SM 122 Test #2 Solutions Integration 21 Feb 2007

1. (a) Complete the statement of the formula for integration by parts.

$$\int u\,dv \quad = \quad \underline{\quad uv \quad - \quad \int v\,du \quad}$$

(b) Our proof in class of the integration by parts formula relied on:

Circle ALL correct capital letters.

(A) $\boxed{\text{Product Rule for derivatives}}$
(B) definition of a definite integral
(C) $\boxed{\text{Fundamental Theorem of Calculus}}$
(D) definition of derivative
(E) Chain Rule

(c) We may evaluate the integral

$$\int_0^1 2x \arctan(x)\,dx$$

using integration by parts with $v = x^2 + 1$.
This is an unusual choice for v, but it will work!
Fill in the three remaining blanks.

$$u = \underline{\arctan(x)} \qquad dv = \underline{2x\,dx}$$

$$du = \underline{\frac{1}{x^2+1}\,dx} \qquad v = \underline{x^2+1}$$

(d) The value of the integral in (c) is

$$\int_0^1 2x \arctan(x)\,dx = (x^2+1)\arctan(x)\Big|_0^1 - \int_0^1 \frac{x^2+1}{x^2+1}\,dx$$

$$= \quad (2\arctan(1) - 0) - x\Big|_0^1 = \frac{\pi}{2} - 1.$$

(A) $\pi/4$
(B) $\pi/2$
(C) $\boxed{\pi/2 \ - \ 1}$
(D) $\pi/4 \ + \ 1$
(E) $\pi + 1$

SM 122 Test #2 Solutions Integration 21 Feb 2007

2. Circle the best CAPITAL LETTER (not the entire answer) for each problem.

(a) What is the best way to evaluate the integral

$$\int_1^e \frac{\ln(x)}{x}\,dx\,?$$

(A) integrate by parts
(B) make a trigonometric substitution
(C) $\boxed{\text{make a non-trigonometric substitution}}$ Let $u = \ln(x)$ with $du = (1/x)\,dx$
(D) use partial fractions
(E) use the definition of the definite integral

(b) **Fact:** The improper integral $\int_0^\infty (1/e^x)\,dx$ converges to 1. Therefore the improper integral $\int_0^\infty [1/(e^x+2)]\,dx$ ...

(A) also converges to 1.
(B) $\boxed{\text{converges to some number less than 1.}}$
(C) converges to some number greater than 1.
(D) diverges.
(E) None of above.

The new integrand $1/(e^x+2)$ is smaller than the given one $1/e^x$. Since the given integral converges, the new integral also converges by Comparison Test.

$\boxed{\text{cf Homework p538 \#51}}$

(c) If $x = \sec(\theta)$, then $\sin(\theta) =$

HINT: DRAW A RIGHT TRIANGLE WITH HYPOTENUSE x

(A) $\sqrt{1-x^2}$
(B) $\sqrt{x^2-1}$
(C) $\boxed{\sqrt{x^2-1}/x}$
(D) $x/\sqrt{x^2-1}$
(E) $1/\sqrt{x^2-1}$

Draw your right triangle with angle θ and hypotenuse x. and adjacent side be 1. So $\sec(\theta) = \dfrac{\text{hyp}}{\text{adj}} = \frac{x}{1}$. Then $\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-1}}{x}$.

(d) Find the value of the constant A in the partial fraction expansion

$$\frac{8x+7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

(A) $A = 0$
(B) $A = 1$
(C) $A = 3$
(D) $\boxed{A = 5}$
(E) $A = -3$

Multiply through by $(x-1)(x+2)$.
$8x+7 = A(x+2) + B(x-1)$.
Now plug in $x = 1$ and get $15 = 3A$. So $A = 5$.

SM 122 Test #2 Solutions Integration 21 Feb 2007

(e) Each edge of a cubical aquarium has length 4 feet. The aquarium is filled halfway to the top with sea water, which has weight density δ lb/ft^3. What is the total hydrostatic force on the bottom surface of the aquarium?

