SM 223 Test #2 [Curves, Partials] Solutions 19 Oct 2009

1. (a) Complete the "limit" definition of the partial derivative of $f(x, y)$ with respect to y at the point (a, b):

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \underline{\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}}$$

(b) Use the table of values for $T(x, y)$ to estimate $T_y(0, 0)$.

- ○ 2
- ● 5/2
- ○ 2/3
- ○ 3
- ○ 5

$T(x,y)$	$y=0$	$y=2$	$y=4$
$x=0$	5	10	16
$x=3$	7	11	15
$x=6$	11	12	15

$$T_y(0,0) = \lim_{h\to 0}\frac{T(0,0+h)-T(0,0)}{h} \approx \frac{T(0,2)-T(0,0)}{2} = \frac{10-5}{2} = \frac{5}{2}$$

(c) The point $(3, 6)$ is shown as a small circle in the contour diagram for $f(x, y)$. Each level curve of f is a straight line of the form $f(x, y) = k$.

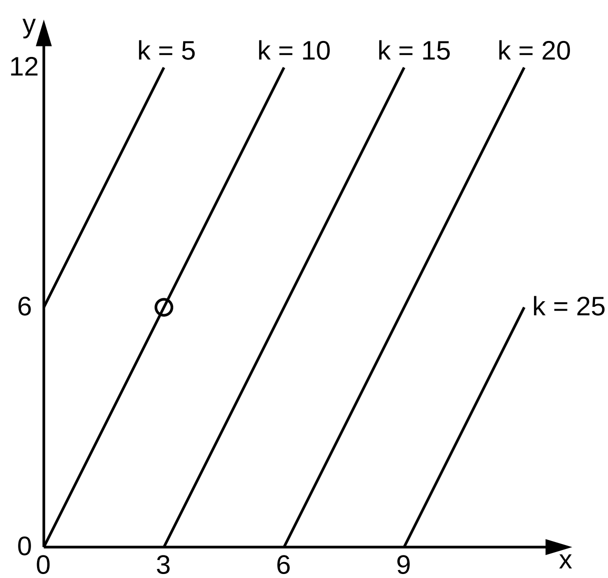
i. Evaluate $f(3, 6)$.

○ 2 ○ 3 ○ 3/2 ○ 5 ● 10

ii. Estimate $f_y(3, 6)$.

● −5/6 ○ 3/2 ○ 2/3 ○ −5 ○ 5/3

$$f_y(3,6) = \lim_{h\to 0}\frac{f(3,6+h)-f(3,6)}{h} \approx \frac{f(3,6+6)-f(3,6)}{6} = \frac{5-10}{6} = -\frac{5}{6}$$



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2. (a) Complete the "limit" definition of the derivative of the vector-valued function $\mathbf{r}(t)$:

$$\mathbf{r}'(t) \quad = \quad \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

(b) Complete the formula for the length of the space curve $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t) \rangle$ for $t = a$ to $t = b$:

$$L \quad = \quad \int_a^b |\mathbf{r}'(t)|\, dt \quad OR \quad \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

(c) The position of a particle at time t is given by

$$\mathbf{r}(t) \quad = \quad \langle 2t,\, \sin(t),\, \cos(t) \rangle \qquad\qquad (0 \le t \le 2\pi)$$

Find the distance traveled by the particle.

○ 4π ○ $\sqrt{14}\,\pi$ ○ $3\sqrt{2}\,\pi$ ● $2\sqrt{5}\,\pi$ ○ $\sqrt{22}\,\pi$

Reason: The velocity is $\mathbf{r}'(t) = \langle 2,\, \cos(t), \sin(t) \rangle$.
The speed is

$$|\mathbf{r}'(t)| = |\langle 2,\, \cos(t), \sin(t) \rangle| = \sqrt{2^2 + (\cos(t))^2 + (-\sin(t))^2} = \sqrt{4+1} = \sqrt{5}.$$

