SM $223_{\text{TSM}}$ Test #3 [8 Nov 2010] Last Name SOLUTIONS Section: ○ 4041 ○ 5041 ○ 6041

## Instructions

0. Failure to follow instructions can result in your losing points.
1. **Do NOT turn the page or begin until instructed to do so.**
2. Print your name and indicate your section above.
3. **Write nothing else on this cover page,** except your signature on the line below to indicate you've read and understood the directions.
4. There are **8** problems altogether. Relative weights are given in the table.
5. For fill-in-the-blank, multiple-choice, matching, and similar problems, write your answer directly on the test paper. There is plenty of space on each page (and the back) for your work. Although your work will not be graded, you might receive part-credit based on how "good" your incorrect answer is.
6. If a problem requests you to show your work, use the space provided to receive credit.
7. **Calculators are forbidden for the last page of the test. Do that page first. You must hand in that page before putting your calculator on your desk.**
8. Unless otherwise indicated, leave answer in exact form; don't approximate $\sqrt{2}$ as 1.41, for instance.

**Signature** ____________________________

DO NOT WRITE ANYTHING ON THIS PAGE BELOW THIS LINE

| Problem | Points | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 30 | |
| 3 | 40 | |
| 4 | 30 | |
| 5 | 30 | |
| 6 | 80 | |
| 7 | 30 | |
| 8 | 30 | |
| **Total** | **300** | |

| | | |
|---|---|---|
| **Test 1** | **300** | |
| **Test 2** | **300** | |
| **Test 3** | **300** | |
| **Quizzes*** | **400** | |
| **12-Week Total** | **1300** | |
| **12-Week Grade** | % | **Grade** |
| 1040 | 80 | *A* |
| 910 | 70 | *B* |
| 780 | 60 | *C* |
| 715 | 55 | *D* |
| < 715 | < 60 | *F* |

* 8 best quizzes (out of 12) times 5

| Test Score | % | Grade |
|---|---|---|
| 240 | 80 | *A* |
| 210 | 70 | *B* |
| 180 | 60 | *C* |
| < 180 | < 60 | *F* |

**SM 223 Test #3 SOLUTIONS [8 Nov 2010]**

1. Complete the formulas.

(a) The gradient of $f(x, y, z)$ is defined as

$$\text{grad}(f) = \nabla f = \underline{\langle f_x,\, f_y,\, f_z \rangle}$$

(b) The formula for the directional derivative of $f$ at point $P$ in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}(f(P)) = \underline{\nabla f(P) \cdot \mathbf{u}}$$

(c) Fill in the blanks with mathematical expressions to complete the statement of the theorem.

*Let $f = f(x, y, z)$ be a function of three variables and let $P$ be a point.*

i. *The maximum rate of change of $f$ at $P$ occurs in the direction* $\underline{\nabla f(P)}$

ii. *The maximum rate of change of $f$ at $P$ equals* $\underline{|\nabla f(P)|}$

(d) Complete the proof of both parts of the Theorem.

**Proof.** Let $\mathbf{u}$ be a unit vector. The rate of change of $f$ at $P$ in the direction $\mathbf{u}$ equals the directional derivative $D_{\mathbf{u}}(f(P))$. From the formula in (b), this directional derivative is

$$D_{\mathbf{u}}(f(P)) = \nabla f(P) \cdot \mathbf{u} = |\nabla f(P)||\mathbf{u}| \cos(\theta) = |\nabla f(P)| \cos(\theta).$$

We have used the fact that the unit vector $\mathbf{u}$ has length 1. Note that

$$\cos(\theta) \leq 1.$$

Therefore

$$D_{\mathbf{u}}(f(P)) = |\nabla f(P)| \cos(\theta) \leq |\nabla f(P)|$$

with equality exactly when $\theta = 0$. However, $\theta = 0$ exactly when $\nabla f(P)$ and $\mathbf{u}$ have the same direction. Thus we have shown that $D_{\mathbf{u}}(f(P)) \leq |\nabla f(P)|$ with equality exactly when the direction $\mathbf{u}$ is in the same direction as $\nabla f(P)$. This proves both parts of the theorem.

