SM 122 Test #1 Solutions Integrals 30 Jan 2007

1. (a) Complete the statement of the Mean Value Theorem for Integrals.

Let $f(x)$ be a continuous function for $a \le x \le b$.

Then there is a number c with $a \le c \le b$ such that $(b-a)f(c) = \int_a^b f(x)\,dx$

(b) Let $f(x)$ be a continuous function on the interval $[a, b]$.
Complete the statement of the Fundamental Theorem of Calculus.

(I) If $g(x) = \int_a^x f(t)\,dt$, then

$$g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(II) If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$$

(c) Fill in the blanks to complete the definition of the definite integral $\int_a^b f(x)\,dx$.
Let $f(x)$ be a continuous function on the interval $[a, b]$. Let the points $a = x_0, x_1, x_2, \ldots, x_n = b$ partition $[a, b]$ into n equal sub-intervals. The length of each sub-interval is

$$\Delta x = \frac{b-a}{n}.$$

Let x_i^* be a sample point in the i-th subinterval. Then the definition of the definite integral is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

(d) The Net Change Theorem was discussed in class and the text.

i. State the Net Change Theorem in words.

$\boxed{\text{The integral of a rate of change equals the net change.}}$

ii. State the Net Change Theorem as a mathematical formula.

YOUR ANSWER SHOULD INVOLVE A DEFINITE INTEGRAL.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

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2. Circle the best CAPITAL LETTER (not the entire answer) for each problem.

(a) Find the average value of the function $f(x) = 2x(x^2-2)^2$ for $0 \le x \le 2$.

(A) 1/3
(B) 2/3
(C) 4/3
(D) $\boxed{8/3}$
(E) 10/3

Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 2x(x^2-1)^2\,dx$
$= \frac{1}{2}\int_{-2}^2 u^2\,du = \int_0^2 u^2\,du = (1/3)u^3|_0^2 = 8/3.$

We made the substitution $u = x^2-2$ with $du = 2x\,dx$.
We also exploited the fact that u^2 is an even function.

(b) The table shows values of a function $f(x)$. Estimate the definite integral $\int_0^{12} f(x)\,dx$ using *left* endpoints and $n = 4$ rectangles.

x	0	3	6	9	12
$f(x)$	2	5	2	3	0

(A) 3
(B) 10
(C) 12
(D) 30
(E) $\boxed{36}$

$\int_0^{12} f(x)\,dx \approx \Delta x[f(0)+f(3)+f(6)+f(9)] = 3[2+5+2+3] = 36$

(c) Give the best substitution for the integral

$$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx$$

YOU ARE NOT BEING ASKED TO EVALUATE THE INTEGRAL

(A) $u = \cos(x)$
(B) $\boxed{u = \sin(x)}$
(C) $u = \sin^2(x)$
(D) $u = 1+\sin^2(x)$
(E) $u = \sqrt{\sin(x)}$

With $u = \sin(x)$ we have $du = \cos(x)\,dx$.
The given integral becomes
$\int \frac{1}{1+u^2}\,du = \arctan(u)+C = \arctan(\sin(x))+C$

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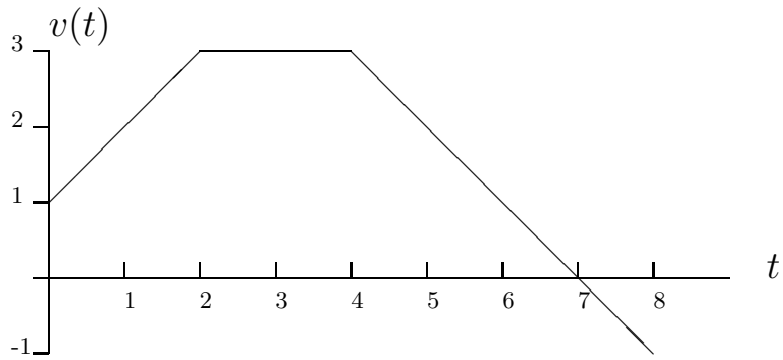
(d) The graph shows the velocity $v(t)$ of a particle on a number line as a function of time from $t = 0$ to $t = 8$. The units are cm/sec.