So the distance traveled is

$$L = \int_a^b \left|\mathbf{r}'(t)\right| dt = \int_0^{2\pi} \sqrt{5}\, dt = 2\sqrt{5}\,\pi.$$

3. This problem has 5 parts, labeled (a)–(e), each worth 6 points. However, you will omit two parts; only three parts will be graded. **You receive full credit for the omitted parts—but only if you fill in two bubbles in the table.**

OMIT:	(a) ○	(b) ○	(c) ○	(d) ○	(e) ○

(a) The position of a particle is

$$\mathbf{r}(t) = \langle e^{-3t}, 4t \rangle.$$

Find the speed of the particle at $t = 0$.

○ 1 ○ 3 ○ 4 ● 5 ○ 7

Reason: The velocity is $\mathbf{r}'(t) = \langle -3e^{-3t}, 4 \rangle$.
The velocity at $t = 0$ is $\mathbf{r}'(0) = \langle -3, 4 \rangle$.
So the speed is $|\langle -3, 4 \rangle| = \sqrt{(-3)^2 + 4^2} = 5$

(b) Suppose that $\mathbf{r}'(t) = \langle 2t,\ 3t^2 \rangle$ and $\mathbf{r}(1) = \langle 1, 4 \rangle$. Find $\mathbf{r}(2)$.

○ $\langle 0, 3 \rangle$ ○ $\langle 2, 8 \rangle$ ○ $\langle 2, 10 \rangle$ ○ $\langle 4, 8 \rangle$ ● $\langle 4, 11 \rangle$

Reason: $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 2t,\ 3t^2 \rangle\,dt = \langle t^2, t^3 \rangle + \mathbf{C}$.
The initial condition gives us $\langle 1, 4 \rangle = \mathbf{r}(1) = \langle 1^2,\ 1^3 \rangle + \mathbf{C}$. So $\langle 0,\ 3 \rangle = \mathbf{C}$.
Thus $\mathbf{r}(t) = \langle t^2,\ t^3 \rangle + \langle 0, 3 \rangle$, and $\mathbf{r}(2) = \langle 2^2, 2^3 \rangle + \langle 0, 3 \rangle = \langle 4, 11 \rangle$

(c) Which are level curves for the function

$$f(x, y) = x^2 - y^2\,?$$

Indicate **all** correct answers.

○ circles ○ ellipses ○ parabolas ● hyperbolas ● intersecting lines

Reason: The level curves are $f(x, y) = k$. Thus we have $x^2 - y^2 = k$. This is a hyperbola for $k \neq 0$. Also, for $k = 0$, we get $x^2 - y^2 = 0$, or $x = \pm y$, which is a pair of intersecting lines.

(d) Which are level surfaces for the function $f(x, y, z) = x^2 - y^2 + z^2$?

The level surfaces are: $f(x, y, z) = x^2 - y^2 + z^2 = k$

- ○ sphere
- ○ ellipsoid (but not a sphere)
- ● cone — $k = 0$ gives: $x^2 + z^2 = y^2$
- ○ plane
- ● hyperboloid of 1 sheet — $k = 1$ gives: $x^2 - y^2 + z^2 = 1$
- ● hyperboloid of 2 sheets — $k = -1$ gives: $-x^2 + y^2 - z^2 = 1$

(e) The function $g(x, y)$ satisfies

$$g_x = 12x^2y^5.$$

Which function could equal g_y? (Only one answer is correct.)

- ○ $g_y = 20x^2y^5$ — $g_{yx} = 40xy^5$
- ○ $g_y = 4x^3y^5$ — $g_{yx} = 12x^2y^5$
- ○ $g_y = 6x^2y^3$ — $g_{yx} = 12xy^3$
- ● $g_y = 20x^3y^4$ — $\underline{g_{yx} = 60x^2y^4}$
- ○ $g_y = 12x^3y^2$ — $g_{yx} = 36x^2y^2$

Reason: We need equality of the mixed partials: $g_{xy} = g_{yx}$.
Now $g_{xy} = \frac{\partial}{\partial y}(g_x) = \frac{\partial}{\partial y}\left(12x^2y^5\right) = 60x^2y^4$.
Only the choice $g_y = 20x^3y^4$ satisfies $g_{yx} = 60x^2y^4$.