2. This problem is about a cylinder with radius $r$ and height $h$.

(a) Complete the formula for the volume of the cylinder. $\underline{V = \pi r^2 h}$

(b) The radius of a certain cylinder is always increasing at the rate of 2 m/hr, but the volume remains constant at $48\pi$ m$^3$.

At a certain instant the radius of the cylinder is 4 m.

What can we say about how the height of the cylinder is changing at this instant?

FILL IN A BUBBLE—AND A BLANK, IF APPROPRIATE.

- ○ We cannot say anything definitive
- ○ The height is not changing
- ○ The height is decreasing at some rate we cannot determine
- ○ The height is increasing at some rate we cannot determine
- ● The height is decreasing at the rate <u>3 m/hr</u>
- ○ The height is increasing at the rate ________

**Solution:** First, observe that since the volume is constant, and the radius is increasing, the height must be decreasing. This narrows down the list of answers to just two possibilities.

Note that when $r = 4$, we have $48\pi = V = \pi(4^2)h$, and so $h = 3$.
We know

$$V(r,h) = \pi r^2 h, \qquad \frac{dr}{dt} = 2, \qquad \frac{dV}{dt} = 0.$$

We seek $\frac{dh}{dt}$ when $r = 4$ and $h = 3$.
By the Chain Rule (draw your own tree diagram),

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}.$$

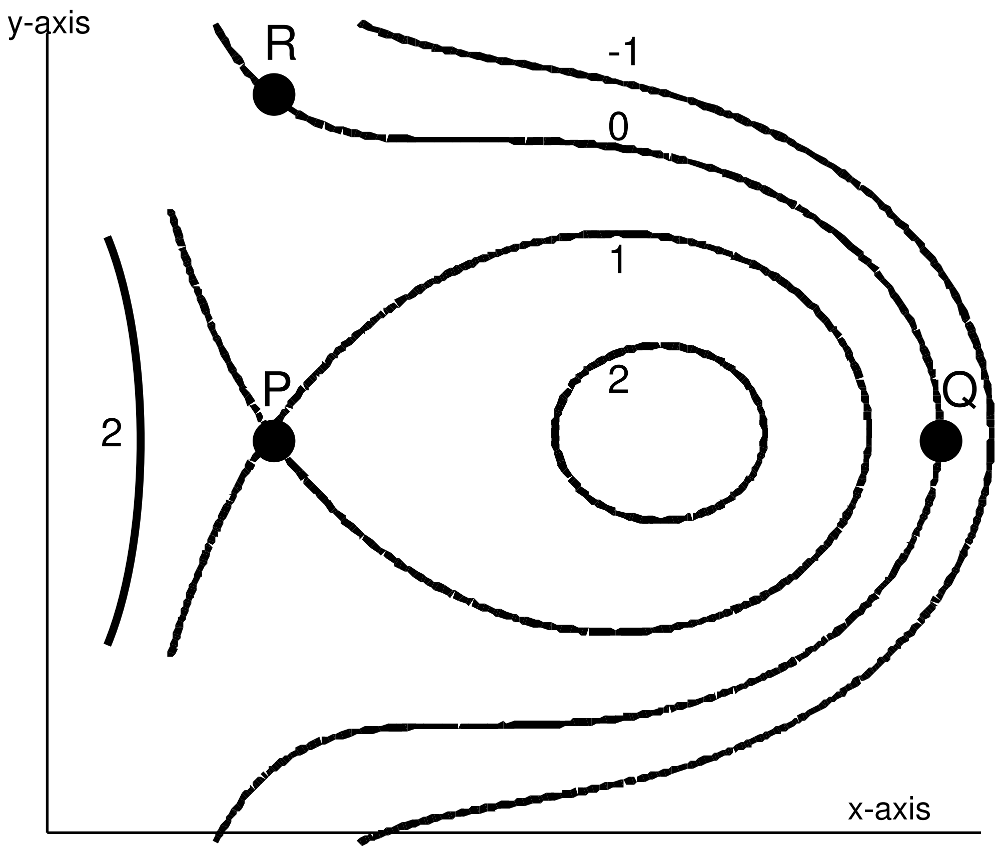
Therefore

$$0 = (2\pi r h)\frac{dr}{dt} + (\pi r^2)\frac{dh}{dt} = (2\pi \cdot 4 \cdot 3)(2) + (\pi \cdot 4^2)\frac{dh}{dt}.$$

Algebra gives $\boxed{\frac{dh}{dt} = -3 \text{ m/hr.}}$ So the height is decreasing at a rate of 3 m/hr.

