H^*(V \setminus \bigcup \mathcal{A})$ factors as $\prod(1 + d_i t)$ where the d_i are the degrees of a minimal system of generators of $Der(\mathcal{A})$.

When $\mathcal{A}$ is free, the degrees of the minimal generators of $Der(\mathcal{A})$ are called the **exponents** of $\mathcal{A}$.

Lattice Type

To each hyperplane arrangement $\mathcal{A}$ we associate a **lattice of subspaces** $L(\mathcal{A})$.



Arrangements $\mathcal{A}$ and $\mathcal{A}'$ with $L(\mathcal{A}) \cong L(\mathcal{A}')$ are **combinatorially equivalent**. Invariants of $\mathcal{A}$ that depend only on $L(\mathcal{A})$ are said to be **combinatorial invariants**.

Freely Combinatorial Invariants

Some invariants are only defined when $\mathcal{A}$ is free. Such an invariant is **freely combinatorial** if it only depends on $L(\mathcal{A})$.

When $\mathcal{A}$ is free the exponents – degrees of the minimal generators of $Der(\mathcal{A})$ – are freely combinatorial invariants and are determined by the Möbius function of the lattice $L(\mathcal{A})$.

Terao's Conjecture and Yuzvinsky's result

Conjecture (Terao)

Freedom is a combinatorial invariant. That is, if $\mathcal{A}$ is free and $L(\mathcal{A}') = L(\mathcal{A})$ then $\mathcal{A}'$ is free too.

Terao's conjecture remains open, though it holds in $\mathbb{P}^1$ and in $\mathbb{P}^2$ for all arrangments with 11 or fewer lines.

Theorem (Yuzvinsky)

The set of free arrangements in $L(\mathcal{A})$ is Zariksi-open.

The Jacobian ideal

Jacobian ideal: $\mathrm{Jac}(\mathcal{A}) = \left(\frac{\partial Q}{\partial x_0}, \ldots, \frac{\partial Q}{\partial x_n}\right)$ with $Q = F_1 \cdots F_d$.

We have an exact sequence:

$$\begin{aligned} 0 \to Der(\mathcal{A}) \to Der(S) \oplus S \to S \to S/\mathrm{Jac}(\mathcal{A}) \to 0. \\ \theta \mapsto (\theta, \theta(Q)/Q) \\ (\theta, P) \mapsto \theta(Q) - PQ \end{aligned}$$

Since $Der(S) \cong S^{n+1}$ is free, many of the algebraic properties of $Der(\mathcal{A})$ are reflected in the ring $S/\mathrm{Jac}(\mathcal{A})$.

Reconstructing $\mathcal{A}$ from Jac($\mathcal{A}$)

Theorem (Wakefield-Yoshinaga, Traves)

The hyperplane arrangement $\mathcal{A}$ itself can be recovered from Jac($\mathcal{A}$). In fact, there is only one degree d hypersurface containing Jac($\mathcal{A}$) with the correct multiplicities.

Hilbert function for free arrangments

Question

Is the Hilbert function of $Der(\mathcal{A})$ combinatorial? Equivalently, is the Hilbert function of $S/Jac(\mathcal{A})$ combinatorial?

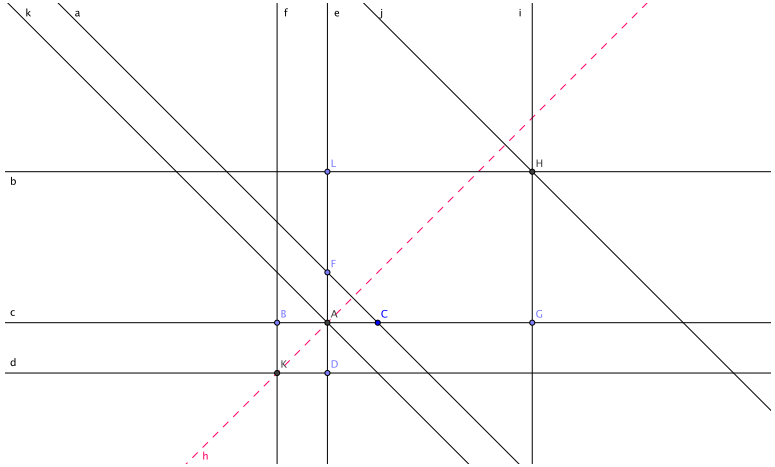
$\mathcal{A}$ and $\mathcal{A}'$ two **free** arrangements with the same lattice type
$\Rightarrow$ $\mathrm{Exp}(\mathcal{A}) = \mathrm{Exp}(\mathcal{A}')$
$\Rightarrow$ $\mathrm{HF}(Der(\mathcal{A})) = \mathrm{HF}(Der(\mathcal{A}'))$

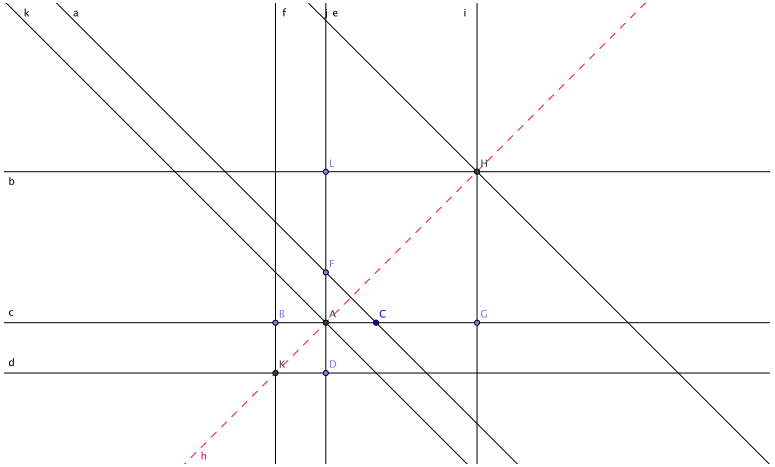
So the Hilbert function of $Der(\mathcal{A})$ is a **freely combinatorial** invariant.

