SM 223_{TSM} Test #4 [7 Dec 2010] Last Name SOLUTIONS Section: ○ 4041 ○ 5041 ○ 6041

Instructions

0. Failure to follow instructions can result in your losing points.
1. **Do NOT turn the page or begin until instructed to do so.**
2. Print your name and indicate your section above.
3. **Write nothing else on this cover page,** except your signature on the line below to indicate you've read and understood the directions.
4. There are **8** problems altogether. Relative weights are given in the table.
5. For fill-in-the-blank, multiple-choice, matching, and similar problems, write your answer directly on the test paper. There is plenty of space on each page (and the back) for your work. Although your work will not be graded, you might receive part-credit based on how "good" your incorrect answer is.
6. If a problem requests you to show your work, use the space provided to receive credit.
7. Calculators are allowed for the entire test.
8. Unless otherwise indicated, leave answer in exact form; don't approximate $\sqrt{2}$ as 1.41, for instance.

Signature ______________________

DO NOT WRITE ANYTHING ON THIS PAGE BELOW THIS LINE

Problem	**Points**	**Score**
1	30	
2	40	
3	30	
4	40	
5	30	
6	30	
7	40	
8	40	
Free	20	20
Total	**300**	
Test Score	%	**Grade**
240	80	*A*
210	70	*B*
180	60	*C*
< 180	< 60	*F*

Test 1	**300**	
Test 2	**300**	
Test 3	**300**	
Test 4	**300**	
Quizzes*	**300**	
Subtotal	**1500**	
– Low Test	**–300**	
16-Week Total	**1200**	
16-Week Grade	%	**Grade**
960	80	*A*
840	70	*B*
720	60	*C*
< 720	< 60	*F*

* N best quizzes (out of 21) times X

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1. Let $f(x, y)$ be a function of two variables defined for all (x, y) in a rectangular region R.

   (a) Complete our 'limit' definition of the double integral of f over R.

$$\iint_R f(x,y)\,dA \quad = \quad \underline{\lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A}$$

   (b) What is our basic geometric interpretation for the double integral

$$\iint_R f(x,y)\,dA\ ?$$

The double integral represents the signed volume of the solid under the surface $z = f(x, y)$ and above the region R in the xy-plane.

   (c) Now suppose that the rectangle R is

$$R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 3\},$$

which has area 6. Also, suppose we know that $7 \le f(x, y) \le 20$ for all (x, y) in R. According to our theorem, what inequalities does the double integral $\iint_R f(x, y)\,dA$ satisfy?

Simply fill in the blank with the correct inequalities; you do not need to justify your answer.

$$7 \cdot 6 = 7 \cdot \text{area of } R \le \iint_R f(x,y)\,dA \le 20 \cdot \text{area of } R = 20 \cdot 6$$

   (d) Now suppose that $\iint_R f(x, y)\,dA = 12$ and that R is the same rectangle as in part (c). What is the average value of f over R?

○ 1 ○ 1/2 ● 2 ○ 6 ○ 12 ○ 13.5 ○ 72

Reason.

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R}\iint_R f(x,y)\,dA = \frac{1}{2\cdot 3}(12) = 2.$$

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2. Fill in the blanks.

(a) In Cartesian coordinates $dA = dx\,dy$ and $dV = dx\,dy\,dz$

In polar coordinates $dA =$ $\underline{\ r\,dr\,d\theta\ }$

In cylindrical coordinates $dV =$ $\underline{\ r\,dz\,dr\,d\theta\ }$

In spherical coordinates $dV =$ $\underline{\ \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta\ }$

(b) Write the point $(x, y, z) = (-1, -\sqrt{3}, 2)$ in cylindrical coordinates.

$$(r, \theta, z) = (\underline{\ 2\ }, \underline{\ 4\pi/3\ }, \underline{\ 2\ })$$

Reason. $r^2 = x^2 + y^2 = (-1)^2 + (-\sqrt{3})^2 = 1 + 3 = 4$. So $r = 2$.
Also, $\tan(\theta) = y/x = \sqrt{3}$. So $\theta = \pi/3$, which is in quadrant I. Since x and y are both negative, we want an angle in quadrant III. So we add pi to get $\theta = \pi + \pi/3 = 4\pi/3$. Of course, $z = 2$.