**SM 223** **Test #3** **SOLUTIONS** **[8 Nov 2010]**

3. The contour diagram for the function $f(x, y)$ is shown with points $P$, $Q$, and $R$. The vector $\mathbf{u}$ is a unit vector in the direction from $Q$ to $R$. Identify each expression as positive, negative, or zero. Fill in a bubble in each row.




| + | − | 0 | expression |
|---|---|---|---|
| ● | ○ | ○ | $f(P)$ |
| ○ | ○ | ● | $f_x(P)$ |
| ● | ○ | ○ | $f_{xx}(P)$ |
| ○ | ● | ○ | $f_{xx}(P)f_{yy}(P) - [f_{xy}(P)]^2$ |

| + | − | 0 | expression |
|---|---|---|---|
| ○ | ● | ○ | $f_x(Q)$ |
| ○ | ○ | ● | $f_y(Q)$ |
| ● | ○ | ○ | $\lvert\nabla f(Q)\rvert - \lvert\nabla f(R)\rvert$ |
| ● | ○ | ○ | $D_{\mathbf{u}}f(Q)$ |

- $f(P) = 1$ from the contour diagram.
- $f_x(P) = 0$ because $P$ is a saddle point and hence a critical point.
- $f_{xx}(P) > 0$ because the terrain is concave up from west-to-east through $P$.
- $f_{xx}(P)f_{yy}(P) - [f_{xy}(P)]^2 < 0$ because saddle points have negative discriminants.
- $f_x(Q) < 0$ because $f$ decreases from west to east thru $Q$.
- $f_y(Q) = 0$ because $f$ achieves a maximum from south to north through $Q$.
- $\lvert\nabla f(Q)\rvert - \lvert\nabla f(R)\rvert > 0$. Since the contours are closer together at $Q$ than at $R$, the terrain is steeper at $Q$ than at $R$. So the magnitude of the gradient is larger at $Q$ than at $R$.
- As we move from $Q$ toward $R$ in this terrain, our first step is uphill.

4. In this problem we find the minimum distance between the point $(8, 6, 0)$ and the cone $z^2 = x^2 + y^2$.

   (a) Express the *square of the distance* between $(8, 6, 0)$ and the point $(x, y, z)$ on the cone as a function $f(x, y)$ of $x$ and $y$.

   IS IS BEST IF YOU DO NOT EXPAND OR OTHERWISE "SIMPLIFY" YOUR ANSWER.

   $$f(x, y) \quad = \quad \underline{(x-8)^2 + (y-6)^2 + (x^2+y^2)}$$

   **Reason.** The square of the distance between $(8, 6, 0)$ and $(x, y, z)$ is

   $$(x-8)^2 + (y-6)^2 + (z-0)^2 = (x-8)^2 + (y-6)^2 + \underbrace{(x^2+y^2)}_{=z^2}.$$

   (b) The minimum distance between $(8, 6, 0)$ and the cone is

   ○ 5 ● $5\sqrt{2}$ ○ 10 ○ 100 ○ $5\sqrt{3}$ ○ none of above; correct is ______

   FILL IN A BUBBLE AND SHOW YOUR WORK BELOW

   **Solution.** Let $(x, y, z)$ be a point on the cone. We want to minimize the distance between $(8, 6, 0)$ and $(x, y, z)$. It suffices to minimize the *square of the distance.* In other words, we want to minimize the function $f(x, y)$.

   We solve the system

   $$\begin{Bmatrix} f_x = 0 \\ f_y = 0 \end{Bmatrix} \qquad \begin{Bmatrix} 2(x-8) + 2x = 0 \\ 2(y-6) + 2y = 0 \end{Bmatrix} \qquad \begin{Bmatrix} x = 4 \\ y = 3 \end{Bmatrix}$$

   So the only critical point is $(x, y) = (4, 3)$. This critical point must correspond to the minimum value of $f$ because there is no maximum distance; points on the cone can be arbitrarily far away from $(8, 6, 0)$. Now

   $$f(4, 3) = (4-8)^2 + (3-6)^2 + (4^2 + 3^2) = 16 + 9 + (16 + 9) = 50.$$

   This is the square of the minimum distance. The minimum distance itself is $\sqrt{50} = 5\sqrt{2}$.

