SM 223_{TSM} Test #1 [20 Sep 2010] **Last Name** SOLUTIONS **Section:** ○ 4041 ○ 5041 ○ 6041

Instructions

0. Failure to follow instruction can result in your losing points.
1. **Do NOT turn the page or begin until instructed to do so.**
2. Print your name and indicate your section above.
3. **Write nothing else on this cover page,** except your signature on the line below to indicate you've read and understood the directions.
4. There are $\boxed{\mathbf{7}}$ problems altogether. Relative weights are given in the table.
5. For fill-in-the-blank, multiple-choice, matching, and similar problems, write your answer directly on the test paper. There is plenty of space on each page (and the back) for your work. Although your work will not be graded, you might receive part-credit based on how "good" your incorrect answer is.
6. If a problem requests you to show your work, use the space provided to receive credit.
7. **Calculators are forbidden for the last page of the test. Do that page first. You must hand in that page before putting your calculator on your desk.**
8. Unless otherwise indicated, leave answer in exact form; don't approximate $\sqrt{2}$ as 1.41, for instance.

Signature ______________________________

DO NOT WRITE ANYTHING ON THIS PAGE BELOW THIS LINE

Problem	Points	Score
1	35	
2	60	
3	60	
4	40	
5	40	
6	40	
7	25	
Total	**300**	

Test Score	%	Grade
240	80	A
210	70	B
180	60	C
< 180	< 60	F

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1. Complete the formulas.

(a) Give a vector formula for the area of the triangle determined by the tail-to-tail 3-dimensional vectors **a** and **b**.

Area of triangle $=$ $\underline{\quad \frac{1}{2}|\mathbf{a} \times \mathbf{b}| \quad}$

(b) Give the parametric equations of the line with anchor point (x_0, y_0, z_0) and direction vector $\langle a, b, c\rangle$.

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

(c) Give an equation of the plane with anchor point (x_0, y_0, z_0) and normal vector $\langle a, b, c\rangle$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(d) In class we explained where the equation of the plane in (c) came from. Which one of the following did we use?

- ○ cross product
- ○ determinant
- ○ Right-Hand Rule
- ○ resultant
- ● dot product

Reason. Here is how we derived the equation of the plane in class: The point $P = (x, y, z)$ lies on the plane exactly when the vector $\overrightarrow{P_0P}$ is orthogonal to the normal vector $\overrightarrow{N} = \langle a, b, c\rangle$. This means the dot product of $\overrightarrow{P_0P}$ and $\overrightarrow{N}$ must be 0:

$$0 = \overrightarrow{P_0P} \cdot \overrightarrow{N} = \langle x - x_0,\ y - y_0,\ z - z_0\rangle \cdot \langle a, b, c\rangle = a(x - x_0) + b(y - y_0) + c(z - z_0).$$

This is the equation of the plane in (c).

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2. Consider the three points

$$P = (3, 3, 4), \qquad Q = (2, 5, 4), \qquad R = (2, 3, 7).$$

Here are some facts you should use to answer the fill-in-the-blank and multiple choice questions:

$$\boxed{\overrightarrow{PQ} = \langle -1, 2, 0\rangle, \quad |\overrightarrow{PQ}| = \sqrt{5}, \quad |\overrightarrow{PR}| = \sqrt{10}, \quad \overrightarrow{PQ}\cdot\overrightarrow{PR} = 1, \quad \overrightarrow{PQ}\times\overrightarrow{PR} = \langle 6, 3, 2\rangle.}$$

(a) Find a unit vector parallel to $\overrightarrow{PQ}$: $\underline{\frac{1}{\sqrt{5}}\langle -1, 2, 0\rangle \quad \text{or} \quad \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle}$

Reason: $\mathbf{u} = \left(\frac{1}{|\overrightarrow{PQ}|}\right)\overrightarrow{PQ} = \frac{1}{\sqrt{5}}\overrightarrow{PQ} = \frac{1}{\sqrt{5}}\langle -1, 2, 0\rangle$

(b) Find a vector in the opposite direction as $\overrightarrow{PQ}$ with length 13: $\underline{-\frac{13}{\sqrt{5}}\langle\ -1\ ,\ 2\ ,\ 0\ \rangle}$

Reason: Simply take -13 times the unit vector in (a).

