SM 122 Test #3 Diff. Eqns, Polar Coords. 5 Apr 2006

1. In this problem we consider two polar curves:

first curve: $r = 4\cos(3\theta)$ second curve: $r = 2$

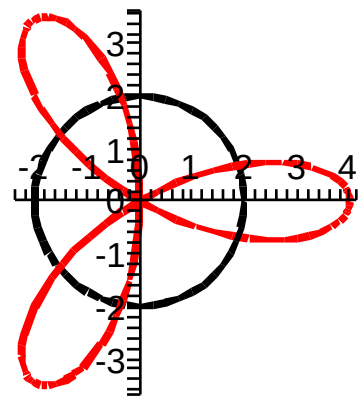
The two curves intersect in a point $P = (r, \theta)$ in the first quadrant.

(a) The first curve is best described as ...

(A) a circle
(B) four spirals
(C) $\boxed{\text{propellor blades}}$ Graph it on your calculator to see a 3-leaf clover, or 3-blade prop.
(D) a 4-leaf clover
(E) four circles

(b) The second curve is best described as ...

(A) $\boxed{\text{a circle}}$ The circle has radius 2.
(B) a spiral
(C) a cardioid
(D) a horizontal line
(E) a vertical line



(c) What is θ for the first quadrant intersection point P?

(A) $\theta = \pi/12$ $(15°)$
(B) $\boxed{\theta = \pi/9 \quad (20°)}$ $4\cos(3\theta) = 2$ gives $\cos(3\theta) = 1/2)$,
(C) $\theta = \pi/6$ $(30°)$ which says that $3\theta = \pi/3$ and $\theta = \pi/9$
(D) $\theta = \pi/4$ $(45°)$
(E) $\theta = \pi/3$ $(60°)$

(d) The y-coordinate of the intersection point P is approximately ...

(A) $\boxed{y \doteq 0.68}$ The polar coordinates of P are $(r, \theta) = (2, \pi/9)$.
(B) $y \doteq 0.34$ So the y-coordinate of P is $r\sin(\theta) = 2\sin(\pi/9) \doteq 0.68$
(C) $y \doteq 0.86$
(D) $y \doteq 1.73$
(E) $y \doteq 0.50$

(e) What is the area in the first quadrant that is inside the first curve but outside the second curve?

(A) $2\pi/9$
(B) $2\pi/3$
(C) $\pi/3$
(D) $\boxed{\sqrt{3}/3 \ + \ 2\pi/9}$
(E) $10\pi/9 \ - \ \sqrt{3}/3$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/9} [(4\cos(3\theta))^2 - 2^2]\, d\theta = \frac{\sqrt{3}}{3} + \frac{2\pi}{9} \quad \text{by calculator}$$

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2. A midshipman is analyzing a series RC-circuit with electromotive force

$$E(t) = 150\sin(3t) \qquad \text{volts.}$$

After applying Kirchhoff's Law and performing some algebraic simplification, she obtains the initial-value problem

$$Q' \quad + \quad 4Q \quad = \quad 50\sin(3t), \qquad\qquad Q(0) = 10 \text{ Coulombs.}$$

(a) The capacitor has capacitance

(A) $\boxed{C = (1/12)\,\mathrm{F}}$ By Kirchhoff's Law $E_R + E_C = E$.
(B) $C = (1/4)\,\mathrm{F}$ So $RQ' + (1/C)Q = 150\sin(3t)$.
(C) $C = 1\,\mathrm{F}$ To get the RHS $= 50\sin(3t)$,
(D) $C = 4\,\mathrm{F}$ she must have divided by 3.
(E) $C = 12\,\mathrm{F}$ So $1/3C = 4$, or $1 = 12C$. Solve to find $C = 1/12$

(b) What integrating factor should the student use?

(A) $\mu(t) = e^{-4t}$
(B) $\mu(t) = 24\sin(3t)$
(C) $\boxed{\mu(t) = e^{4t}}$ The coefficient of Q is 4, and $\mu(t) = e^{\int 4\,dt} = e^{4t}$
(D) $\mu(t) = 8\sin(3t)$
(E) $\mu(t) = e^{4t}\sin(3t)$

(c) Solve the initial-value problem.
Put your answer in the blank below.

Show your work.

$$e^{4t}Q' + e^{4t}\cdot 4Q = e^{4t}\cdot 50\sin(3t)$$

$$[e^{4t}Q]' = e^{4t}\cdot 50\sin(3t)$$

$$\int [e^{4t}Q]'\,dt = \int e^{4t}\cdot 50\sin(3t)\,dt$$

$$e^{4t}Q = e^{4t}\left(8\sin(3t) - 6\cos(3t)\right) \;+\; C \qquad \text{by calculator}$$

$$Q(t) = 8\sin(3t) - 6\cos(3t) \;+\; Ce^{-4t}$$

$$10 = Q(0) = 0 - 6 + C. \qquad \text{So } C = 16$$

$$Q(t) \quad = \quad \underline{\quad 8\sin(3t) - 6\cos(3t) \;+\; 16e^{-4t} \quad}$$

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3. **Multiple Choice.**

(a) A series circuit has a 50 ohm resistor, a 25 henry inductor, and a 100 volt electromotive force. If we separate variables, then we obtain ...