4. Fill in a bubble in each row.

TRUE	FALSE	
○	●	The graph of $f(x, y, z)$ is a surface in three dimensions.
●	○	The contour diagram of $f(x, y, z)$ is a family of surfaces in three dimensions.
●	○	Most familiar functions of two variables satisfy $f_{xy} = f_{yx}$.
○	●	If a particle's speed is constant, then its acceleration vector is **0**.
●	○	If a particle's velocity is constant, then its acceleration vector is **0**.
●	○	For uniform circular motion, the velocity vector is perpendicular to the position vector.

5. A particle's position at time t is given by

$$\mathbf{r}(t) = (3\cos(2t))\,\mathbf{i} \;+\; (3\sin(2t))\,\mathbf{j} \;+\; (4t)\,\mathbf{k}.$$

(a) At what time is the particle at the point $P = (0, 3, \pi)$?

○ $t = 0$ ○ $t = \pi/8$ ● $t = \pi/4$ ○ $t = \pi/2$ ○ $t = \pi$

Reason: Set $(0, 3, \pi) = (x, y, z) = (3\cos(2t), 3\sin 2t, 4t)$.
From the z-coordinates, we have $\pi = 4t$. So $t = \pi/4$.
And $t = \pi/4$ works for the x- and y-coordinates, too.

(b) The tangent line to the curve at point $P = (0, 3, \pi)$ has equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t).$$

Fill in one bubble in each column for x, y, and z. Leave the other bubbles empty.

$x(t)$	$y(t)$	$z(t)$	function
○	○	○	$3 + 6t$
○	○	○	$3 - 6t$
○	○	○	$4 - 3t$
○	○	○	$4 + 3t$
○	○	○	$6t$
●	○	○	$-6t$
○	○	○	$\pi + t$
○	○	○	$\pi - t$
○	○	●	$\pi + 4t$
○	○	○	$\pi - 4t$
○	●	○	3
○	○	○	π
○	○	○	4

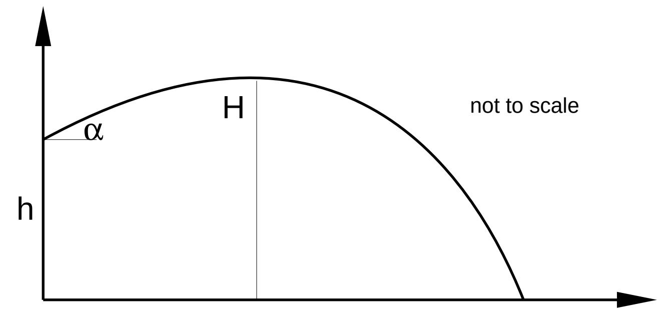
Reason: The derivative is $\mathbf{r}'(t) = (-6\sin(2t))\,\mathbf{i} + (6\cos(2t))\,\mathbf{j} + (4)\,\mathbf{k}$.
So the tangent vector at $t = \pi/4$ is

$$\mathbf{r}'(\pi/4) = -6\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}.$$

Thus our tangent line has anchor point $(0, 3, \pi)$ and direction vector $\langle -6, 0, 4\rangle$.
This line is $x = 0 - 6t; \quad y = 3 + 0t; \quad z = \pi + 4t$.