(c) Identify each surface.

i. $\rho = 16$

●sphere ○paraboloid ○cylinder ○circle ○cone ○plane

ii. $\phi = \pi/4$

○sphere ○paraboloid ○cylinder ○circle ●cone ○plane

iii. $\rho\sin(\phi)\sin(\theta) = 20$ **Reason.** $y = 20$

○sphere ○paraboloid ○cylinder ○circle ○cone ●plane

(d) A point has cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ). Fill in the bubble for *every* correct statement.

○$z = \rho\cos(\theta)$ ○$z = \rho\sin(\phi)$ ●$r^2 + z^2 = \rho^2$

○$r = \rho$ ○$r = \rho^2\sin(\phi)$ ●$r^2 = \rho^2\sin^2(\phi)$

Reason. $r^2 + z^2 = (x^2 + y^2) + z^2 = \rho^2$.
$r^2 = \rho^2 - z^2 = \rho^2 - (\rho\cos(\phi))^2 = \rho^2(1 - \cos^2(\phi)) = \rho^2\sin^2(\phi)$

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3. (a) Let $f(x, y)$ be a joint probability density function. List the two key properties f satisfies.

$f(x) \geq 0$ ______ $\iint_{xy\text{-plane}} f(x, y)\, dA = 1$ ______

THE REST OF THIS PROBLEM DEALS WITH THE FOLLOWING SCENARIO.

We are throwing a darts at a square 2-by-2 target. The landing point (x, y) is given by the random variables X and Y with joint probability density function

$$f(x, y) = \begin{cases} Cx(3-y) & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(b) The lines $x = 1$ and $y = 1$ partition the 2-by-2 target into four square quadrants, each of size 1-by-1. Which of the four square quadrants is most likely to be hit?

○ upper left ○ upper right ○ lower left ● lower right
○ all four regions are equally likely to be hit

Reason. The function $f(x, y) = Cx(3 - y)$ is largest when x is big and y is small. This occurs in the lower right corner of the target.

(c) What is the significance of the value of the following double integral?

$$\int_0^2 \int_0^2 Cxy(3-y)\, dx\, dy = \iint_{xy\text{-plane}} y \cdot f(x, y)\, dA = \mu_2$$

This is the *mean* value (or *expected value* or *average*) of ______

the y-coordinate for the landing point (x, y) of the dart. ______

(d) Find the value of the constant C.

● $C = 1/8$ ○ $C = 1/4$ ○ $C = 1$ ○ $C = 4$ ○ $C = 8$

Reason. To have a joint probability density function we must have

$$1 = \iint_{xy\text{-plane}} f(x, y)\, dA = \iint_{\text{target}} f(x, y)\, dA = \int_0^2 \int_0^2 Cx(3-y)\, dy\, dx = 8C.$$

So $1 = 8C$. Therefore $C = \frac{1}{8}$.

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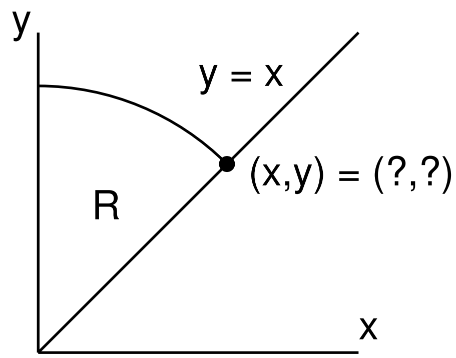
4. The region R lies in the first quadrant and is inside the circle $x^2 + y^2 = 4$ and above the line $y = x$. See the figure.