(A) $\underline{\int \frac{dI}{4-2I} = \int dt}$ Kirchhoff: $E_L + E_R = E$; $L\frac{dI}{dt} + RI = E$

(B) $\int \frac{dI}{2-4I} = \int dt$ $25\frac{dI}{dt} + 50I = 100$; $\frac{dI}{dt} + 2I = 4$

(C) $\int \frac{dI}{25+50I} = \int 100\,dt$ $\frac{dI}{dt} = 4 - 2I$; $\frac{dI}{4-2I} = dt$

(D) $\int \frac{dI}{25-50I} = \int 100\,dt$

(E) $\int \frac{dI}{50-25I} = \int 100\,dt$

(b) What integrating factor should we use to solve the differential equation

$$(1+x^2)y' + 2xy = 2006x?$$

(A) $\mu(x) = e^{x^2}$ Put in standard form: $y' + \frac{2x}{1+x^2}y = 2006x$

(B) $\mu(x) = e^{\arctan(x)}$ $\mu(x) = \exp\left(\int \frac{2x}{1+x^2}\,dx\right)$

(C) $\boxed{\mu(x) = 1 + x^2}$ $= \exp\left(\ln(1+x^2)\right) = 1 + x^2$

(D) $\mu(x) = e^{x^2 \arctan(x)}$

(E) $\mu(x) = x^2$

(c) For what value(s) of the parameter k is $y = e^{kx}$ a solution to

$$y'' + 2y' - 3y = 0 \quad ?$$

CIRCLE **ALL** CORRECT CAPITAL LETTERS

(A) $\boxed{k = -3}$ If $y = e^{kx}$, then $y' = ke^{kx}$ and $y'' = k^2e^{kx}$

(B) $k = -1$ Now plug into differential equation:

(C) $k = 0$ $k^2e^{kx} + 2ke^{kx} - 3e^{kx} = 0$. Cancel the exponential.

(D) $\boxed{k = 1}$ You get $0 = k^2 + 2k - 3 = (k+3)(k-1)$.

(E) $k = 3$ So $k = -3$ and $k = 1$. **OR** you could try all five possible answers

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(d) The mass in grams of a radioactive substance is given by

$$m(t) = 6e^{-0.07t},$$

where t is measured in days.

The half-life of the substance is approximately ...

(A) 1 day
(B) 3 days
(C) 6 days
(D) $\boxed{\text{10 days}}$ $6 = m(0)$; So we solve $(6/2) = 6e^{-0.07t}$ for t and get $t \approx 10$.
(E) 13 days

(e) A bacteria population starts with 500 bacteria and grows at a rate proportional to the population itself. After 2 hours there are 1000 bacteria. How many bacteria are there at 6 hours?

(A) 2500
(B) 3000
(C) $\boxed{4000}$ The pop. doubles every 2 hrs. So at 6 hrs the pop. is $500 \cdot (2)(2)(2) = 4000$
(D) 4500 **OR** write $P(t) = P_0 e^{kt} = 500e^{kt}$. Then $1000 = P(2) = 500e^{k \cdot 2}$. Solve to get $k = \frac{1}{2}\ln(2)$.
(E) 6000 Then $P(6) = 500e^{6 \cdot \frac{1}{2}\ln(2)} = 500e^{\ln(2^3)} = 500 \cdot 8 = 4000$

(f) The function $y = g(x)$ satisfies $g'(x) = 2x$ and $g(0) = 5$. What is $g(2)$?

(A) 1
(B) 3
(C) 5
(D) 7 Integration gives $g(x) = x^2 + C$. Also, $5 = g(0) = 0^2 + C$.
(E) $\boxed{9}$ So $C = 5$, and thus $g(x) = x^2 + 5$. Therefore $g(2) = 2^2 + 5 = 9$.

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4. Throughout this problem we consider the differential equation

$$y' = y^2 - 2x$$

together with the initial condition $y(3) = 2$.

(This means that the graph of the solution $y = f(x)$ passes through the point $(x_0, y_0) = (3, 2)$.)

(a) If we use Euler's method with step-size $h = 0.1$ to approximate $y(3.1)$, then we find

(A) $y(3.1) \approx 1.6$ $\quad x_0 = 3$ and $y_0 = 2$

(B) $\boxed{y(3.1) \approx 1.8}$ $\quad y(3.1) \approx y_1 = y_0 + h(y_0^2 - 2x_0) = 2 + 0.1(2^2 - 2 \cdot 3) = 1.8$

(C) $y(3.1) \approx 2.8$

(D) $y(3.1) \approx 2.9$

(E) $y(3.1) \approx 3.0$

(b) The differential equation is

(A) both linear and separable

(B) linear, but not separable

(C) separable, but not linear

(D) $\boxed{\text{neither linear, nor separable}}$

(E) none of the above.

(c) What is the value of $y'(3)$?

(A) $\boxed{y'(3) = -2}$ $\quad y'(3) = 2^2 - 2 \cdot 3 = -2$ since $(x, y) = (3, 2)$ is on the curve $y = y(x)$.

