SM 223 Test #1 [Vectors] Solutions 21 Sep 2009

1. This problem deals with the point $P = (1, 2, 12)$ and the plane

$$2x + 2y + z = 36.$$

(a) Give parametric equations for the line through P perpendicular to the plane.

$$x = 1 + 2t \qquad y = 2 + 2t \qquad z = 12 + t$$

Reason: Take $P = (1, 2, 12)$ as the anchor point of the line. Also, since the line is perpendicular to the plane, a direction vector of the line is a normal vector the plane, which is $\langle 2, 2, 1 \rangle$.

(b) Find the point on the plane that is closest to P.

Put your answer here: $(\; 5 \; , \; 6 \; , \; 14 \;)$

Show your work below.

Solution: The shortest distance is measured along the line through P perpendicular to the plane. This is the line we found in (a). Substitute the parametric equations from (a) into the plane to find the point of intersection. This will be the closest point on the plane to P.

$$2x + 2y + z = 36$$

$$2(2t + 1) + 2(2t + 2) + (t + 12) = 36$$

$$9t + 18 = 36$$

$$t = 2$$

With $t = 2$ our parametric equations give $(x, y, z) = (5, 6, 14)$.

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2. Each equation defines a surface in three dimensions.
Identify each surface by writing the **best** capital letter [A]–[L] in the blank.
A capital letter may be used more than once or not at all.

A:	plane	D:	paraboloid	G:	hyperboloid of 2 sheets	J:	parabolic cylinder
B:	sphere	E:	ellipsoid	H:	hyperbolic paraboloid	K:	two parallel planes
C:	cone	F:	circular cylinder	I:	hyperboloid of 1 sheet	L:	two intersecting planes

__D__ $z = x^2 + y^2$

__H__ $z = x^2 - y^2$

__F__ $x^2 + y^2 = 12$ The variable z is absent; also x and y define a circle.

__D__ $x^2 + y^2 = 12 - z$ $z = 12 - (x^2 + y^2)$, an inverted paraboloid

__G__ $-x^2 + y^2 - z^2 = 12$

__J__ $z = (4 + y)(4 - y)$ $z = 16 - y^2$ x is absent; also y and z define a parabola

__K__ $z^2 = 1$ $z = 1$ or $z = -1$

__C__ $z^2 - x^2 = y^2$ $z^2 = x^2 + y^2$

__A__ $\langle 1, -2, 3\rangle \cdot \langle x, y, z\rangle = 12$ $x - 2y + 3z = 12$

__I__ $\det \begin{bmatrix} x^2 & y^2 & z^2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 12$ $x^2 - y^2 + z^2 = 12$

3. Identify each curve by writing the **best** capital letter (A)–(J) in the blank.
A capital letter may be used more than once or not at all.

A:	circle	F:	line
B:	semi-circle	G:	line segment
C:	quarter-circle	H:	parabola
D:	ellipse	I:	spiral
E:	semi-ellipse	J:	helix

__A__ $\mathbf{r}(t) = \langle \cos(t),\, \sin(t) \rangle$

__I__ $\mathbf{r}(t) = \langle t\cos(t),\, t\sin(t) \rangle$

__H__ $\mathbf{r}(t) = \langle \cos(t),\, \sin^2(t) \rangle$ $x^2 + y = 1$, which is the parabola $y = 1 - x^2$.

__D__ $\mathbf{r}(t) = \langle 3\cos(2t),\, 4\sin(2t) \rangle \quad (0 \le t \le \pi)$

__J__ $\mathbf{r}(t) = \langle \cos(2t),\, \sin(2t),\, 2t \rangle$

__A__ $\mathbf{r}(t) = \langle \cos(2t),\, \sin(2t),\, 2 \rangle$ The circle is in the plane $z = 2$

__A__ $\mathbf{r}(t) = \langle 12\sin(t),\, 12\cos(t) \rangle$

__G__ $\mathbf{r}(t) = \langle t,\, 2t,\, 3t \rangle \quad (0 \le t \le 12)$

4. TRUE or FALSE.
Fill in a bubble in each row.
All statements deal with three dimensions.

TRUE	FALSE	
●	○	Two lines parallel to a third line must be parallel to each other.
○	●	Two lines perpendicular to a third line must be parallel to each other.
●	○	Two planes parallel to a third plane must be parallel to each other.
●	○	Two lines perpendicular to a plane are parallel to each other.
●	○	Two planes perpendicular to a line are parallel to each other.
○	●	Two lines either intersect or are parallel.
●	○	A plane and a line either intersect or are parallel.