Hilbert function for non-free arrangements

Example (Wakefield-Yoshinaga)

A non-free example:As the rightmost vertical and slanted lines translate left the Hilbert **polynomial** changes. So the Hilbert function of $Der(\mathcal{A})$ is not a **combinatorial** invariant.





Hilbert function for families

Question

If $\mathcal{A}(t)$ is a parameterized flat family where the generic fibre is a free arrangement, then how does the Hilbert function of the induced fibers $Der(\mathcal{A}(t))$ vary?

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If Terao's conjecture holds then the Hilbert function cannot change unless the lattice type changes. This suggests a weak conjecture:

Conjecture (Weak)

HF($Der(\mathcal{A}(t))$) changes only when the lattice type changes.

Hilbert function for families

Question

If $\mathcal{A}(t)$ is a parameterized flat family where the generic fibre is a free arrangement, then how does the Hilbert function of the induced fibers $Der(\mathcal{A}(t))$ vary?

If Terao's conjecture holds then the Hilbert function cannot change unless the lattice type changes. This suggests a weak conjecture:

Conjecture (Weak)

HF($Der(\mathcal{A}(t))$) changes only when the lattice type changes.

Conjecture (Strong)

The flat limit of free arrangements is free.

Hypothetical counterexamples to this conjecture and to Terao's conjecture must place the Jacobian scheme $\mathrm{Jac}(\mathcal{A}(t))$ in special position.

Multi-arrangements

A multi-arrangement $\mathcal{A}$ is a list of hyperplanes together multiplicites: $\{(\mathbb{V}(F_1), m_1), \ldots, (\mathbb{V}(F_d), m_d)\}$.

Derivations: $\mathbf{Der}(\mathcal{A}) = \{\theta \in \mathrm{Der}(S) : \text{for each index } i, \theta(\mathbf{F_i}) \in (\mathbf{F_i^{m_i}})\}$ is a submodule of $\mathrm{Der}(S)$.

$\mathcal{A}$ is **free** if $\mathbf{Der}(\mathcal{A})$ is a free S-module.

Muli-arrangments on $\mathbb{P}^1$

Theorem (Wakefield)

Multi-arrangments on $\mathbb{P}^1$ are always free. If $m = \sum m_i$ then the exponents are of the form $\lfloor m/2 \rfloor - i$ and $\lceil m/2 \rceil + i$. In the generic case $i = 0$.

Example (Ziegler)

Consider 4 points on $\mathbb{P}^1$ with multiplicities 3,3,1 and 1. Then $m = 8$ and we expect that in most arrangements the exponents are 4 and 4. WLOG the two triple points are at 0 and ∞ and that the two other points are at $x = a$ and $x = b$.

Ziegler's example: change of exponents

When is a derivation θ in $Der(\mathcal{A})_3$?

The conditions $x^3|\theta(x)$ and $y^3|\theta(y)$ force $\theta = Ax^3\partial_x - By^3\partial_y$. Then $\theta \in Der(\mathcal{A})$ if and only if

$$\begin{cases}\theta(x-ay)|_{x=a,y=1} = 0 \\ \theta(x-by)|_{x=b,y=1} = 0\end{cases} \iff \begin{cases}a^3A + aB = 0 \\ b^3A + bB = 0\end{cases} \iff \begin{bmatrix}a^3 & a \\ b^3 & b\end{bmatrix}\begin{bmatrix}A \\ B\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

So $\theta \in Der(\mathcal{A}) \iff ab(a-b)(a+b) = 0$. The exponents are (3,5) only when $a+b=0$.

New generators in Ziegler's example

What explains the new derivation in degree 3?
Saturation: when $a+b=0$, $\theta_{41} = x\theta_3$ and $\theta_{42} = y\theta_3$, so $\theta_3 \in Der(\mathcal{A})$.

What explains the new generator in degree 5?
$\theta_5 = x^2y^2(x\partial_x + y\partial_y)$ and

$$-(a+b)x\theta_{41} - y\theta_{42} + x\theta_{42} = (a+b)\theta_5$$

so $\theta_5 \in S\theta_{41} + S\theta_{42}$ precisely when $a+b \neq 0$.

Symmetry

Ziegler's Example: exponents change when symmetry group changes $\Rightarrow$ exponents of multi-arrangements are **not combinatorial**.

Question

Why does symmetry play such a special role in determining the exponents of multiarrangements? How is Exp($\mathcal{A}$) related to the symmetry group?

The equations in the moduli space for our multi-arrangement that cut out the degeneracy locus (arrangements with more distant exponents) are invariant under a permutation action (relabeling of points).