6. A projectile is fired from a cliff of height h with initial speed v_0 at an angle α above horizontal. The x- and y-axes are in their usual orientations with the origin at the base of the cliff. Distance is measured in meters, and time in seconds. The projectile reaches its maximum height H. This experiment occurs on a planet where the acceleration due to gravity is $\mathbf{g} = \langle 0, -g\rangle$. The equations of motion for the projectile are:

$$\begin{aligned} x(t) &= 40t \\ y(t) &= -5t^2 + 30t + 80 \end{aligned}$$



(a) Fill in the correct bubble for each parameter. Give the angle α to the nearest degree.

h	g	v_0	α	H
○ 30 m	○ 5 m/s^2	○ 40 m/s	○ 27°	○ 80 m
○ 40 m	● 10 m/s^2	● 50 m/s	○ 37°	○ 100 m
○ 60 m	○ 20 m/s^2	○ 60 m/s	○ 47°	○ 120 m
○ 70 m	○ 30 m/s^2	○ 70 m/s	○ 57°	● 125 m
● 80 m	○ 40 m/s^2	○ 80 m/s	○ 67°	○ 130 m

Our general equations for motion are

$$x(t) = (v_0 \cos(\alpha))\, t \qquad \text{and} \qquad y(t) = -\frac{1}{2}\, g t^2 + (v_0 \sin(\alpha))\, t + h.$$

$\boxed{h :}$ $h = y(0) = 80$

$\boxed{g :}$ Match the quadratic coefficients in the expression for y to see that $-5t^2 = -\frac{1}{2}gt^2$. So $5 = \frac{1}{2}g$ and $g = 10$.
OR: Compute acceleration in the y-direction as $y''(t) = -10$.

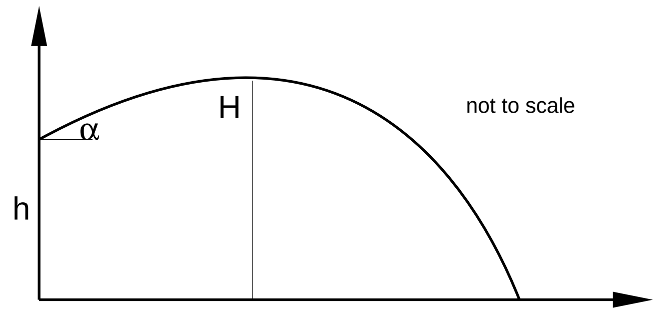
$\boxed{v_0 :}$ The velocity vector is $\langle x'(t), y'(t)\rangle = \langle 40, -10t + 30\rangle$.
So the initial velocity vector is $\langle x'(0), y'(0)\rangle = \langle 40, 30\rangle$.
The initial speed is $v_0 = |\langle 40, 30\rangle| = \sqrt{40^2 + 30^2} = 50$.

$\boxed{\alpha :}$ We have $\tan(\alpha) = \frac{v_0 \sin(\alpha)}{v_0 \cos(\alpha)} = \frac{30}{40}$. So $\alpha = \arctan(\frac{30}{40}) \approx 37°$.

$\boxed{H :}$ The maximum height occurs when $0 = y'(t) = -10t + 30$, i.e, at time $t = 3$. So $H = y(3) = -5(3)^2 + 30(3) + 80 = 125$.

(b) Now suppose a radio tower with height 50 meters is located 280 meters down-range. Does the projectile fall short of the tower, hit the tower, or sail over the tower?

$$
\begin{aligned}
x = x(t) &= 40t \\
y = y(t) &= -5t^2 + 30t + 80
\end{aligned}
$$



FILL IN A BUBBLE. ALSO INDICATE THE VALUE OF D

- ○ Falls short: The projectile lands D meters in front of the base of tower.
- ● Hits: The projectile hits the tower D meters from the top.
- ○ Sails over: The projectile lands D meters beyond the base of tower.

● $D = 5$ ○ $D = 10$ ○ $D = 20$ ○ $D = 40$ ○ $D = 45$

Reason: The correct range occurs when $280 = x(t) = 40t$, that is, at $t = 7$. At that time we have $y(7) = 45$. So the projectile hits the tower 5 meters from the top of the 50 meter tower.