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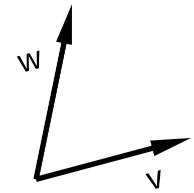
5. The sketch shows two vectors $\mathbf{v}$ and $\mathbf{w}$. The vectors satisfy

$\lvert\mathbf{v}\rvert = 4$	$\lvert\mathbf{w}\rvert = 6$	$\mathbf{v}\cdot\mathbf{w} = 8$



(a) Draw the vector projection of $\mathbf{v}$ onto $\mathbf{w}$ in the sketch at a suitable position.

(b) Find $\mathbf{v}\cdot\mathbf{v}$.

○ 0 ○ 2 ○ 4 ○ 8 ● 16

Reason: $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$

(c) Find $\mathbf{w}\cdot\mathbf{v}$.

○ -8 ● 8 ○ 0 ○ -24 ○ 24

Reason: $\mathbf{w}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{w} = 8$; the order doesn't matter for the dot product.

(d) Find the scalar projection of $\mathbf{v}$ along $\mathbf{w}$.

○ 1/3 ○ 1/2 ○ 1 ● 4/3 ○ 2

Reason: $\text{comp}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|} = \frac{8}{6} = \frac{4}{3}$

(e) The angle θ formed by the two vectors satisfies $\cos(\theta) =$

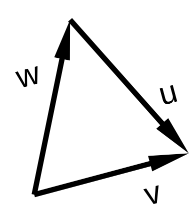
○ 1/4 ● 1/3 ○ 1/2 ○ 2/3 ○ 3/4

Reason: $\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{8}{(4)(6)} = \frac{1}{3}$

(f) Which expression is equal to the vector $\mathbf{u}$ in the diagram?

○ $\mathbf{v}+\mathbf{w}$ ● $\mathbf{v}-\mathbf{w}$ ○ $\mathbf{w}-\mathbf{v}$ ○ $\left(\frac{1}{|\mathbf{v}|}\right)\mathbf{v}$ ○ $\left(\frac{1}{|\mathbf{w}|}\right)\mathbf{w}$

Reason: From the tip-to-tail situation we see that $\mathbf{w}+\mathbf{u} = \mathbf{v}$.
By algebra $\mathbf{u} = \mathbf{v}-\mathbf{w}$.



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6. Consider the three points

$$P = (1, 3, 3), \qquad Q = (4, 5, 9), \qquad R = (2, 5, 5).$$

Here are some facts you should use to answer the fill-in-the-blank and multiple choice questions:

$$\boxed{\mathbf{PQ} = \langle 3, 2, 6\rangle, \quad |\mathbf{PQ}| = 7, \quad \mathbf{PQ}\cdot\mathbf{PR} = 19, \quad \mathbf{PQ}\times\mathbf{PR} = \langle -8, 0, 4\rangle.}$$

(a) Find a unit vector parallel to **PQ**: $\frac{1}{7}\langle 3, 2, 6\rangle$ or $\langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7}\rangle$

Reason: $\mathbf{u} = \left(\frac{1}{|\mathbf{PQ}|}\right)\mathbf{PQ} = \frac{1}{7}\mathbf{PQ} = \frac{1}{7}\langle 3, 2, 6\rangle$

(b) Find a vector parallel to **PQ** with length 14: $\langle\ 6\ ,\ 4\ ,\ 12\ \rangle$

Reason: Simply take 14 times the unit vector in (a).
Or observe that **PQ** has length 7 and multiply by 2.