(c) Give a vector equation of the line through P and Q:

$$\underline{\mathbf{r}(t) = \langle 3, 3, 4\rangle + t\langle -1, 2, 0\rangle}$$

Reason: We may take $P = (3, 3, 4)$ as our anchor point.
A direction vector is $\overrightarrow{PQ} = \langle -1, 2, 0\rangle$.
Now plug into the formula $\mathbf{r}(\mathbf{t}) = \langle x_0, y_0, z_0\rangle + t\langle a,\ b,\ c\rangle$.

(d) Give an equation of the plane through P, Q, and R:

$$\underline{6(x-3) + 3(y-3) + 2(z-4) = 0 \qquad \text{or} \qquad 6x + 3y + 2z = 35}$$

Reason: We may take $P = (3, 3, 4)$ as our anchor point.
A normal vector is $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle 6, 3, 2\rangle$.
Now plug into the basic plane equation $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.

(e) Find the area of $\triangle PQR$.

● 3.5 ○ 7 ○ 1 ○ 6.5 ○ 13

Reason: Area of triangle $= \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \frac{1}{2}\sqrt{6^2 + 2^2 + 1^2} = \frac{1}{2}\sqrt{49} = 3.5$

(f) Find the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ to the nearest 5 degrees.

○ 75° ● 80° ○ 85° ○ 90° ○ 95°

Reason. $\cos(\theta) = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{1}{\sqrt{5}\sqrt{10}}$. So $\theta = \arccos(1/\sqrt{50}) \approx 82° \approx 80°$.

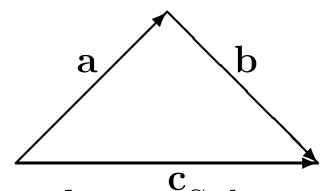
3. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form an isosceles right triangle with

$$|\mathbf{a}| = |\mathbf{b}| = \sqrt{2}, \qquad |\mathbf{c}| = 2,$$

as shown in the figure.

(a) Which relationship is true?

○ $\mathbf{b} = \mathbf{a} + \mathbf{c}$
○ $\mathbf{b} = \mathbf{a} - \mathbf{c}$
● $\mathbf{b} = \mathbf{c} - \mathbf{a}$
○ $\mathbf{c} = \sqrt{2}\,(\mathbf{a} + \mathbf{b})$
○ $\mathbf{c} = 2\,(\mathbf{a} + \mathbf{b})$



Reason. From the head-to-tail relationship, we have $\mathbf{a} + \mathbf{b} = \mathbf{c}$. Subtract $\mathbf{a}$ from both sides to solve for $\mathbf{b}$.

(b) $\mathbf{c} \cdot \mathbf{c} =$

○ 0 ○ $\sqrt{2}$ ○ 2 ○ $2\sqrt{2}$ ● 4

Reason: $\mathbf{c} \cdot \mathbf{c} = |\mathbf{c}|^2 = 2^2 = 4.$

(c) A unit vector in the same direction as $\mathbf{c}$ is

● $\frac{1}{2}\mathbf{c}$ ○ $\frac{1}{\sqrt{2}}\mathbf{c}$ ○ $2\,\mathbf{c}$ ○ $\sqrt{2}\,\mathbf{c}$ ○ $\mathbf{c}$

Reason: $\mathbf{u} = \left(\frac{1}{|\mathbf{c}|}\right)\mathbf{c} = \frac{1}{2}\,\mathbf{c}.$

(d) $\mathbf{a} \cdot \mathbf{b} =$

● 0 ○ $\sqrt{2}$ ○ 2 ○ $2\sqrt{2}$ ○ 4

Reason: The vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, so their dot product is 0.

(e) The vector $\mathbf{a} \times \mathbf{c}$

○ is the zero vector ○ is a unit vector ○ points out of the page
● points into the page ○ none of above

Reason: Apply the Right-Hand Rule.

