SM 223_{TSM} Test #2 [15 Oct 2012] Last Name SOLUTIONS Section: ○ 4023 ○ 5023 ○ 6023

Instructions

0. Failure to follow instructions can result in your losing points.
1. **Do NOT turn the page or begin until instructed to do so.**
2. Print your name and indicate your section above.
3. **Write nothing else on this cover page,** except your signature on the line below to indicate you've read and understood the directions.
4. There are $\boxed{\textbf{10}}$ problems altogether. Point values are given in the table.
5. For fill-in-the-blank, multiple-choice, matching, and similar problems, write your answer directly on the test paper. There is plenty of space on each page (and the back) for your work. Although your work will not be graded, you might receive part-credit based on how "good" your incorrect answer is.
6. If a problem requests you to show your work, use the space provided to receive credit.
7. **Calculators are forbidden for the last page of the test only. Do that page first. You must hand in that page before using your calculator.**
8. Unless otherwise indicated, leave answer in exact form; don't approximate $\sqrt{2}$ as 1.41, for instance.

Signature ______________________________

DO NOT WRITE ANYTHING ON THIS PAGE BELOW THIS LINE

Problem	Points	Score
1	30	
2	30	
3	30	
4	30	
5	30	
6	30	
7	30	
8	30	
9	30	
10	30	
Total	**300**	

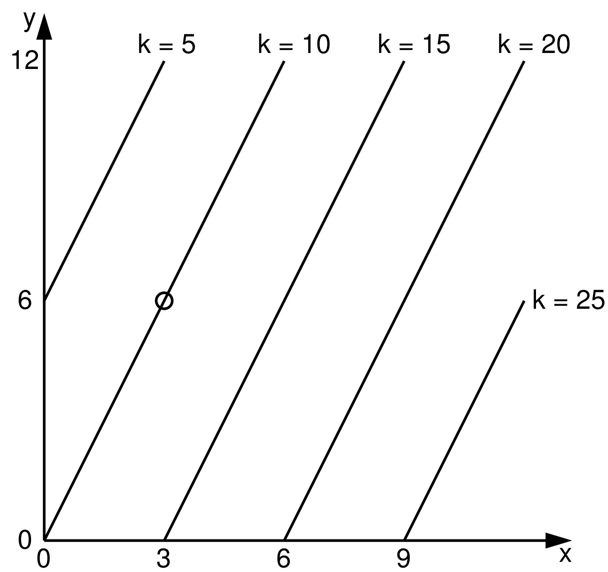
Test Score	%	Grade
240	80	A
210	70	B
180	60	C
< 180	< 60	F

SM 223 **Test #2** $\boxed{\textbf{SOLUTIONS}}$ **[15 Oct 2012]**

1. (a) Complete the "limit" definition of the partial derivative of $f(x, y)$ with respect to y at the point (a, b):

$$\frac{\partial f}{\partial y}(a, b) = f_y(a, b) = \underline{\qquad \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} \qquad}$$

(b) The point $(3, 6)$ is shown as a small circle in the contour diagram for $f(x, y)$. Each level curve of f is a straight line of the form $f(x, y) = k$.



i. Evaluate $f(3, 6)$.

○ 2 ○ 3 ○ 3/2 ○ 5 ● 10

ii. Estimate $f_y(3, 6)$.

● $-5/6$ ○ $3/2$ ○ $2/3$ ○ -5 ○ $5/3$

$$f_y(3, 6) = \lim_{h \to 0} \frac{f(3, 6+h) - f(3, 6)}{h} \approx \frac{f(3, 6+6) - f(3, 6)}{6} = \frac{5 - 10}{6} = -\frac{5}{6}$$

(c) Use the table of values for $T(x, y)$ to estimate $T_y(0, 0)$.