(a) The indicated point of intersection is $(x, y) = (\underline{\sqrt{2}}, \underline{\sqrt{2}})$

Solution 1. Put $y = x$ in the equation of the circle to get $x^2 + x^2 = 4$, which gives $x^2 = 2$ and $x = \pm\sqrt{2}$. Of course, we only want the positive value. So the intersection point is $(x, y) = (\sqrt{2}, \sqrt{2})$.

Solution 2. It is clear from the diagram that the polar coordinates of the point of intersection are $(r, \theta) = (2, \pi/4)$. So the x-coordinate of the point is $x = r\cos(\theta) = 2\cos(\pi/4) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$ with a similar computation for y.

(b) Fill in the blanks for the double integral in Cartesian coordinates.

$$\iint_R x^2 y \, dA = \int_{\underline{0}}^{\underline{\sqrt{2}}} \int_{\underline{x}}^{\underline{\sqrt{4-x^2}}} \underline{\quad x^2 y \quad} \, d\,\underline{y}\, d\,\underline{x}$$

(c) Fill in the blanks for the double integral in polar coordinates.

$$\iint_R x^2 y \, dA = \int_{\underline{\pi/4}}^{\underline{\pi/2}} \int_{\underline{0}}^{\underline{2}} \underline{\quad (r\cos\theta)^2 (r\sin\theta)\, r \quad} \, d\,\underline{r}\, d\,\underline{\theta}$$

5. The solid E is one-eighth of a solid ball with radius 4. It lies in the first octant (with all coordinates non-negative) and is bounded by by the sphere

$$x^2 + y^2 + z^2 = 16.$$

The density at a point is equal to the distance from that point to the xy-plane. Fill in the blanks for the multiple integral expressions for the volume and mass of E.

(a) Volume in Cartesian coordinates:

$$\text{Volume} = \iiint_E 1\,dV = \int_{0}^{4} \int_{0}^{\sqrt{16-x^2}} \int_{0}^{\sqrt{16-x^2-y^2}} \underline{\quad 1 \quad}\, dz\, dy\, dx$$

(b) Volume in cylindrical coordinates:

$$\text{Volume} = \iiint_E 1\,dV = \int_{0}^{\pi/2} \int_{0}^{4} \int_{0}^{\sqrt{16-r^2}} \underline{\quad r \quad}\, dz\, dr\, d\theta$$

(c) mass in spherical coordinates:

$$\text{mass} = \iiint_E \text{density}\,dV = \int_{0}^{\pi/2} \int_{0}^{\pi/2} \int_{0}^{4} \underline{\quad \overbrace{(\rho\cos(\phi))}^{=z=\text{density}}\ \rho^2 \sin(\phi) \quad}\, d\rho\, d\phi\, d\theta$$

6. Throughout this problem we consider the iterated integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{x^4}\, dx\, dy.$$

(a) The region of integration has three corners, one which is $(0,0)$. Fill in *two* bubbles for the other two corners.

● $(2,0)$ ○ $(8,0)$ ○ $(0,2)$ ○ $(0,8)$ ● $(2,8)$ ○ $(8,2)$ ○ $(4,4)$

Reason. The region of integration is

$$R: \quad \left\{ \begin{array}{l} y = 0 \text{ to } y = 8 \\ x = \sqrt[3]{y} \text{ to } x = 2 \end{array} \right\}$$

Note that the curve $x = \sqrt[3]{y}$ is $x^3 = y$.

(b) Rewrite the iterated integral in the reverse order of integration:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{x^4}\, dx\, dy \quad = \quad \int_{\underline{0}}^{\underline{2}} \int_{\underline{0}}^{\underline{x^3}} \underline{\quad 4e^{x^4} \quad}\, dy\, dx$$

(c) Evaluate the iterated integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{x^4}\, dx\, dy.$$

Fill in the correct bubble and show your work for full or partial credit.