HINT: Recall that weight density $= \delta = \rho g$.

(A) 2δ lb $\boxed{\text{cf Homework, p 569 \#1b}}$

(B) 4δ lb

(C) 8δ lb No calculus needed! The bottom surface is at constant depth (2 feet)

(D) 16δ lb

(E) $\boxed{32\delta \text{ lb}}$ Force = Pressure $\cdot$ Area $= (\rho g)(\text{depth})\cdot 4^2 = (\delta)\cdot 2\cdot 4^2 = 32\delta$.

(f) The improper integral

$$\int_0^\infty \frac{2}{1+x^2}\,dx = \lim_{b\longrightarrow\infty}\int_1^b \frac{2}{1+x^2}\,dx = \lim_{b\longrightarrow\infty} 2\arctan(x)\Big|_0^b$$

$$= \lim_{b\longrightarrow\infty}(2\arctan(b)-0) = 2(\pi/2) = \pi$$

(A) diverges

(B) converges to $\pi/4$

(C) converges to $\pi/2$

(D) $\boxed{\text{converges to } \pi}$

(E) converges to 2π

(g) If the continuous function g satisfies $g(2) = 5$, $g(0) = 4$, and $g'(2) = 3$, then

$$\int_0^2 x\cdot g''(x)\,dx = x\cdot g'(x)\Big|_0^2 - \int_0^2 g'(x)\,dx = 2g'(2)-(g(2)-g(0)) = 2\cdot 3-(5-4) = 5$$

(A) 1 Integrate by parts with $u = x$ and $dv = g''(x)\,dx$.

(B) $\boxed{5}$ Then $du = dx$ and $v = g'(x)$.

(C) 6 Also use the FTC to justify $\int g'(x)\,dx = g(x)$.

(D) 7

(E) 9 $\boxed{\text{cf Homework, p 481 \#63}}$

(h) Evaluate the integral.

$$\int_0^{\pi/4}\cos^2(2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/4}(1+\cos(4\theta))\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin(4\theta)}{4}\right)\Big|_0^{\pi/4} = \frac{\pi}{8}$$

(A) π $\boxed{\text{cf Homework, p 488 \#7}}$

(B) $\pi/2$

(C) $\pi/4$

(D) $\boxed{\pi/8}$

(E) $3\pi/8$

SM 122 Test #2 Solutions Integration 21 Feb 2007

3. Circle the best CAPITAL LETTER (not the entire answer) for each problem.

(a) Find the values of the constants $\boxed{h}$ and $\boxed{k}$ in the recurrence formula:

$$\int x^n e^{2x}\,dx \quad = \quad \boxed{\text{h}} \cdot x^n e^{2x} \quad - \quad \boxed{\text{k}} \int x^{n-1} e^{2x}\,dx$$

CIRCLE ONE CAPITAL LETTER FROM EACH COLUMN

(A)	$h = 1/n$	(A)	$\boxed{k = n/2}$
(B)	$\boxed{h = 1/2}$	(B)	$k = 2n$
(C)	$h = 1$	(C)	$k = 1/(n+1)$
(D)	$h = 2$	(D)	$k = 1/(2n+2)$
(E)	$h = n$	(E)	$k = 1/(2n)$

Integrate by parts with $u = x^n, dv = e^{2x}\,dx$; $du = nx^{n-1}\,dx$ and $v = \frac{1}{2}e^{2x}$:

$$\int x^n e^{2x}\,dx \quad = \quad \frac{1}{2} \cdot x^n e^{2x} \quad - \quad \frac{n}{2} \int x^{n-1} e^{2x}\,dx$$

