SM 223 Test #4 [Lagrange/Multiple Integrals] Solutions 9 Dec 2009

1. Throughout this problem we consider the function $f(x, y)$ of two variables defined for all (x, y) in a rectangle

$$R = \{(x, y) \big| \ a \le x \le b, \ c \le y \le d\}.$$

We partition the interval $[a, b]$ into m equal sub-intervals, and the interval $[c, d]$ into n equal sub-intervals.

(a) Complete our 'limit' definition of the double integral of f over R.

$$\int\int_R f(x, y)\, dA \quad = \quad \underline{\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A}$$

(b) In your definition what is the point (x_{ij}^*, y_{ij}^*) called? sample point

(c) In your definition what is ΔA?

○ $\frac{mn}{abcd}$ ○ $\frac{abcd}{mn}$ ● $\frac{(b-a)(d-c)}{mn}$ ○ $\frac{mn}{(b-a)(d-c)}$ ○ $\frac{mn}{(d-a)(c-b)}$

(d) Complete the formula for the average value of f over R:

$$\text{Average value of } f \text{ over } R \quad = \quad \frac{1}{\text{area of } R} \int\int_R f(x, y)\, dA$$

(e) The definite integral $\int_a^b g(x)\, dx$ has a geometric interpretation: It is the (signed) area under the curve $y = g(x)$ over the interval $[a, b]$. State the corresponding geometric interpretation for the double integral

$$\int\int_R f(x, y)\, dA.$$

The double integral represents the signed volume of the solid under the surface $z = f(x, y)$ and above the region R in the xy-plane.

2. We are constructing a rectangular box with three faces in the coordinate planes. One vertex is at the origin, and the opposite vertex (x, y, z) is on the plane

$$5x + 2y + z = 60.$$

We rejoice at the opportunity to use Lagrange multipliers to find the dimensions of the box with maximum volume.

(a) The volume of the box in terms of x, y, and z is $V(x, y, z) =$ $\underline{\quad xyz \quad}$

(b) What is our constraint? $\underline{\quad g(x, y, z) = 5x + 2y + z = 60 \quad}$

(c) List the four equations in four unknowns that arise from the method of Lagrange multipliers.

$\underline{\quad yz = \lambda \cdot 5 \quad}$

$\underline{\quad xz = \lambda \cdot 2 \quad}$

$\underline{\quad xy = \lambda \cdot 1 \quad}$

$\underline{\quad 5x + 2y + z = 60 \quad}$

The first three equations come from $\nabla V(x, y, z) = \lambda \nabla g(x, y, z)$.
The fourth equation is the constraint.

3. **Multiple Choice.**

(a) A lamina has center of mass $(\overline{x}, \overline{y}) = (3, 5)$ and moment $M_x = 30$ about the x-axis. Find the mass.

● $m = 6$ ○ $m = 10$ ○ $m = 15$ ○ $m = 90$ ○ $m = 150$

Reason. $5 = \overline{y} = \frac{M_x}{m} = \frac{30}{m}$. So $m = 6$.

(b) What solid E has volume given by the triple integral

$$\iiint_E 1\, dV \quad = \quad \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{12}} 1\, dz\, dy\, dx \text{ ?}$$

○ circle ○ sphere ○ hemisphere ● cylinder ○ cube

Reason: The inequalities $-2 \leq x \leq 2$ and $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ define the circle of radius 2 centered at the origin in the xy-plane. This circle is the base of the solid E. The height is the surface $z = \sqrt{12}$, which is a plane. Thus E is a cylinder.

(c) Use a Midpoint approximation with $m = n = 2$ to estimate

$$\int_0^4 \int_0^4 f(x, y)\, dy\, dx$$

for $f(x, y) = x^2 y$.

○ 40 ○ 80 ○ 140 ● 160 ○ 200

Reason: $$\int_0^4 \int_0^4 f(x, y)\, dy\, dx \approx [f(1,1) + f(3,1) + f(1,3) + f(3,3)]\Delta A$$

$$= \left(1^2 \cdot 1 + 3^2 \cdot 1 + 1^2 \cdot 3 + 3^2 \cdot 3\right) \left(\frac{4-0}{2} \cdot \frac{4-0}{2}\right) = (1 + 9 + 3 + 27)\, 4 = 160$$

(d) Evaluate the double integral,

$$\int\int_R 12(x^2+y^2)\,dA,$$

where R be the ring-shaped region between the two circles with polar equations $r=1$ and $r=2$.