○ 2
● 5/2
○ 2/3
○ 3
○ 5

$T(x, y)$	$y = 0$	$y = 2$	$y = 4$
$x = 0$	5	10	16
$x = 3$	7	11	15
$x = 6$	11	12	15

$$T_y(0, 0) = \lim_{h \to 0} \frac{T(0, 0+h) - T(0, 0)}{h} \approx \frac{T(0, 2) - T(0, 0)}{2} = \frac{10 - 5}{2} = \frac{5}{2}$$

2. The function $g(x, y)$ satisfies

$$g(20, 14) = 7, \qquad g_x(20, 14) = 2, \qquad g_y(20, 14) = -3.$$

(a) Find an equation of the tangent plane to the graph $z = g(x, y)$ at the point $(20, 14, 7)$.

Answer: $z - 7 = 2\big(x - 20\big) - 3\big(y - 14\big)$

The equation of the tangent plane to g at point $(20, 14)$ is

$$z - g(20, 14) = g_x(20, 14)\big(x - 20\big) + g_y(20, 14)\big(y - 14\big)$$

which is

$$z - 7 = 2\big(x - 20\big) + (-3)\big(y - 14\big).$$

(b) Use a linear approximation to estimate $g(20.1, 14.2)$.

Answer: $g(20.1, 14.2) \approx 6.6$

PUT YOUR WORK BELOW TO RECEIVE FULL OR PARTIAL CREDIT

The linear approximation asserts that for $(x, y) \approx (20, 14)$ we have

$$\begin{aligned} g(x, y) &\approx g(20, 14) + g_x(20, 14)\big(x - 20\big) + g_y(20, 14)\big(y - 14\big) \\ &= 7 + 2(x - 20) - 3(y - 14). \end{aligned}$$

Therefore

$$g(20.1, 14.2) \approx 7 + 2(20.1 - 20) - 3(14.2 - 14) = 7 + 0.2 - 0.6 = 6.6$$

3. (a) The general formula (involving partial derivatives) for the differential dV of the function $V = V(\ell, w, h)$ is

$$\underline{dV = \left(\frac{\partial V}{\partial \ell}\right) d\ell + \left(\frac{\partial V}{\partial w}\right) dw + \left(\frac{\partial V}{\partial h}\right) dh}$$

(b) A rectangular brick has length ℓ, width w, and height h. The volume of the brick is given by the function

$$V = V(\ell, w, h) = \underline{\quad \ell w h \quad}$$

(c) Suppose the brick has length 10 inches, width 6 inches, and height 5 inches and rests on the ground with a 10-by-6 inch face downward. We apply a coat of paint 0.01 inches thick to the five exposed faces of the brick. Use differentials to estimate the volume of paint used.

○ 1.3 in^3 ○ 1.4 in^3 ○ 2.0 in^3 ● 2.2 in^3 ○ 2.8 in^3

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Reason. We have

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial \ell}\right) d\ell + \left(\frac{\partial V}{\partial w}\right) dw + \left(\frac{\partial V}{\partial h}\right) dh \\ &= (wh)\, d\ell + (\ell h)\, dw + (\ell w)\, dh \\ &= (6 \cdot 5)\, 0.02 + (10 \cdot 5)\, 0.02 + (10 \cdot 6)\, 0.01 \\ &= 0.6 + 1.0 + 0.6 \\ &= 2.2 \text{ in}^3 \end{aligned}$$

4. The velocity of a cross country runner during part of Saturday's N^*-meet was

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 2t-4\rangle \qquad \text{for } 1 \le t \le 4.$$

Time t is measured in minutes, and distance is in some unspecified units of distance. The positive z-axis points straight up.