(b) After an appropriate trig substitution

$$\int_0^1 \frac{y^3}{\sqrt{4-y^2}}\,dy \quad = \int_0^{\pi/6} \frac{(2\sin(\theta))^3}{2\cos(\theta)}(2\cos(\theta))\,d\theta = 8\int_0^{\pi/6} \sin^3(\theta)\,d\theta$$

(A) $8\int_0^{\pi/6} \sin^2(\theta)\cos(\theta)\,d\theta$

(B) $16\int_0^{\pi/6} \sin^3(\theta)\cos(\theta)\,d\theta$

(C) $\boxed{8\int_0^{\pi/6} \sin^3(\theta)\,d\theta}$

(D) $4\int_0^{\pi/6} [\sin^3(\theta)/\cos(\theta)]\,d\theta$

(E) none of above

Let $y = 2\sin(\theta)$. Then $dy = 2\cos(\theta)\,d\theta$.
The limits become $0 \le \theta \le \pi/6$.
Also, $\sqrt{4-y^2} = \sqrt{4-4\sin^2(\theta)} = 2\cos(\theta)$.

$\boxed{\text{cf Quiz \#10}}$

(c) Evaluate the integral.

$$\int_0^{\pi/6} \sin^3(3\theta)\,d\theta = \int_0^{\pi/6} \sin^2(3\theta)\sin(3\theta)\,d\theta = \int_0^{\pi/6} \left(1-\cos^2(3\theta)\right)\sin(3\theta)\,d\theta$$

$$= \int_1^0 \left(1-u^2\right)\frac{-1}{3}\,du = +\frac{1}{3}\int_0^1 (1-u^2)\,du = \frac{1}{3}\left(u - \frac{1}{3}u^3\right)\Big|_0^1 = \frac{2}{9}$$

(A) $\boxed{2/9}$

(B) $2/3$

(C) 1

(D) 2

(E) 4

We let $u = \cos(3\theta)$. So $du = -3\sin(3\theta)\,d\theta$
Also, $\theta = 0$ gives $u = 1$; and $\theta = \pi/6$ gives $u = 0$.

$\boxed{\text{cf Quiz \#10}}$

SM 122 Test #2 Solutions Integration 21 Feb 2007

(d) A rectangular lamina has vertices $(0,0)$, $(0,4)$, $(2,4)$, and $(2,0)$.
The constant density is $\rho = 8$ gm/unit^2.
Find the moment about the x-axis for the lamina.

HINT: WHAT IS THE MASS? WHAT IS $\overline{y}$?

(A) $M_x = 8$ By symmetry $\overline{y} = 2$.
(B) $M_x = 16$ Also, the mass is area times density $= 4 \cdot 2 \cdot 8 = 64$ gm.
(C) $M_x = 32$ We also know that $\overline{y} = \frac{M_x}{m} = \frac{M_x}{64}$
(D) $M_x = 64$ So $2 = M_x/64$, which tells us $M_x = 128$.
(E) $\boxed{M_x = 128}$

(e) A lamina has constant density ρ gm/cm^2 and is bounded above by $y = f(x) = 1/x^3$, below by the positive x-axis, and on the left by the vertical line $x = 1$; the lamina extends forever to the right. Thus the mass of the lamina is given by the improper integral

$$m = \rho \int_1^\infty \frac{1}{x^3}\, dx$$

Find a similar definite integrals for the moments about the axes.

CIRCLE ONE CAPITAL LETTER FROM EACH COLUMN

(A)	$M_y = \rho \int_1^\infty \frac{1}{x}\, dx$	(A)	$M_x = (\rho/2) \int_1^\infty \frac{1}{x^2}\, dx$
(B)	$\boxed{M_y = \rho \int_1^\infty \frac{1}{x^2}\, dx}$	(B)	$M_x = (\rho/2) \int_1^\infty \frac{1}{x^3}\, dx$
(C)	$M_y = \rho \int_1^\infty \frac{1}{x^3}\, dx$	(C)	$M_x = (\rho/2) \int_1^\infty \frac{1}{x^4}\, dx$
(D)	$M_y = \rho \int_1^\infty \frac{1}{x^4}\, dx$	(D)	$M_x = (\rho/2) \int_1^\infty \frac{1}{x^5}\, dx$
(E)	$M_y = \rho \int_1^\infty \frac{1}{x^5}\, dx$	(E)	$\boxed{M_x = (\rho/2) \int_1^\infty \frac{1}{x^6}\, dx}$