(B) $y'(3) = 0$

(C) $y'(3) = 2$

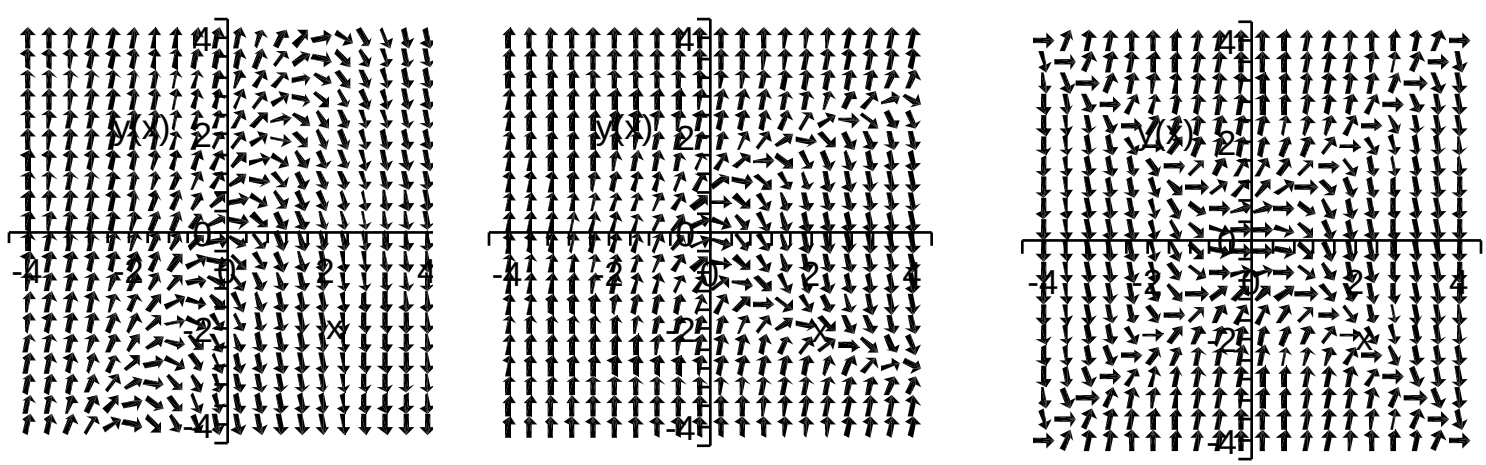
(D) $y'(3) = 4$

(E) $y'(3) = 5$

(d) The correct direction field for the differential equation is __II__

Fill in a Roman numeral

Ignore the arrows on each slope-segment. The axes go from $-4 \le x \le 4$ and $-4 \le y \le -4$.



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5. **Word Problems.** Circle the best capital letter and **show your work!**

(a) A tank contains 10 kg of salt dissolved in 500 L of water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 20 liters per minute. The solution is kept thoroughly mixed. Liquid drains from the tank at the rate of 20 L/min.

Let $y(t)$ denote the number of kg of salt in the tank after t minutes.
We know that $y(0) = 10$, for instance.

Which differential equation does $y(t)$ satisfy?
YOU ARE NOT BEING ASKED TO SOLVE THE DIFFERENTIAL EQUATION.

(A) $\boxed{y' = \frac{25-y}{25}}$
(B) $y' = \frac{20-y}{500}$
(C) $y' = \frac{10-y}{500}$
(D) $y' = \frac{15-y}{500}$
(E) $y' = 1 - \frac{y}{100}$

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = \left(\frac{20\ \text{L}}{1\ \text{min}}\right)\left(\frac{0.05\,\text{kg}}{1\ \text{L}}\right) - \left(\frac{20\ \text{L}}{1\ \text{min}}\right)\left(\frac{y(t)\,\text{kg}}{500\ \text{L}}\right)$$

$$= 1 - \frac{20y}{500} = 1 - \frac{y}{25} = \frac{25-y}{25}$$

(b) On a 90-degree day last summer, a Plebe filled his water bottle with 40-degree water from a secret refrigerator in his room at 0800. Of course, the water immediately started warming up in accordance with Newton's Law of Cooling (actually Newton's Law of Warming, in this case). At 0810 the water was already 60 degrees.

What was the water temperature at 0820?

(A) 70 degrees
(B) $\boxed{\text{72 degrees}}$
(C) 75 degrees
(D) 80 degrees
(E) 85 degrees

Let $y(t)$ be the water temperature at time t, where $t = 0$ is 0800.
We know that $\frac{dy}{dt} = k(90 - y)$, and $y(0) = 40$, and $y(10) = 60$
Separation of variables and integration yields $y(t) = 90 - Ae^{kt}$.
The condition $40 = y(0) = 90 + A$ gives $A = 50$. So $y(t) = 90 - 50e^{kt}$.
The condition $60 = y(10) = 90 - 50e^{k\cdot 10}$ gives $k = \frac{1}{10}\ln(3/5)$.
So $y(t) = 90 - 50\left(\frac{3}{5}\right)^{t/10}$.
Finally, $y(20) = 90 - 50\left(\frac{3}{5}\right)^{20/10} = 72$