○ 3π ○ 18π ○ 36π ○ 56π ● 90π

Reason: Rewrite the integral in polar coordinates:

$$\int\int_R 12(x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 12r^2\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} 3r^4\Big|_{r=1}^{r=2}\,d\theta = 3(2^4-1^4)\int_0^{2\pi} d\theta = 3(16-1)\cdot 2\pi = 90\pi$$

(e) What shape is the region of integration for the double integral

$$\int_0^1\int_{-x}^{x} x^2\,dy\,dx\ ?$$

○ parabola ○ paraboloid ● right triangle ○ non-right triangle ○ trapezoid

Reason. We have $0\le x\le 1$ and $-x\le y\le x$. These inequalities define a right triangle with vertices $(0,0)$, $(1,1)$, and $(1,-1)$. Note that the integrand x^2 has no effect on the region of integration.

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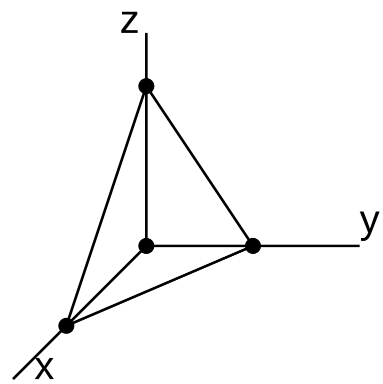
4. Fill in the blanks for the multiple integral expressions for the volume of the tetrahedron bounded by the three coordinate planes and the plane

$$3x + 2y + z = 12.$$

YOU ARE NOT BEING ASKED TO EVALUATE THE INTEGRALS



It is helpful to compute the intercepts $(4, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 12)$.

$$\text{volume} = \int_{0}^{4} \int_{0}^{(12-3x)/2} \int_{0}^{12-3x-2y} 1 \; dz \, dy \, dx$$

$$\text{volume} = \int_{0}^{12} \int_{0}^{(12-z)/2} \int_{0}^{(12-2y-z)/3} 1 \; dx \, dy \, dz$$

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5. A quarter-circular lamina in the first quadrant is bounded by the x- and y-axes and the circle

$$x^2 + y^2 = 4.$$

The density at the point (x, y) is

$$\rho(x, y) \quad = \quad \text{distance from } (x, y) \text{ to the origin}$$

in grams per square unit.

Matching. Write a capital letter in each blank to indicate the correct double integral. Not every capital letter is used.

__A__ mass m of lamina

__F__ area A of lamina

__E__ moment M_x of lamina about the x-axis

__B__ moment M_y of lamina about the y-axis

__G__ moment of inertia I_x of lamina about the x-axis

__D__ moment of inertia I_O of lamina about the origin

Note: density $= \rho = \sqrt{x^2 + y^2} = r$.

A: $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

B: $\displaystyle\int_0^{\pi/2} \int_0^2 r\cos(\theta) \cdot r \cdot r\, dr\, d\theta$

C: $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} x^2\sqrt{x^2+y^2}\, dy\, dx$

D: $\displaystyle\int_0^{\pi/2} \int_0^2 r^4\, dr\, d\theta$

E: $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} y\sqrt{x^2+y^2}\, dy\, dx$

F: $\displaystyle\int_0^{\pi/2} \int_0^2 r\, dr\, d\theta$

G: $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} y^2\sqrt{x^2+y^2}\, dy\, dx$

H: $\displaystyle\int_0^{\pi/2} \int_0^2 r^3\, dr\, d\theta$

6. A student's joy is the product of the amount of money M (in dollars) he has and the amount C (in liters) of cola he has. So

$$\text{Joy} \quad = \quad J(M,C) \quad = \quad MC.$$

Currently he has \$10 and no cola. Each liter of cola costs \$0.40. The purpose of this problem is to determine how much cola the student should buy to maximize his joy. We will use Lagrange multipliers.