Reasoning: $M_y = \rho \int_a^b x \cdot f(x)\, dx$. Also, $M_x = (\rho/2) \int_a^b [f(x)]^2\, dx$.
Apply these formulas with $f(x) = 1/x^3$.

SM 122 Test #2 Solutions Integration 21 Feb 2007

(f) The integral

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{2} \frac{1}{x^2}\,dx = \boxed{\text{I}} + \boxed{\text{II}}$$

(A) is proper Integrand undefined at $x = 0$, so the integral is improper.

(B) $\boxed{\text{is improper and diverges}}$ Integral $\boxed{\text{II}} = \lim_{a\to 0^+} \int_a^2 (1/x^2)\,dx$

(C) is improper and converges to 1/2 $= \lim_{a\to 0^+} (-1/x)\Big|_a^2$

(D) is improper and converges to 1 $= \lim_{a\to 0^+} [(-1/2) - (-1/a)] = \infty$

(E) is improper and converges to 3/2 Since integral $\boxed{\text{II}}$ diverges, so does original integral;

Note: integral $\boxed{\text{I}}$ also diverges.

4. Fill in the blanks.

(a) Integrate.

$$\int \sec^2(10t)\,dt = \underline{\frac{1}{10}\tan(10t) + C}$$

$$\int \cos(\pi y)\,dy = \underline{\frac{1}{\pi}\sin(\pi y) + C}$$

$$\int 10^x\,dx = \underline{\frac{1}{\ln 10}10^x + C}$$

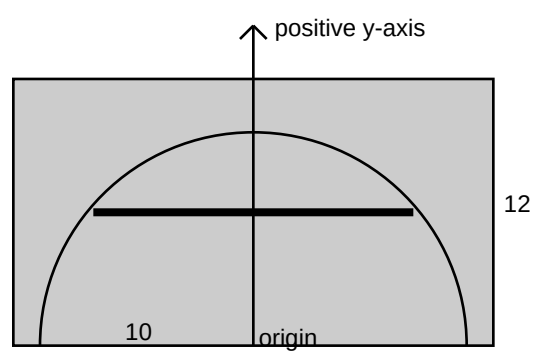
SM 122 Test #2 Solutions Integration 21 Feb 2007

(b) Give the correct form of the partial fraction decomposition for $\frac{10x+9}{x^4+4x^2}$

DO NOT SOLVE FOR THE CONSTANTS A, B, ETC.

$$\frac{10x+9}{x^4+4x^2} = \frac{10x+9}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$

(c) Give a definite integral for the hydrostatic force on the semi-circular disk with radius 10 m submerged in a tank that is 12 m deep, as shown. Express your integral in terms of g and ρ, the mass density of the liquid (in units of kg/m^3). The horizontal rectangle at position y is included to help you. Do not simplify or evaluate.

$$\text{Hydrostatic force} \quad = \quad \int_{\underline{0}}^{\underline{10}} \quad \underbrace{\rho g(12-y)}_{pressure} \underbrace{2\sqrt{10^2-y^2}\ dy}_{area}$$

Reasoning: The slice at position y has area $2\sqrt{10^2-y^2}\,dy$.
The depth of the slice is $12-y$.
This follows from the Pythagorean theorem.
(Can you spot the right triangle?)
For this slice: Force = Pressure $\cdot$ Area = $[\rho g(\text{depth})] \cdot [2\sqrt{10^2-y^2}\,dy]$.