(a) Her speed at $t = 2$ was

- ● $\sqrt{17}$ units/min **Reason.** $|\mathbf{r}'(2)| = |\langle 1, 4, 0\rangle| = \sqrt{1+16+0} = \sqrt{17}$
- ○ 5 units/min
- ○ $\sqrt{5}$ units/min
- ○ 6 units/min
- ○ $\sqrt{21}$ units/min

(b) At time $t = 2$ she was running

- ○ uphill
- ○ downhill
- ● neither uphill nor downhill **Reason.** The z-component of velocity is 0

(c) Suppose that the runner's position at time $t = 2$ was $\mathbf{r}(2) = \langle 3, 6, 5\rangle$. Give parametric equations of the tangent line to her running path at $t = 2$.

$$\begin{aligned} x(t) &= \underline{\quad 3 + 1t \quad} \\ y(t) &= \underline{\quad 6 + 4t \quad} \\ z(t) &= \underline{\quad 5 + 0t \quad} \end{aligned}$$

Reason. The anchor point of the line is $(3, 6, 5)$ and a direction vector is $\mathbf{r}'(2) = \langle 1, 4, 0\rangle$.

(d) Continue to suppose that the runner's position at time $t = 2$ was $\mathbf{r}(2) = \langle 3, 6, 5\rangle$. Then her position at $t = 1$ was $\mathbf{r}(1) =$

○ $\langle 1, 1, 6\rangle$ ● $\langle 2, 3, 6\rangle$ ○ $\langle 1, 2, 6\rangle$ ○ $\langle 1, 3, 6\rangle$ ○ $\langle 2, 2, 6\rangle$

Reason. We integrate velocity to find position:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\, dt = \int \langle 1, 2t, 2t-4\rangle\, dt = \langle t, t^2, t^2 - 4t\rangle + \mathbf{C}.$$

To find the constant vector $\mathbf{C}$ we use the condition $\mathbf{r}(2) = \langle 3, 6, 5\rangle$.

$$\langle 3, 6, 5\rangle = \mathbf{r}(2) = \langle 2, 2^2, 2^2 - 4\cdot 2\rangle + \mathbf{C} = \langle 2, 4, -4\rangle + \mathbf{C}.$$

So $\langle 1, 2, 9\rangle = \mathbf{C}$, and we can write $\mathbf{r}(t) = \langle t, t^2, t^2 - 4t\rangle + \langle 1, 2, 9\rangle$. Therefore $\mathbf{r}(1) = \langle 1, 1^2, 1^2 - 4\cdot 1\rangle + \langle 1, 2, 9\rangle = \langle 2, 3, 6\rangle$.

5. When x thousand dollars is spent on labor, and y thousand dollars is spent on advertising, a bottled water company sells

$$B(x,y) \quad = \quad 10x^{1/2}y^{3/2}$$

thousand gallons of water per month.

(a) How many gallons of water does the company sell when it spends \$25,000 on labor and \$4,000 on advertising? In other words, compute $B(25,4)$.

● 400,000 ○ 200,000 ○ 500,000 ○ 800,000 ○ 4,000,000

Reason: $B(25,4) = 10{\cdot}25^{1/2}{\cdot}4^{3/2}$ thousand $= 10{\cdot}5{\cdot}8$ thousand $= 400,000$.

(b) Compute $B_y(25,4)$.

● 150 ○ 8 ○ 30 ○ 50 ○ 4

Reason: $B_y(25,4) = \dfrac{\partial}{\partial y}\left(10x^{1/2}y^{3/2}\right) = 10x^{1/2}\cdot\dfrac{3}{2}y^{1/2} = 15\sqrt{x}\sqrt{y}.$

So $B_y(25,4) = 15\sqrt{25}\sqrt{4} = 15\cdot 5\cdot 2 = 150.$

(c) **Fact:** $B_x(25,4) = 8$.
Interpret this fact in a complete English sentence or two.

Suppose the company is currently spending \$25,000 on labor and \$4,000 on advertising. Then for every \$1,000 more it spends on labor (keeping advertising expenditures constant), 8,000 more gallons of water are sold per month.

