A Maple package for the decomposition of certain tensor products and restrictions of representations using crystal graphs

David Joyner, Roland Martin

April 12, 1997

Abstract

This document is an introduction to the Maple package **crystal**, version 2.

1. Introduction

The **crystal**, version 2 package is a significant revision of **crystal**, version 1. Indeed, the names of most procedures have changed. This package makes much use of John Stembridge's packages **coxeter** and **weyl** [S]. This package also makes use of the standard maple package **plots**.

This document is an introduction to **crystal**, version 2, a Maple package which helps decompose

(a) the tensor product of two irreducible finite dimensional representations of a simple Lie algebra, and

(b) the restriction of an irreducible finite dimensional representations of a simple Lie algebra to a semisimple subalgebra,

into irreducible constituents using crystal graphs. We will not discuss background on Lie algebras or their representations. We refer to [H] for background. A crystal graph is defined by

Definition 1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h} \subset \mathfrak{g}$ denote a Cartan subalgebra, and let (π, V) denote a finite dimensional complex representation of $\mathfrak{g}$. The* **crystal graph** *is a colored digraph $\Gamma = (V, E)$, where*

- *the set V of vertices are given by the set of all one-dimensional irreducible subrepresentations of the restriction of (π, V) to $\mathfrak{h}$,*

- *the set E of edges is defined by the following condition: we say that $W \in V$ is connected to $W' \in V$ by an edge in E if and only if there is a simple root vector X of $\mathfrak{g}$ such that $\pi(X)W = W'$. In this case, we label of "color" this edge by X (or by the simple root associated to X).*

Basically this package amounts to implementing a theorem of Kashiwara [K]: the irreducible constituents of such a tensor product are in a natural 1-1 correspondence with the connected components of the crystal graph of the tensor product. The **crystal** package contains programs to

(a) compute the crystal graph of an irreducible representation, associated to a fundamental weight, provided it is multiplicity free when restricted to the Cartan subalgebra,

(b) compute the crystal graph of the tensor product of two irreducible representations, each of which is obtained from (a),

(c) display the crystal graph of the tensor product obtained in (b) (using our own routines, which are better suited for our purposes than the **networks** package draw command),

(d) display the crystal graph of a fundamental representation obtained in (a),

(e) computing all the weights occurring in a given irreducible representation.

The procedures in crystal are graphprodrep, graphrep, weight_system, linsubgraphrep, showgraph, showgraphprod, and dynkin_diagram. The effect of these procedures are described in the crystal help files. The remainder of this document is devoted to explaining how to install crystal and its help files onto your system. We shall restrict ourselves to the case of a PC using windows 3.1 or better with MAPLEV4 installed on the 'c:' drive. Modifications for other machines and operating systems hopefully will not be too complicated.

2. Installation

Create directories

```
c:\maplev4\share\algebra\crystal
c:\maplev4\share\algebra\crystal\cryhelp
```

Copy all crystal .mws files into the crystal directory. Save the crystal worksheets into the help database file maple.hdb in the cryhelp directory using MAPLE's *save as database* in the *help* menu. In a new session of MAPLEV4, type

```
with(share);
readshare(coxeter,algebra);
libname:=libname,`c:\\MAPLEV4/share/algebra/crystal/cryhelp`;
read(`c:/maplev4/share/algebra/crystal/crystal25.mpl`);
?crystal
```

This should bring up the main crystal help page which describes the procedures in this package.

3. Copyleft

Copyleft ©1996 by David Joyner, Roland Martin

Permission is granted to anyone to to use, modify, or redistribute this software freely, subject to the following restrictions:

1. The authors accepts no responsibility for any consequences of this software and makes no guarantee that the software is free of defects.

2. The origin of this software must not be misrepresented, either by explicit claim or by omission.

3. This notice and the copyleft must be included in all copies or altered versions of this software.

4. This software may not be included or redistributed as part of any package to be sold for profit without the explicit written permission of the authors.

References

[H] J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Springer Verlag, Berlin-New York, 1972.

[K] Masaki Kashiwara, "Crystalizing the q-analog of universal enveloping algebras," Comm. Math. Physics, 133 (1990)249-260

[S] John Stembridge, "A Maple package for roots systems and finite coxeter groups", preprint, 1992, available as TeX documentation for the maple share package **coxeter**

David Joyner
wdj@nadn.navy.mil
Roland Martin
Math. Dept. US Naval Academy, Annapolis, MD 21402
rem@sma.usna.navy.mil