(c) Give an equation of the plane through P, Q, and R:

$-8(x-1) + 0(y-3) + 4(z-3) = 0$ or $-2x + z = 1$

Reason: We may take $P = (1, 3, 3)$ as our anchor point is P.
A normal vector is $\mathbf{PQ}\times\mathbf{PR} = \langle -8, 0, 4\rangle$.
Now plug into the basic plane equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

(d) Find the work done by the constant force **PQ** in moving an object on a straight line from P to R.

Note: The distance is in meters and force is in Newtons. So the work is in Joules.

○ 29 J ● 19 J ○ 80 J ○ $\sqrt{80}$ J ○ 4 J

Reason: Work = (Force) dot (displacement) = $\mathbf{PQ}\cdot\mathbf{PR} = 19$.

(e) Find the area of $\triangle PQR$.

○ $\sqrt{80}$ ● $\frac{1}{2}\sqrt{80}$ ○ $\sqrt{19}$ ○ $\frac{1}{2}\sqrt{19}$ ○ 12

Reason: Area of triangle $= \frac{1}{2}|\mathbf{PQ}\times\mathbf{PR}| = \frac{1}{2}\sqrt{(-8)^2 + 0^2 + 4^2} = \frac{1}{2}\sqrt{80}$

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7. This problem has 8 parts, labeled (a)–(h). However, you get to omit two parts. Only 6 of the parts will be graded. Each part is worth 10 points. You MUST say which two parts you want to omit by filling in two bubbles.

OMIT: (a) ○ (b) ○ (c) ○ (d) ○ (e) ○ (f) ○ (g) ○ (h) ○

(a) Consider the sphere

$$x^2 + (y-3)^2 + (z+4)^2 = 14.$$

i. Where is the center of the sphere?

○ $(0,-3,4)$ ● $(0,3,-4)$ ○ $(0,0,0)$ ○ $(0,3,4)$ ○ $(0,-3,-4)$

ii. The point $P = (1,5,-1)$ is on the sphere. What point on the sphere is farthest from P?

○ $(0,3,-4)$ ○ $(1,8,-5)$ ○ $(-1,-8,5)$ ○ $(2,4,1)$ ● $(-1,1,-7)$

Reason: The center point $(0,3,-4)$ is the midpoint between the point $P = (1,5,-1)$ and the desired point $Q = (x,y,z)$. So

$$(0,3,-4) = \left(\frac{1+x}{2}, \frac{5+y}{2}, \frac{-1+z}{2}\right).$$

Therefore $(x,y,z) = (1,-1,7)$.

(b) Which curves arise as (x-, y-, or z-) traces for the surface

$$z = x^2 - y^2\,?$$

Indicate **all** correct answers.

○ circles ○ ellipses ● parabolas ● hyperbolas ● intersecting lines

Reason: The traces $x = k$ and $y = k$ give the parabolas $z = k^2 - y^2$ and $z = x^2 - k^2$. The trace $z = k$ gives $x^2 - y^2 = k$, which is a hyperbola for $k \neq 0$. Also, for $k = 0$ the z-trace is $x^2 - y^2 = 0$, or $x = \pm y$, which is a pair of intersecting lines.

(c) Simplify each expression.

i. $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$

● $\mathbf{0}$ ○ $\mathbf{k}$ ○ $-\mathbf{k}$ ○ $\mathbf{j}$ ○ $-\mathbf{j}$

ii. $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

○ $\mathbf{0}$ ○ $\mathbf{k}$ ○ $-\mathbf{k}$ ○ $\mathbf{j}$ ● $-\mathbf{j}$

(d) Let $P = (3, -4, 12)$.

i. How far is P from the origin?

○ 5 ○ 11 ● 13 ○ 15 ○ 19

Reason: $\text{dist}(O, P) = \sqrt{3^2 + (-4)^2 + 12^2} = \sqrt{169} = 13$.

ii. How far is P from the xy-plane?

○ 5 ○ 7 ○ 11 ● 12 ○ $\sqrt{19}$

Reason: The z-coordinate measure displacement from the xy-plane: $\text{dist}(O, xy\text{-plane}) = |12| = 12$.

iii. How far is P from the z-axis?