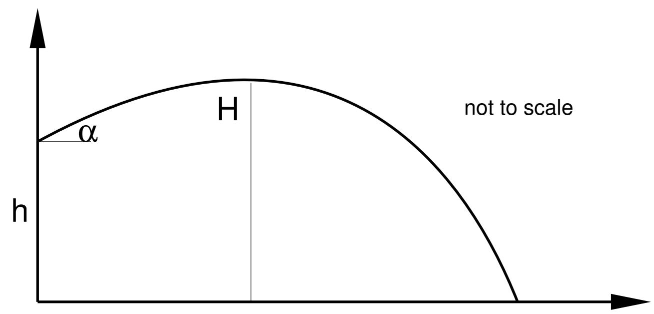
(d) Suppose the company is currently spending \$25,000 on labor and \$4,000 per month on advertising. If the company has budgeted an additional \$1,000 to spend next month on either labor or advertising, what should it do to improve sales the most?

○ spend \$1,000 on labor
● spend \$1,000 on advertising

Reason: $B_y(25,4) = 150 > 8 = B_x(25,4)$. The company sells more water by increasing advertising rather than labor.

6. A projectile is fired from a cliff of height h with initial speed v_0 at an angle α above horizontal. The x- and y-axes are in their usual orientations with the origin at the base of the cliff. Distance is measured in meters, and time in seconds. The projectile reaches a maximum height H and hits the ground with speed v^*. This experiment occurs on a planet where the acceleration due to gravity is $\mathbf{g} = \langle 0, -g \rangle$. Do not assume that $g = 9.8$ m/sec^2. The equations of motion for the projectile are:

$$\begin{aligned} x = x(t) &= 150t \\ y = y(t) &= -20t^2 + 200t + 1500 \\ &= -20(t+5)(t-15). \end{aligned}$$



Fill in the correct bubble for each parameter. Give the angle α to the nearest degree and the speed at impact v^* to the nearest m/s.

h	g	v_0	α	H	v^*
○ 20 m	○ 5 m/s^2	○ 200 m/s	○ 33°	○ 1000 m	○ 423 m/s
○ 150 m	○ 10 m/s^2	● 250 m/s	○ 43°	○ 1200 m	● 427 m/s
○ 200 m	○ 15 m/s^2	○ 350 m/s	● 53°	○ 1500 m	○ 442 m/s
● 1500 m	○ 20 m/s^2	○ 700 m/s	○ 63°	● 2000 m	○ 474 m/s
○ 3000 m	● 40 m/s^2	○ 750 m/s	○ 73°	○ 2500 m	○ 486 m/s

Our general equations for motion are

$$x(t) = (v_0 \cos(\alpha))\, t \qquad \text{and} \qquad y(t) = -\frac{1}{2} g t^2 + (v_0 \sin(\alpha))\, t + h.$$

$\boxed{h}$ $h = y(0) = 1500$ m.

$\boxed{g}$ Match the quadratic coefficients in the expression for y to see that $-20t^2 = -\frac{1}{2}gt^2$. So $20 = \frac{1}{2}g$ and $g = 40$.
OR: Compute acceleration in the y-direction as $y''(t) = -40$ m/s^2.