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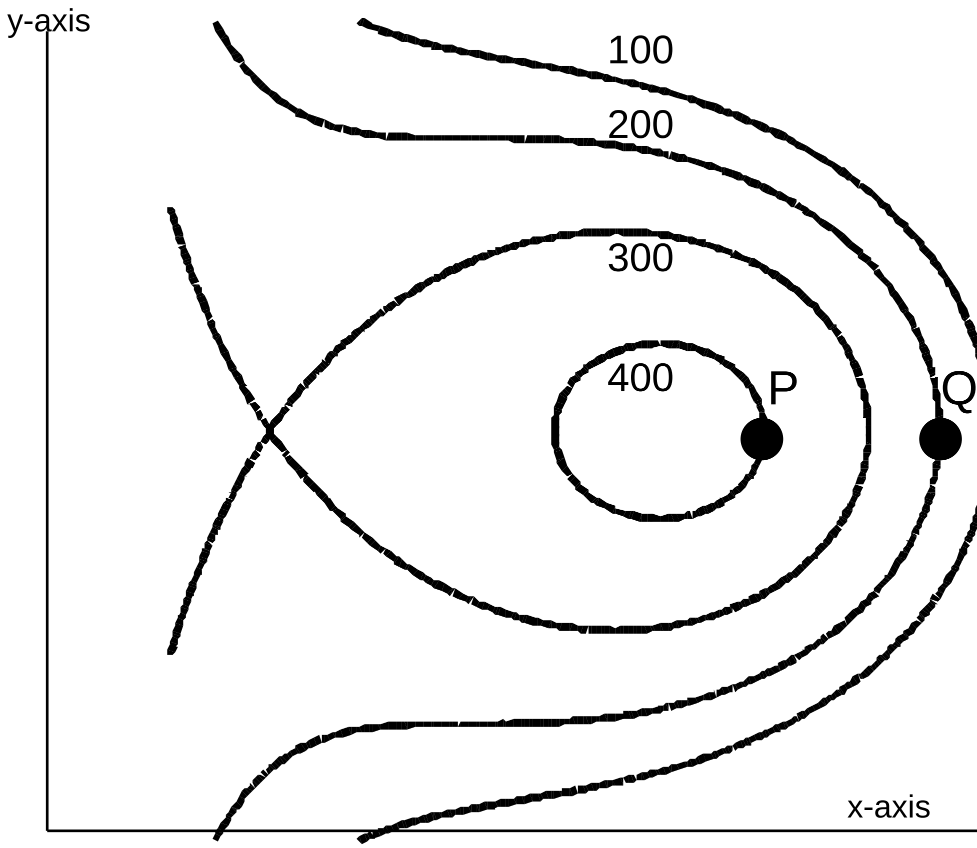
7. The contour diagram for the function $f(x, y)$ is shown, along with two points P and Q.



Fill in a bubble in each column.

$f(P) - f(Q)$	$f_x(Q)$	$f_y(Q)$	$f_{xx}(Q)$
● positive	○ positive	○ positive	○ positive
○ negative	● negative	○ negative	● negative
○ zero	○ zero	● zero	○ zero
○ undefined	○ undefined	○ undefined	○ undefined

- $f(P) - f(Q) = 400 - 200 > 0$.
- The function decreases as we move from left to right through Q.
- Along a vertical line through Q, the function is at a maximum right at Q.
- The function is concave down in the "east-west" direction through Q since the contours are closer together to the right of Q.

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8. Throughout this problem we let

$$f(x,y) = e^{x^2 y}.$$

Fill in the correct bubble. Also, show your work!

(a) Compute $f_x(5,2)$.