At which of the following times is the particle farthest from its position at $t = 0$? CIRCLE ALL CORRECT CAPITAL LETTERS

(A) 0
(B) 2
(C) 4
(D) $\boxed{7}$ Up until $t = 7$ the particle moves forward;
(E) 8 after $t = 7$ it moves backward.

(e) For the situation depicted in the graph above, find the *total distance* traveled by the particle between for $6 \le t \le 8$.

(A) 0 cm By examining the area under the velocity curve
(B) 1/2 cm we see the particle travels 1/2 cm forward for $6 \le t \le 7$
(C) $\boxed{1 \text{ cm}}$ and 1/2 cm backward for $7 \le t \le 8$.
(D) 2 cm
(E) 4 cm

(f) A 50-foot rope hangs from the top of a 100-foot building. The rope weighs 10 lbs. How much work is done in pulling the rope to the top of the building?

(A) 200 ft-lbs Work $= \int_0^{50} \left(\frac{10}{50}\right) x\, dx = \frac{1}{5} \cdot \frac{1}{2} x^2 \big|_0^{50} = 250$ ft-lbs
(B) $\boxed{250 \text{ ft-lbs}}$
(C) 500 ft-lbs The work to lift a slice of length dx at distance x from the top is
(D) 1000 ft-lbs work = force·distance = weight $\cdot x = \left(\frac{10}{50}\right) dx \cdot x$
(E) 2000 ft-lbs

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(g) Find $G'(3)$ if

$$G(x) = \int_1^{e^x} \ln(t)\,dt.$$

(A) $\boxed{3e^3}$
(B) $3e^3 - 1$
(C) $\ln(3)$
(D) 3
(E) $\frac{1}{3}\left(e^3 - 1\right)$

By FTC part (1) and the Chain Rule,
$G'(x) = \ln(e^x) \cdot \frac{d}{dx}(e^x) = x \cdot e^x$.
So $G'(3) = 3e^3$.

(h) Find the area of the lens-shaped region bounded by the parabola $y = x^2$ and the line $y = x$.

(A) 1/2
(B) 1/3
(C) 1/4
(D) $\boxed{1/6}$
(E) 1/12

Area $= \int_0^1 (x - x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(i) A force of 60 N is applied to stretch a spring from its natural length of 20 cm to a length of 30 cm. What is the spring constant k?

(A) 0.06 N/meter
(B) 0.6 N/meter
(C) 6 N/meter
(D) 60 N/meter
(E) $\boxed{\text{600 N/meter}}$

By Hooke's Law $F = kx$.
So $60 = k(0.1)$ and $k = 600$.
Note: The displacement is 10 cm, which is 0.1 meters

(j) Find the work done while stretching the spring in the previous part of this problem.

(A) $\boxed{\text{3 Joules}}$
(B) 6 Joules
(C) 10 Joules
(D) 12 Joules
(E) 20 Joules

Work $= \int_0^{0.1} kx\,dx = \frac{k}{2}x^2\big|_0^{0.1} = \frac{k}{2}(0.1)^2 = \frac{600}{2}0.01 = 3$

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(k) Evaluate the integral.

$$\int_0^2 2^t\,dt \quad = \quad \frac{1}{\ln 2}\cdot 2^t\Big|_0^2 \quad = \quad \frac{1}{\ln 2}\left(2^2-2^0\right) \quad = \quad \frac{3}{\ln 2}.$$

(A) $3\ln(2)$
(B) $\boxed{3/\ln(2)}$
(C) $3/(\ln(2)+1)$
(D) 3
(E) $3\ln(2)-1$

(l) The continuous function g satisfies $\int_0^4 g(x)\,dx = 16$.
Find

$$\int_0^2 x\cdot g(x^2)\,dx \quad = \quad \frac{1}{2}\int_0^4 g(u)\,du \quad = \quad \left(\frac{1}{2}\right)\cdot 16 = 8$$

(A) 4
(B) $\boxed{8}$
(C) 12
(D) 16
(E) 32

We made the substitution $u = x^2$ with $du = 2x\,dx$.
If $x = 0$, then $u = 0$. If $x = 2$, then $u = 4$.