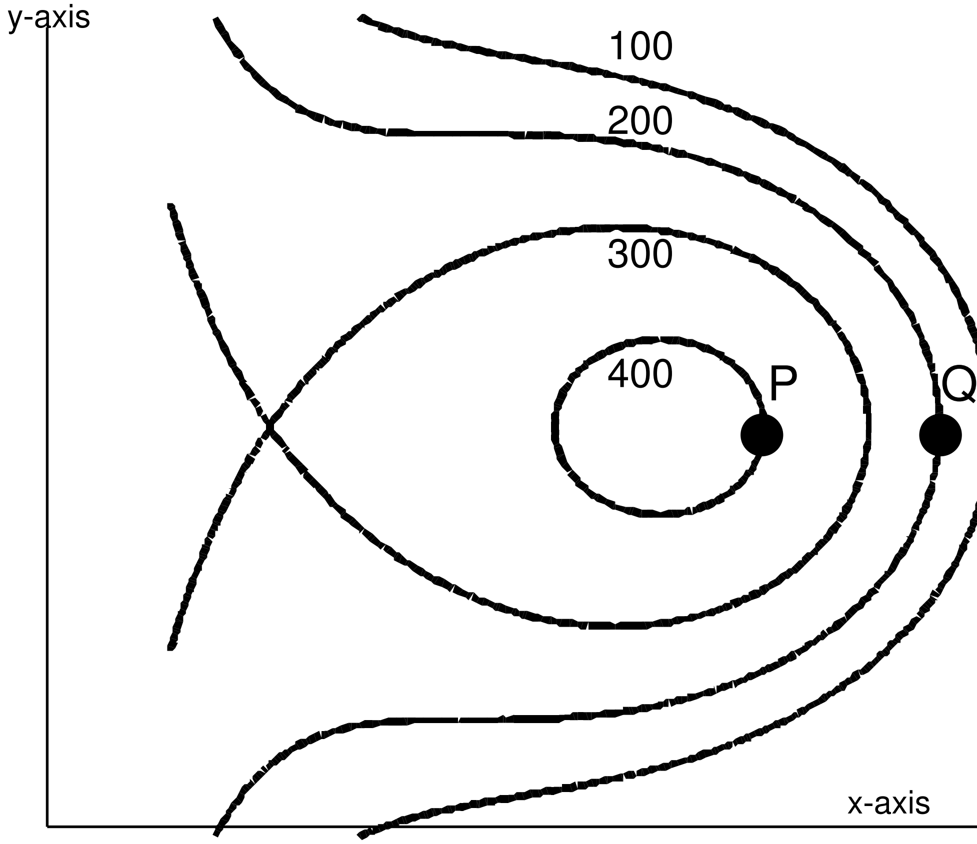
$\boxed{v_0}$ The velocity vector is $\langle x'(t), y'(t) \rangle = \langle 150,\ -40t + 200 \rangle$.
So the initial velocity vector is $\langle x'(0), y'(0) \rangle = \langle 150,\ 200 \rangle$.
The initial speed is $v_0 = |\langle 150, 200 \rangle| = \sqrt{150^2 + 200^2} = 250$ m/s.

$\boxed{\alpha}$ We have $\tan(\alpha) = \frac{v_0 \sin(\alpha)}{v_0 \cos(\alpha)} = \frac{200}{150}$. So $\alpha = \arctan(\frac{200}{150}) \approx 53°$.

$\boxed{H}$ The maximum height occurs when $0 = y'(t) = -40t + 200$, i.e, at time $t = 5$ s. So $H = y(5) = -20(5+5)(5-15) = -20 \cdot 10 \cdot (-10) = 2000$ m.

$\boxed{v^*}$ We set $y(t) = 0$ to find the time of impact $t^* = 15$ s. The velocity at impact is $\langle x'(t), y'(t) \rangle|_{t=t^*} = \langle 150, -40t + 200 \rangle|_{t=15} = \langle 150, -400 \rangle$. So the speed at impact is $v^* = |\langle 150, -400 \rangle| = \sqrt{150^2 + (-400)^2} \doteq 427$ m/s.

7. Suppose $f(x, y)$ gives elevation (in meters) in hostile terrain as a function of position (x, y). The contour diagram for f is shown with two points P and Q.



(a) Fill in the best bubble in each column.

$f(P) - f(Q)$	$f_x(Q)$	$f_y(Q)$	$f_{xx}(Q)$
● positive	○ positive	○ positive	○ positive
○ negative	● negative	○ negative	● negative
○ zero	○ zero	● zero	○ zero
○ undefined	○ undefined	○ undefined	○ undefined

- $f(P) - f(Q) = 400 - 200 > 0$.
- The function decreases as we move from "west-to-east" through Q.
- Along a vertical ("south-to-north") line through Q, the function is a maximum right at Q.
- The function is concave down in the "west-to-east" direction through Q because $f_x(Q)$ is negative, and the contours are closer together to the right of Q than to the left of Q.