● $20e^{50}$ ○ $25e^{50}$ ○ $50e^{50}$ ○ $100e^{50}$ ○ none of above; correct is ________

$$f_x = \frac{\partial}{\partial x}\left(e^{x^2 y}\right) = e^{x^2 y} \cdot \frac{\partial}{\partial x}\left(x^2 y\right) = e^{x^2 y} \cdot 2xy = 2xy \cdot e^{x^2 y}.$$

$$\text{So} \qquad f_x(5,2) = 2 \cdot 5 \cdot 2 \cdot e^{5^2 \cdot 2} = 20\, e^{50}.$$

(b) Compute $f_y(5,2)$.

○ $20e^{50}$ ● $25e^{50}$ ○ $50e^{50}$ ○ $100e^{50}$ ○ none of above; correct is ________

$$f_y = \frac{\partial}{\partial y}\left(e^{x^2 y}\right) = e^{x^2 y} \cdot \frac{\partial}{\partial y}\left(x^2 y\right) = e^{x^2 y} \cdot x^2 = x^2 \cdot e^{x^2 y}.$$

$$\text{So} \qquad f_y(5,2) = 5^2 e^{5^2 \cdot 2} = 25\, e^{50}.$$

(c) Compute $f_{xy}(5,2)$.

○ $800e^{50}$ ○ $820e^{50}$ ○ $1000e^{50}$ ● $510e^{50}$ ○ $2012e^{50}$

$$f_{xy} = \frac{\partial}{\partial y}\left(f_x\right) = \frac{\partial}{\partial y}\left(2xy \cdot e^{x^2 y}\right) = 2xy \cdot \frac{\partial}{\partial y}\left(e^{x^2 y}\right) + e^{x^2 y} \cdot \frac{\partial}{\partial y}\left(2xy\right)$$

$$= 2xy\left(e^{x^2 y} \cdot x^2\right) + e^{x^2 y} \cdot 2x = \left(2x^3 y + 2x\right) e^{x^2 y}.$$

$$\text{So} \qquad f_{xy}(5,2) = \left(2 \cdot 5^3 \cdot 2 + 2 \cdot 5\right) e^{50} = 510\, e^{50}.$$

9. When x thousand dollars is spent on labor, and y thousand dollars is spent on advertising, a bottled water company sells

$$B(x, y) = 10x^{1/2}y^{3/2}$$

thousand gallons of water per month.

(a) How many gallons of water does the company sell when it spends \$25,000 on labor and \$4,000 on advertising?
Hint: $4^{3/2} = \left(4^{1/2}\right)^3 = 2^3 = 8$

● 400,000 ○ 200,000 ○ 500,000 ○ 800,000 ○ 4,000,000

Reason: $B(25, 4) = 10{\cdot}25^{1/2}{\cdot}4^{3/2}$ thousand $= 10{\cdot}5{\cdot}8$ thousand $= 400,000$.

(b) Compute $B_y(25, 4)$.

● 150 ○ 8 ○ 30 ○ 50 ○ 4

Reason: $$B_y(25, 4) = \frac{\partial}{\partial y}\left(10x^{1/2}y^{3/2}\right) = 10x^{1/2} \cdot \frac{3}{2}y^{1/2} = 15\sqrt{x}\sqrt{y}.$$

So $$B_y(25, 4) = 15\sqrt{25}\sqrt{4} = 15 \cdot 5 \cdot 2 = 150.$$

(c) **Fact:** $B_x(25, 4) = 8$.
Interpret this fact in a complete English sentence or two.

Suppose the company is currently spending \$25,000 on labor
and \$4,000 on advertising. Then for every \$1,000 more it spends
on labor (keeping advertising expenditures constant),
8,000 more gallons of water are sold per month.

(d) Suppose the company is currently spending \$25,000 on labor and \$4,000 per month on advertising. If the company has budgeted an additional \$1,000 to spend next month on either labor or advertising, what should it do to improve sales the most?

○ spend \$1,000 on labor
● spend \$1,000 on advertising

Reason: $B_y(25, 4) = 150 > 8 = B_x(25, 4)$. The company sells more water by increasing advertising rather than labor.