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3. Throughout this problem we consider the region below the curve $y = 1/x$ and above the x-axis for $1 \le x \le 2$.
Circle the CAPITAL LETTER (not the entire answer) for each problem.

(a) Find the area of the region.

(A) $\boxed{\ln(2)}$ Area $= \int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_1^2 = \ln(2) - \ln(1) = \ln(2)$
(B) $1/2$
(C) $\ln(\sqrt{2})$
(D) 1
(E) $1/\sqrt{2}$

(b) Suppose the region is rotated around the x-axis.
Which integral represents the volume of the solid?

(A) $\pi \int_1^2 \frac{1}{x}\,dx$ Volume of disk $= \pi r^2\,dx = \pi\left(\frac{1}{x}\right)^2\,dx$
(B) $\boxed{\pi \int_1^2 \frac{1}{x^2}\,dx}$ Total volume $= \int_1^2 \pi\left(\frac{1}{x}\right)^2\,dx$
(C) $2\pi \int_1^2\,dx$
(D) $2\pi \int_1^2 \frac{1}{x^2}\,dx$
(E) none of the above

(c) Suppose the region is rotated around the y-axis.
Which integral represents the volume of the solid?

(A) $2\pi \int_1^2 \frac{1}{x}\,dx$ Vol. cyl. shell $= 2\pi r h\,dx = 2\pi x\left(\frac{1}{x}\right)\,dx = 2\pi\,dx$
(B) $2\pi \int_1^2 \frac{1-x}{x}\,dx$
(C) $\boxed{2\pi \int_1^2\,dx}$
(D) $2\pi \int_1^2 \frac{1}{x^2}\,dx$
(E) none of the above

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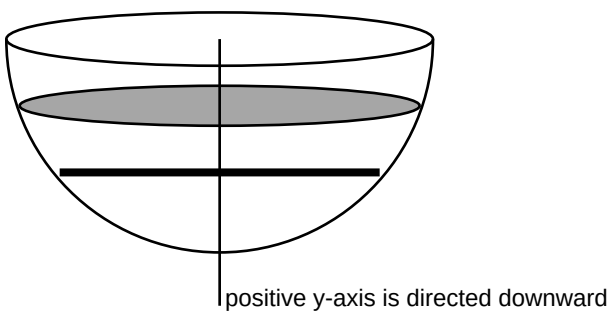
4. A hemispherical crater with radius 10 meters has been found on a planet where the acceleration due to gravity is g m/sec^2. The crater is filled to a depth of 8 m with a liquid of mass density ρ kg/m^3.



(a) Put the origin at the center of the top of the crater, and direct the y-axis downward, as shown. Which expression is the volume of the slice at position y (represented by the thick horizontal segment in the sketch)?
Circle the CAPITAL LETTER (not the entire answer).

(A) $\pi y^2\, dy$ The slice is a circular cylinder with thickness dy
(B) $2\pi y^2\, dy$ and volume πradius$^2\, dy$
(C) $2\pi y^2 \sqrt{10^2 - y^2}\, dy$ Draw a right triangle, apply Pythagoras to see
(D) $\pi \sqrt{10^2 - y^2}\, dy$ that the square of the radius equals $10^2 - y^2$.
(E) $\boxed{\pi \left(10^2 - y^2\right)\, dy}$

(b) Give a definite integral for the work needed to pump the liquid to the top of the crater. Do not simplify or evaluate.

$$\text{Work} \quad = \quad \int_{\underline{2}}^{\underline{10}} \underline{\quad \rho g \pi \left(10^2 - y^2\right) y \quad}\, dy$$

Reasoning: The weight of the slice of the water at position y is:

$$\text{Weight of slice} = \text{mass} \cdot \text{gravity} = \text{mass density} \cdot \text{volume} \cdot \text{gravity}$$

$$= \rho \cdot \pi \left(10^2 - y^2\right)\, dy \cdot g$$

This slice is lifted through a distance of y meters.
Also y ranges from 2 to 10 according to how we've set up our axis.