● 5 ○ 7 ○ 11 ○ 12 ○ $\sqrt{19}$

Reason: $\text{dist}(O, z\text{-axis}) = \sqrt{3^2 + (-4)^2} = 5$.

(e) At what point does the line

$$\frac{x}{3} = \frac{y+2}{3} = \frac{z+2}{1}$$

intersect the xy-plane?

○ $(0, -2, 0)$ ○ $(0, 2, 0)$ ○ $(3, 1, 0)$ ○ $(3, 3, 0)$ ● $(6, 4, 0)$

Reason: The xy-plane is $z = 0$. We take $z = 0$ in the symmetric equations of the line and get

$$\frac{x}{3} = \frac{y+2}{3} = \frac{0+2}{1} = 2.$$

Now we must have $x = 6$ and $y = 4$.

(f) In how many points does the line with parametric equations

$$x = 3t + 2, \qquad y = 4t + 1, \qquad z = 5t + 2$$

intersect the hyperboloid of 1 sheet

$$x^2 + y^2 - z^2 = 1\,?$$

○ 0 ○ 1 ○ 2 ○ 4 ● infinitely many

Reason: Substitute the parametric equations into the equation of the hyperboloid to find the values of t for the intersection:

$$(3t+2)^2 + (4t+1)^2 - (5t+2)^1 = 1; \qquad (9+16-25)t^2 + (12+8-20)t + (4+1-4) = 1.$$

This gives $1 = 1$. So the line intersects the hyperboloid for all t.

(g) Let

$$\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{b} = 3\mathbf{i} + \mathbf{k}.$$

Find the vector projection of **a** onto **b**.

Put your answer on the blank line.
Put your work in the space below.

Answer: $\frac{7}{10}\langle 3, 0, 1\rangle$

Reason: $\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \left(\frac{\langle 3,0,1\rangle \cdot \langle 2,-2,1\rangle}{\langle 3,0,1\rangle\cdot\langle 3,0,1\rangle}\right)\mathbf{b} = \left(\frac{7}{10}\right)\mathbf{b}.$

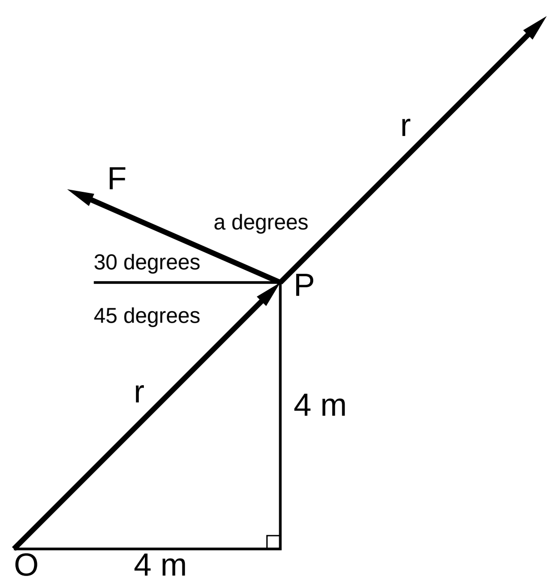
(h) The force **F** with magnitude 3 Newtons is applied at point P and generates a torque about the origin O. (See the diagram.) The magnitude of the torque $\vec{\tau}$ has the form

$$|\vec{\tau}| = A\sin(a).$$

Find A and a.

- ○ $A = 12$
- ● $A = 12\sqrt{2}$
- ○ $A = 48$
- ○ $A = 7\sqrt{2}$
- ○ $A = 4\sqrt{2}$

- ○ $a = 30°$
- ○ $a = 45°$
- ○ $a = 90°$
- ● $a = 105°$
- ○ $a = 135°$



The torque is $\vec{\tau} = \mathbf{r}\times\mathbf{F}$. The vector **r** is **OP**, which has length $4\sqrt{2}$. Slide the vector **r** up to P so it's in the tail-to-tail position with **F**. The angle formed is $a = 180 - 30 - 45 = 105$ degrees. So

$$|\vec{\tau}| = |\mathbf{r}\times\mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin(a) = (4\sqrt{2})(3)\sin(105°) = 12\sqrt{2}\sin(105°)$$