○ $4e^{16}$ ○ $4e^{16}-4$ ● $e^{16}-1$ ○ e^{16} ○ none of above; correct is ________

Reason. The integral is easier to evaluate if we reverse the order of integration:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{x^4}\, dx\, dy \quad = \quad \int_0^2 \int_0^{x^3} 4e^{x^4}\, dy\, dx \quad = \quad \int_0^2 4e^{x^4} y\Big|_{y=0}^{y=x^3} dx$$

$$= \quad \int_0^2 4e^{x^4} x^3\, dx \quad = \quad e^{x^4}\Big|_{x=0}^{x=2} \quad = \quad e^{16} - 1$$

Note. Once you've reversed the order of integration, your calculator will evaluate the resulting double integral exactly.

7. (a) Evaluate the iterated integral.

SHOW YOUR WORK BELOW AS THOUGH YOU DID NOT HAVE A CALCULATOR.

$$\int_0^3 \int_0^4 x^2 y \, dy \, dx.$$

○ 12 ○ 18 ○ 24 ○ 36 ● 72

Reason:

$$\int_0^3 \int_0^4 x^2 y \, dy \, dx = \int_0^3 \left. \frac{x^2 y^2}{2} \right|_{y=0}^{y=4} dx = 8 \int_0^3 x^2 \, dx = \left. \frac{8x^3}{3} \right|_{x=0}^{x=3} = \frac{8 \cdot 3^3}{3} = 8 \cdot 3^2 = 72.$$

(b) Evaluate the double integral, where R be the ring-shaped region between the two circles with polar equations $r = 1$ and $r = 2$.

$$\iint_R \frac{12}{\sqrt{x^2 + y^2}} \, dA$$

○ 3π ○ 6π ○ 12π ○ 18π ● 24π ○ $12\pi \ln(2)$

Reason: Rewrite the integral in polar coordinates:

$$\iint_R \frac{12}{\sqrt{x^2 + y^2}} \, dA = \int_0^{2\pi} \int_1^2 \frac{12}{r} \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_1^2 12 \, dr \, d\theta = 24\pi$$

(c) Use a Midpoint approximation with $m = n = 2$ to estimate

$$\int_0^4 \int_0^4 f(x, y) \, dy \, dx$$

for $f(x, y) = xy$.

○ 4 ○ 8 ○ 16 ○ 52 ● 64

Reason.

$$\int_0^4 \int_0^4 f(x, y) \, dy \, dx \approx f(1,1)\Delta A + f(3,1)\Delta A + f(3,1)\Delta A + f(3,3)\Delta A = (1 + 3 + 3 + 9)\, 4 = 64$$

8. A student is computing the total number of calories in each of six prospective desserts. He chooses an appropriate coordinate system and applies the basic triple integral formula

$$\text{total number of calories in dessert with shape } E = \iiint_E \delta \, dV,$$

where δ is an estimated "caloric density function." Write a capital letter in each blank to match each triple integral expression to a dessert. Note: The units of length are not necessarily the same for each dessert.

YOU ARE NOT BEING ASKED TO EVALUATE ANY TRIPLE INTEGRALS!

(A) $\displaystyle\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^5 13\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$ lower half of a sphere: grapefruit

(B) $\displaystyle\int_0^{2\pi}\int_2^5\int_0^3 7r\,dz\,dr\,d\theta$ cylindrical coordinates with a hole: life-saver

(C) $\displaystyle\underbrace{\int_0^{\pi/3}\int_0^8\int_0^3 17r\,dz\,dr\,d\theta}_{\text{lemon}} + \underbrace{\int_0^{\pi/3}\int_0^8\int_3^7 9r\,dz\,dr\,d\theta}_{\text{meringue}}$

(D) $\displaystyle\int_0^1\int_0^{2\pi}\int_0^{\pi} 20\,dx\,dy\,dz$ brownie uses Cartesian coordinates

(E) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^3 11\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$ sphere: scoop of ice cream

(F) $\displaystyle\int_0^{\pi/4}\int_0^{\pi}\int_0^3 9\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$ $\frac{1}{4}$-sphere: orange wedge

__D__ brownie
__E__ scoop of ice cream
__A__ half a grapefruit
__F__ orange wedge
__B__ life-saver
__C__ lemon meringue pie