(a) What is the constraint?

● $M+0.4C=10$ ○ $0.4M+C=10$ ○ $M+\frac{C}{0.4}=10$ ○ $\frac{M}{0.4}+C=10$

(b) List the three equations in three unknowns that arise from Lagrange multipliers.

$C = \lambda \cdot 1$

$M = \lambda \cdot 0.4$

$M + 0.4C = 10$

Reason: Lagrange tells us to solve the system

$$\{\nabla J = \lambda \nabla g(M,C), \qquad \text{and} \qquad g(M,C) = 10\}$$

where $g(M,C) = M + 0.4C$.
This gives us $\langle C, M\rangle = \lambda\langle 1, 0.4\rangle$ and $M + 0.4C = 10$, which leads to the three equations listed above.

(c) Solve the system to determine how much cola the student should buy. Put your final answer in a box.
Since $C = \lambda$, the second equation becomes $M = 0.4C$.
Plug this into the constraint to get $0.4C + 0.4C = 10$.
Solve to find $\boxed{C = 12.5 \text{ liters.}}$

SM 223 Test#4 9 Dec 2009 Last Name <u>SOLUTIONS</u> **Period** ○ **1st** ○ **2nd** ○ **3rd**

7. (a) Fill in the correct bubble and show your work for full or partial credit. Evaluate the iterated integral

$$\int_0^3 \int_0^4 x^2 y \, dy \, dx.$$

○ 12 ○ 18 ○ 24 ○ 36 ● 72

Reason:

$$\int_0^3 \int_0^4 x^2 y \, dy \, dx = \int_0^3 \left. \frac{x^2 y^2}{2} \right|_{y=0}^{y=4} dx = 8 \int_0^3 x^2 \, dx = \left. \frac{8x^3}{3} \right|_{x=0}^{x=3} = \frac{8 \cdot 3^3}{3} = 8 \cdot 3^2 = 72.$$

(b) A sprinkler distributes water in a circular pattern. It supplies water to a depth of $f(r)$ feet per hour at each point at distance r from the sprinkler.

i. Interpret the value of

$$\frac{1}{\pi \cdot 12^2} \int_0^{2\pi} \int_0^{12} f(r) \, dr \, d\theta$$

in words understandable by a middle school student.

<u>The average depth of water supplied by</u>

<u>the sprinkler over a circle of radius 12 feet</u>

ii. Which one of the following functions is reasonable for $f(r)$?

○ r^2 ○ $12r$ ○ $e^{r/12}$ ○ e^{12r} ● $e^{-r/12}$

Reason. The other functions grow arbitrarily large as r increases, which is not reasonable because less water should be distributed far away from the sprinkler.

8. Throughout this problem we consider the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx.$$

(a) The region of integration is roughly triangular. One of the three corners is $(0,0)$. Fill in *two* bubbles for the other two corners.

○ $(2,0)$ ○ $(4,0)$ ● $(0,2)$ ○ $(0,4)$ ○ $(2,4)$ ● $(4,2)$

Reason. The region of integration is

$$R: \quad \left\{ \begin{array}{l} x = 0 \text{ to } \; x = 4 \\ y = \sqrt{x} \text{ to } \; y = 2 \end{array} \right\}$$

(b) Rewrite the iterated integral in the reverse order of integration:

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx \quad = \quad \int_{\underline{0}}^{\underline{2}} \int_{\underline{0}}^{\underline{y^2}} \underline{\quad e^{y^3} \quad}\, dx\, dy$$

(c) Evaluate the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx.$$

Fill in the correct bubble and show your work for full or partial credit.

○ e^8 ○ $e^8 - 1$ ● $\frac{e^8-1}{3}$ ○ $3e^8$ ○ none of above

Reason. The integral is easier to evaluate if we reverse the order of integration:

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx = \int_0^2 \int_0^{y^2} e^{y^3}\, dx\, dy = \int_0^2 e^{y^3} x\Big|_{x=0}^{x=y^2}\, dy$$

$$= \int_0^2 e^{y^3} y^2\, dy = \frac{e^{y^3}}{3}\Bigg|_{y=0}^{y=2} = \frac{e^8 - 1}{3}$$