5. We are constructing a open-top aquarium with glass sides and a slate base.
   The volume of the aquarium must be 5000 $\text{cm}^3$.
   Glass costs 3 cents/$\text{cm}^2$, and slate costs 7 cents/$\text{cm}^2$.
   The height of the aquarium is $z$, and the base has length $x$ and width $y$.

   (a) Express the total cost of the aquarium in terms of $x$, $y$, and $z$.

   $$C(x, y, z) = \underline{\quad 7xy + 6xz + 6yz \quad}$$

   **Reason.** Total cost = cost of base + cost of sides = $7xy + 2(3xz) + 2(3yz)$ (in cents).

   (b) What is our constraint on $x$, $y$, and $z$?

   $$\underline{\quad V(x, y, z) = xyz = 5000 \quad}$$

   (c) List the four equations in four unknowns that arise when the method of Lagrange multipliers is used to minimize the cost of the aquarium.

   Do not solve the system.

   $\underline{\quad 7y + 6z = \lambda \cdot yz \quad}$

   $\underline{\quad 7x + 6z = \lambda \cdot xz \quad}$

   $\underline{\quad 6x + 6y = \lambda \cdot xy \quad}$

   $\underline{\quad xyz = 5000 \quad}$

   The first three equations come from $\nabla C(x, y, z) = \lambda \nabla V(x, y, z)$.
   Note that

   $$\nabla C = \langle 7y + 6z,\ 7x + 6z,\ 6x + 6y \rangle$$

   and

   $$\nabla V = \langle yz,\ xz,\ xy \rangle.$$

   The constraint is the fourth equation in the system.

6. This problem has 8 parts, labeled (a)–(h), each worth 10 points. However, you can omit as many parts as you want. You get 5 points for each omitted part.

**You must fill the bubble in the table for any part(s) you want to omit.**

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OMIT: ○(a) ○(b) ○(c) ○(d) ○(e) ○(f) ○(g) ○(h)

---

(a) Let $f(x, y)$ satisfy

$$f_x(20, 10) = f_y(20, 10) = 0,$$

$$f_{xx}(20, 10) = 2 \qquad f_{yy}(20, 10) = 5, \qquad f_{xy}(20, 10) = -3.$$

How should we classify the point (20, 10)?

○ not a critical point
○ a saddle point
● a relative minimum
○ a relative maximum
○ none of the above

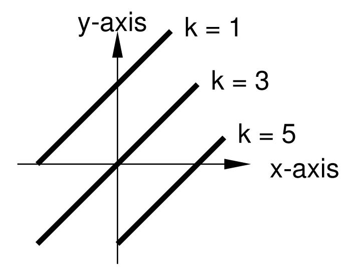
**Reason:** Since $f_x(20, 10) = f_y(20, 10) = 0$, the point (20, 10) is definitely a critical point. The discriminant of $f$ at (20, 10) is

$$D(20, 10) = f_{xx}(20, 10) f_{yy}(20, 10) - (f_{xy}(20, 10))^2 = (2)(5) - (-3)^2 = 1 > 0.$$

The discriminant is positive, so we have more work to do. We have $f_{xx}(20, 10) = 2 > 0$. So (20, 10) is a relative minimum by the Second Derivative Test.

(b) The contour map shows the level curves $f(x, y) = k$ for $k = 1, 3, 5$. Which vector could be the gradient of $f$ at the origin?

○$\mathbf{i} + \mathbf{j}$ ○$2\mathbf{i} + 2\mathbf{j}$ ○$3\mathbf{i} + 3\mathbf{j}$ ●$\mathbf{i} - \mathbf{j}$ ○$-\mathbf{i} + \mathbf{j}$ ○$-\mathbf{i} - \mathbf{j}$




**Reason:** The gradient vector must be perpendicular to the level curve through the origin, which eliminates all the given vectors except $\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + \mathbf{j}$. Also, the gradient must point in the direction of maximum rate of increase of $f$ at the origin. From the labels on the level curves, $f$ increases in the direction of Quadrant IV at the origin, so the only choice is $\mathbf{i} - \mathbf{j}$.

(c) How many critical points does the function $g(x,y) = (x+y)^2$ have?