(f) $\frac{1}{2}|\mathbf{a} \times \mathbf{c}| =$

○ $\frac{1}{2}$ ○ $\frac{1}{\sqrt{2}}$ ○ 2 ○ $\sqrt{2}$ ● 1

Solution #1: $\frac{1}{2}|\mathbf{a} \times \mathbf{c}| = \frac{1}{2}|\mathbf{a}||\mathbf{c}|\sin(45^\circ) = \frac{1}{2}(\sqrt{2})(2)\frac{1}{\sqrt{2}} = 1.$
Solution #2: $\frac{1}{2}|\mathbf{a} \times \mathbf{c}| = \text{area of triangle} = \frac{1}{2}\,(\text{base})\,(\text{height})) = \frac{1}{2}(2)(1) = 1.$

4. Multiple choice and fill-in-the-blank.

(a) Give the parametric equations of the line through the points

$$(1, 2, 3) \qquad \text{and} \qquad (20, 13, -12).$$

Answer: $x = 1 + 19t \qquad y = 2 + 11t \qquad z = 3 - 15t$

Reason: Use the anchor point $(1, 2, 3)$ and direction vector $\langle 20-1,\ 13-2,\ -12-3\rangle = \langle 19, 11, -15\rangle$.

(b) Give an equation of the plane through the origin and parallel to the plane

$$x + 2y - 3z = 2012.$$

Answer: $x + 2y - 3z = 0$

Solution: Choose anchor point $(x_0, y_0, z_0) = (0, 0, 0)$ and normal vector $\langle a, b, c\rangle = \langle 1, 2, -3\rangle$.

Then substitute into the basic formula for a plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

(c) This problem deals with the sphere

$$(x-1)^2 + (y+4)^2 + (z-3)^2 = 25.$$

i. Find the center and radius of the sphere.

center = $(\ 1\ ,\ -4\ ,\ 3\ )$ radius = 5

ii. Which one of the listed planes is tangent to the sphere and parallel to the xy-plane?

● $z = 8$ ○ $z = 3$ ○ $z = 5$ ○ $z = 0$ ○ $x - 4y + 3z = 5$

Reason: The plane must be of the form $z = c$ for some constant c. The radius of the sphere is 5. So the desired plane will be 5 units above the center or 5 units below the center. Since the z-coordinate of the center of the sphere is 3, the desired planes are $z = 3 + 5 = 8$ and $z = 3 - 5 = -2$. Only $z = 8$ is among the listed choices.

(d) Which vector is parallel to the line of intersection of the two planes

$$3x - 2y + z = 1 \qquad \text{and} \qquad 2x + y - 3z = 3\,?$$

○ $\langle 3, -2, 1\rangle$
○ $\langle 2, 1, -3\rangle$
○ $\langle 3, -2, 1\rangle + \langle 2, 1, -3\rangle$
○ $\langle 3, -2, 1\rangle - \langle 2, 1, -3\rangle$
● $\langle 3, -2, 1\rangle \times \langle 2, 1, -3\rangle$

Reason: The cross product of the normal vectors to the planes is a vector that is perpendicular to both normal vectors—and hence parallel to the line of intersection.

5. Multiple choice and fill-in-the-blank.

(a) How far is $P = (8, 4, -1)$ from the xy-plane?

● 1 ○ $\sqrt{12}$ ○ $\sqrt{80}$ ○ 9 ○ 11

Reason. Since the z-coordinate of P is -1, the point is 1 unit from the xy-plane.

(b) Which curves arise as (x-, y-, or z-) traces for the surface

$$z = x^2 + 4y^2\,?$$

INDICATE ALL CORRECT ANSWERS

○ circles ● ellipses ● parabolas ○ hyperbolas ○ lines

Solution: x-traces: $z = k^2 + 4y^2$ (parabolas);
y-traces: $z = x^2 + 4k^2$ (parabolas);
z-traces: $k^2 = x^2 + 4y^2$ (ellipses).