(b) i. Can a prone sniper at P see the entrance to an underground cave at Q?
○yes ●no

ii. Explain. The terrain between P and Q is concave down. So the direct line-of-sight from P to Q would pass underground.

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8. Throughout this problem we let

$$f(x,y) = e^{x^2y}.$$

Fill in the correct bubble. Also, show your work!

(a) Compute $f_x(5,2)$.

● $20e^{50}$ ○ $25e^{50}$ ○ $50e^{50}$ ○ $100e^{50}$ ○ none of above; correct is ______

$$f_x = \frac{\partial}{\partial x}\left(e^{x^2y}\right) = e^{x^2y}\cdot\frac{\partial}{\partial x}\left(x^2y\right) = e^{x^2y}\cdot 2xy = 2xy\cdot e^{x^2y}.$$

$$\text{So} \qquad f_x(5,2) = 2\cdot 5\cdot 2\cdot e^{5^2\cdot 2} = 20\,e^{50}.$$

(b) Compute $f_y(5,2)$.

○ $20e^{50}$ ● $25e^{50}$ ○ $50e^{50}$ ○ $100e^{50}$ ○ none of above; correct is ______

$$f_y = \frac{\partial}{\partial y}\left(e^{x^2y}\right) = e^{x^2y}\cdot\frac{\partial}{\partial y}\left(x^2y\right) = e^{x^2y}\cdot x^2 = x^2\cdot e^{x^2y}.$$

$$\text{So} \qquad f_y(5,2) = 5^2e^{5^2\cdot 2} = 25\,e^{50}.$$

(c) Compute $f_{xy}(5,2)$.

○ $800e^{50}$ ○ $820e^{50}$ ○ $1000e^{50}$ ● $510e^{50}$ ○ $2012e^{50}$

$$f_{xy} = \frac{\partial}{\partial y}\left(f_x\right) = \frac{\partial}{\partial y}\left(2xy\cdot e^{x^2y}\right) = 2xy\cdot\frac{\partial}{\partial y}\left(e^{x^2y}\right) + e^{x^2y}\cdot\frac{\partial}{\partial y}\left(2xy\right)$$

$$= 2xy\left(e^{x^2y}\cdot x^2\right) + e^{x^2y}\cdot 2x = \left(2x^3y + 2x\right)e^{x^2y}.$$

$$\text{So} \qquad f_{xy}(5,2) = \left(2\cdot 5^3\cdot 2 + 2\cdot 5\right)e^{50} = 510\,e^{50}.$$

9. Complete the formulas.

   (a) The "limit" definition of the derivative of the vector-valued function $\mathbf{r}(t)$:

$$\mathbf{r}'(t) \quad = \quad \underline{\lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}}$$

   (b) The length of the space curve $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ for $t = a$ to $t = b$:

$$L \quad = \quad \underline{\int_a^b |\mathbf{r}'(t)|\, dt \quad OR \quad \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt}$$

   (c) The tangent plane to the surface $z = f(x,y)$ at the point $(x,y,z) = (a,b,f(a,b))$:

$$\underline{z - f(a,b) = f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)}$$

   (d) Linear approximation to the function $f(x,y)$ near (a,b):

$$\underline{\text{If } (x,y) \approx (a,b), \text{ then } f(x,y) \approx \quad f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)}$$

10. Fill in a bubble in each row.

TRUE	FALSE	
○	●	The graph of $f(x,y,z)$ is a surface in three dimensions.
●	○	The contour diagram of $f(x,y,z)$ is a family of surfaces in three dimensions.
○	●	If a particle's speed is constant, then its acceleration vector is $\mathbf{0}$.
●	○	If a particle's velocity is constant, then its acceleration vector is $\mathbf{0}$.
●	○	For uniform circular motion, the velocity vector is perpendicular to the position vector.