○ 0 ○ 1 ○ 2 ○ 3 ○ more than 3, but finite ● infinitely many

**Reason:** Every point $(x,y)$ on the line $y = -x$ is a critical point of $g$:

$$\left\{\begin{array}{l} g_x = 0 \\ g_y = 0 \end{array}\right\} \quad \left\{\begin{array}{l} 2(x+y) = 0 \\ 2(x+y) = 0 \end{array}\right\} \quad \left\{\begin{array}{l} y = -x \\ y = -x \end{array}\right\}$$

(d) How many critical points does the function $F(x,y) = e^x \cos(y)$ have?

● 0 ○ 1 ○ 2 ○ 3 ○ more than 3, but finite ○ infinitely many

**Reason:**

$$\left\{\begin{array}{l} F_x = 0 \\ F_y = 0 \end{array}\right\} \quad \left\{\begin{array}{l} e^x \cos(y) = 0 \\ -e^x \sin(y) = 0 \end{array}\right\} \quad \left\{\begin{array}{l} \cos(y) = 0 \\ \sin(y) = 0 \end{array}\right\}$$

We divided by $e^x$, which is never 0. Note that no value of $y$ can satisfy $\cos(y) = \sin(y) = 0$ since we have the identity $\cos^2(y) + \sin^2(y) = 1$.

(e) The function $f(x,y) = -20x + 13y$ has no critical points. What is the minimum value of $f$ for $(x,y)$ in (or on the boundary of) the triangular region with vertices $(0,0)$, $(1,0)$, and $(1,1)$?

○ 0 ● −20 ○ −7 ○ −13 ○ −33 ○ none of above

**Reason:** Because $f$ has no critical points the minimum value must occur on the boundary of the region. The boundary has three pieces.

- Along the segment $y = 0$ for $0 \le x \le 1$ the function is $f_1(x) = -20x$, which has minimum value $-20$ at $x = 1$.
- Along the segment $x = 1$ for $0 \le y \le 1$ the function is $f_2(y) = -20 + 13y$, which has minimum value $-20$ at $y = 0$.
- Finally, along the segment $y = x$ for $0 \le x \le 1$, the function is $f_3(x) = -20x + 13x = -7x$, which has minimum value $-7$ at $x = 1$.

So the minimum value of $f$ is $-20$ at $(x,y) = (1,0)$.

(f) Suppose $w = f(x, y, z, t)$, and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$. What is the *total* number of nodes in the corresponding tree diagram? (The nodes are the points we draw to represent the variables.)

○ 7
○ 9
○ 11
● 13 **Reason.** See text p 904, Example 4 and Figure 3.
○ 15

(g) Suppose that

$$z = x^2 y + 3xy^4, \qquad \text{and} \qquad x = \sin(2t), \qquad \text{and} \qquad y = \cos(t).$$

Find $\frac{dz}{dt}$ when $t = 0$.

○ 0
○ 2
○ 4
● 6 **Reason.** See text p 902, Example 1.
○ 8 Note that when $t = 0$, we have $x = 0$ and $y = 1$.

(h) Find the tangent plane to the surface $y = x^2 - z^2$ at the point $(4, 7, 3)$.

● $8(x-4) - (y-7) - 6(z-3) = 0$
○ $8(x-4) + (y-7) - 6(z-3) = 0$
○ $-4(x-4) - 7(y-7) - 3(z-3) = 0$
○ $4(x-4) + 0(y-7) + 3(z-3) = 0$
○ $8(x-4) - 7(y-7) - 6(z-3) = 0$
○ none of above

**Reason:** The plane is of the form $A(x-4) + B(y-7) + C(z-3) = 0$. The normal vector to the level surface $F(x, y, z) = x^2 - y - z^2 = 6$ is

$$\langle A, B, C \rangle = \nabla F(4, 7, 3) = \langle 2x, -1, -2z \rangle \Big|_{(x,y,z)=(4,7,3)} = \langle 8, -1, -6 \rangle.$$

So the plane is $8(x-4) - (y-7) - 6(z-3) = 0$.