(c) Give the *best* name for each surface.

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{16} = 1$$ hyperboloid of 1 sheet

$$\frac{x}{9} - \frac{y}{4} + \frac{z}{16} = 1$$ plane

$$x^2 + y^2 - z^2 = 0$$ cone

$$z = y^2$$ parabolic cylinder

$$z = \sqrt{1 - x^2 - y^2}$$ hemisphere

Reason: Square both sides and rearrange to get $x^2 + y^2 + z^2 = 1$. This is a sphere with radius 1. However the original equation is of the form $z = \sqrt{}$, which implies that z can never be negative. So we only get the upper half of the sphere.

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NO CALCULATOR ALLOWED FOR THIS PAGE

FILL IN YOUR NAME AND SECTION NUMBER.
Detach this sheet from the rest of the test.
Solve the problems without using your calculator.
After you hand in this sheet of paper, you can use your calculator.

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6. Let

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = 2\mathbf{j} - \mathbf{k}$$

Put your answers in the blanks and show your work in the space provided.

(a) $\mathbf{a} - \mathbf{b} =$ $\underline{\ \mathbf{i} - 4\mathbf{j} + 3\mathbf{k}\ }$

Reason. $\mathbf{a} - \mathbf{b} = \langle 1, -2, 2\rangle - \langle 0, 2, -1\rangle = \langle 1-0,\ -2-2,\ 2-(-1)\rangle = \langle 1, -4, 3\rangle = \mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

(b) $|\mathbf{a}| =$ $\underline{\ 3\ }$

Reason. $|\mathbf{a}| = |\langle 1, -2, 2\rangle| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

(c) $\mathbf{a} \cdot \mathbf{b} =$ $\underline{\ -6\ }$

Reason. $\mathbf{a} \cdot \mathbf{b} = \langle 1, -2, 2\rangle \cdot \langle 0, 2, -1\rangle = (1)(0) + (-2)(2) + (2)(-1) = -6$

(d) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} =$ $\underline{\ 0\ }$

Solution #1. The cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$. So the given dot product must be zero!

Solution #2 (Outline). Expand the cross product using the determinant. Then dot the resulting vector with $\mathbf{a}$. You get 0.

7. This problem is about the non-zero vectors

$$\mathbf{a} = \langle a_1,\, a_2,\, a_3 \rangle \qquad \text{and} \qquad \mathbf{b} = \langle b_1,\, b_2,\, b_3 \rangle.$$

(a) If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{b}$ is a scalar multiple of $\mathbf{a}$, say, $\mathbf{b} = s\,\mathbf{a}$ for some scalar s, and so we can write

$$\langle b_1,\, b_2,\, b_3 \rangle = \langle sa_1,\, sa_2,\, sa_3 \rangle.$$

FILL IN THE BLANKS, COMPLETE THE COMPUTATION OF THE CROSS PRODUCT, AND SIMPLIFY YOUR ANSWER

$$\mathbf{a} \times \mathbf{b} = \langle a_1, a_2, a_3 \rangle \times \langle \underline{\ sa_1\ },\ \underline{\ sa_2\ },\ \underline{\ sa_3\ } \rangle = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ \underline{sa_1} & \underline{sa_2} & \underline{sa_3} \end{bmatrix}$$

$$= \mathbf{i}\,\underbrace{(a_2(sa_3) - a_3(sa_2))}_{0} - \mathbf{j}\,\underbrace{(a_1(sa_3) - a_3(sa_1))}_{0} + \mathbf{k}\,\underbrace{(a_1(sa_2) - a_2(sa_1))}_{0}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \langle 0, 0, 0 \rangle$$

(b) Use your preceding work (or mathematical insight) to complete the statement of the theorem succinctly.

Theorem. If $\mathbf{a}$ and $\mathbf{b}$ are parallel 3-dimensional vectors, then

$$\mathbf{a} \times \mathbf{b} = \underline{\quad \langle 0, 0, 0 \rangle \quad}$$

Solution: Part (a) proves this statement!