SM $223_{\text{TSM}}$ **Test #3 [8 Nov 2010]** **Last Name** SOLUTIONS **Section:** ○ 4041 ○ 5041 ○ 6041

NO CALCULATOR ALLOWED FOR THIS PROBLEM

7. For this problem we consider the point $P = (4, -1)$ and the function

$$f(x, y) = 20\sqrt{x} - 3y^2.$$

SHOW YOUR WORK TO RECEIVE FULL OR PARTIAL CREDIT!

(a) Find the gradient of $f$ at $P$.

● $\langle 5, 6\rangle$ ○ $\langle 5, -6\rangle$ ○ $\langle 10, -6\rangle$ ○ $\langle 10, 6\rangle$ ○ none of above; correct is ______

**Reason:**

$$\nabla f(P) = \langle f_x, f_y\rangle\big|_{(x,y)=(4,-1)} = \langle 10/\sqrt{x},\ -6y\rangle\big|_{(x,y)=(4,-1)} = \langle 5, 6\rangle.$$

(b) Find the maximum rate of change of $f$ at $P$.

○ $\sqrt{37}$ ○ $\sqrt{47}$ ○ $\sqrt{51}$ ● $\sqrt{61}$ ○ $\sqrt{136}$

**Reason:** The maximum rate of change of $f$ at $P$ is

$$|\nabla f(P)| = |\langle 5, 6\rangle| = \sqrt{5^2 + 6^2} = \sqrt{61}.$$

(c) Find the directional derivative of $f$ at $P$ in the direction from $P = (4, -1)$ to $Q = (1, 3)$.

○ $\frac{39}{5}$ ○ 39 ● $\frac{9}{5}$ ○ 9 ○ 0 ○ $-\frac{3}{\sqrt{13}}$ ○ none of above; correct is ______

**Reason:** The vector from $P$ to $Q$ is

$$\overrightarrow{PQ} = \langle 1 - 4,\ 3 - (-1)\rangle = \langle -3, 4\rangle,$$

which has length $\sqrt{(-3)^2 + 4^2} = 5$. Therefore a unit vector in the same direction as $\overrightarrow{PQ}$ is

$$\mathbf{u} = \frac{1}{|\overrightarrow{PQ}|}\overrightarrow{PQ} = \frac{1}{5}\langle -3, 4\rangle.$$

The desired directional derivative is

$$D_{\mathbf{u}}f(P) = \nabla f(P)\cdot\mathbf{u} = \langle 5, 6\rangle \cdot \frac{\langle -3, 4\rangle}{5} = \frac{-15 + 24}{5} = \frac{9}{5}.$$

**SM 223** **Test #3** **SOLUTIONS** **[8 Nov 2010]**

NO CALCULATOR ALLOWED FOR THIS PROBLEM

8. A farmer will install fences to form three adjacent rectangular pens next to a straight river, as shown. Each pen has dimensions $x$ feet by $y$ feet. No fencing is required along the river.

In this problem you will use Lagrange multipliers to determine maximum total area the farmer can enclose if he has a total of 120 feet of fence.


(a) Express the total area enclosed in terms of $x$ and $y$.

$$A(x, y) = \underline{\quad 3xy \quad}$$

(b) What is the constraint?

● $3x + 4y = 120$ ○ $3x + y = 120$ ○ $x + 4y = 120$ ○ $x + \frac{y}{4} = 120$

(c) List the three equations in three unknowns that arise from Lagrange multipliers.

$3y = \lambda \cdot 3$

$3x = \lambda \cdot 4$

$3x + 4y = 120$

**Reason:** We must solve the system $\{\nabla A = \lambda \nabla g(x, y)$ and $g(x, y) = 120\}$, where $g(x, y) = 3x+4y$. This gives us $\langle 3y, 3x\rangle = \lambda\langle 3, 4\rangle$ and $3x+4y = 120$, which leads to the three equations listed above.

(d) Solve the system to determine the maximum total area the farmer can enclose.

WRITE YOUR ANSWER IN THE BLANK. SHOW YOUR WORK FOR FULL OR PARTIAL CREDIT.

**Answer:** $A = 900$ ft$^2$

**Solution.** The first equation implies that $y = \lambda$. Then the second equation becomes $3x = 4y$. Use the constraint to get $3x+3x = 120$. Therefore $x = 20$. Also, $y = 15$. Thus the maximum area is $A = 3 \cdot 20 \cdot 15 = 900$ ft$^2$.