L–FUNCTIONS FOR G_2

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Abstract. This note presents an application of the Casselman–Shalika formula to L–functions for the exceptional group G_2.

§0. Introduction

The purpose of this paper is to collect some foundational material on L–functions on the split form of the exceptional group G_2. We shall be concerned with two topics:

Firstly, we shall be concerned with the phenomenon known as "triality", which yields an embedding of G_2 into the double cover Spin(8) of the split form of SO(8). In view of Langlands' conjectured principle of functoriality, the associated L-map ${}^L{G_2}^o(\mathbb{C}) \hookrightarrow {}^L PSO(8)^o(\mathbb{C})$ should induce a "lifting" or transfer of tempered automorphic representations from $G_2(\mathbb{A})$ to $PSO(8,\mathbb{A})$ in such a way that the Langlands L-functions associated to the 8-dimensional ("Cayley") representation of ${}^L{G_2}^o(\mathbb{C}) \cong G_2(\mathbb{C})$, an Euler product of degree 7 with a conjectured functional equation, agrees with the corresponding Euler product for the transferred automorphic representation of $PSO(8,\mathbb{A})$, times a Dirichlet L-function. It seems likely that transfer is "endoscopic", hence amendable to study by means of the (stabilized) trace formula for G_2 and the triality-twisted trace formula for PSO(8), and for this, of course, a good understanding of triality would be necessary. Incidentally, in view of the known functional equations for the L-functions on SO(8) (cf. I. Piatetski-Shapiro and S. Rallis [PSR]), this transfer would imply the functional equations for the L-functions on G_2. We shall determine the triality map explicitly at the root level, and also use it to give an explicit Chevalley basis for the Lie algebra g_2, which should be useful in global calculations.

Secondly, we shall make explicit the Casselman-Shalika formula for unramified Whittaker functions on G_2 over a non-archimedian local field. By integrating, we shall show how to evaluate a certain integral of the Whittaker function associated with a generic cuspidal automorphic representation. The final result, Theorem 3.13, is suggestive of a Rankin-Selberg convolution, but unfortunately we have thus far been unable to use this integral

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to prove the functional equations of the L-functions. However, we include this as a model of how such local calculations are to be carried out starting with the Casselman-Shalika formula. For example, the analogous integral on $GSp(4)$, already studied by I. Piatetski-Shapiro, M. Novodovorsky, and D. Soudry, may be evaluated by a similar calculation, as well as a number of integrals on other groups such as GL(n).

In section 1, we recall some relevant background on $SO(8)$ and G_2. In section 2, we work out the Casselman–Shalika formula in our case in the form we need here. In section 3, we use the results of section 2 and the Weyl character formula to calculate the L–function.
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§1. Background on G_2 and SO(8).

1.1. SO(8).

Let J denote the 8×8 matrix with 1's on the skew–diagonal and 0's elsewhere:

$$J := \begin{pmatrix} 0 & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & 0 \end{pmatrix}.$$

Define the 8×8 **special orthogonal group** by

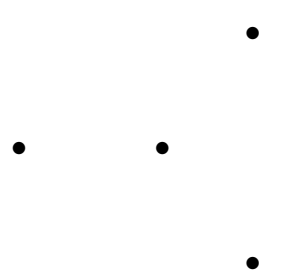
$$SO(8) := \{A \in GL(8) \mid A \cdot J \cdot^t A = J\}^o,$$

so that $SO(8)$ is semi–simple, connected, neither simply connected nor adjoint, non–compact, and split over $\mathbb{Q}$. Its maximal $\mathbb{Q}$-split torus consists of all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \text{ with } A := \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix}, \; B := \begin{pmatrix} a_4^{-1} & & & \\ & a_3^{-1} & & \\ & & a_2^{-1} & \\ & & & a_1^{-1} \end{pmatrix}. \tag{1.1}$$

The compact real form of $SO(8)$ is obtained, of course, by replacing J by the 8×8 identity matrix.

The simply–conected covering group of $SO(8)$ is $SO(8)_{sc} = \mathrm{Spin}(8)$, which has center $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and since $SO(8)$ has center $\mathbb{Z}/2\mathbb{Z}$, Spin(8) is a 2-fold cover of $SO(8)$ and a 4-fold cover of $PSO(8)$. The outer automorphisms of Spin(8) are in one–to–one correspondence with the automorphisms of the Dynkin diagram for $SO(8)$:



The Lie algebra

$$so(8) := \{A \in M_8 \mid AJ + J^t A = 0\}$$

may be expressed as the Lie algebra of 8×8 matrices of the form

$$A = \begin{pmatrix} t_1 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & 0 \\ a_{21} & t_2 & a_{23} & a_{24} & a_{25} & a_{26} & 0 & -a_{17} \\ a_{31} & a_{32} & t_3 & a_{34} & a_{35} & 0 & -a_{26} & -a_{16} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -a_{71} & -a_{61} & -a_{51} & -a_{41} & -a_{31} & -a_{21} & -t_1 \end{pmatrix}. \tag{1.2}$$

If $i \neq j$ then define E_{ij} to be the matrix A as above with $a_{ij} = 1$ and all other entries $=$ 0. Define E_{ii} to be the matrix A as above with $t_i = 1$ and all other entries $= 0$. Then

$$\begin{aligned} &\{E_{11},\, E_{22},\, E_{33},\, E_{44}, \\ &\quad E_{12},\, E_{13}, \ldots, E_{17}, E_{23}, \ldots, E_{26}, E_{34},\, E_{35}, \\ &\quad {}^tE_{12},\, {}^tE_{13}, \ldots, {}^tE_{17}, {}^tE_{23}, \ldots, {}^tE_{26}, {}^tE_{34}, {}^tE_{35}\} \end{aligned} \tag{1.3}$$

forms a basis for the 28-dimensional vector space $so(8)$.

1.2. The Cayley algebra and G_2.

The **split Cayley algebra** is the 2×2 matrix algebra

$$C := \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{Q},\ a, b \in V \right\}, \tag{1.4}$$

where $V \cong \mathbb{Q}^3$ is the cross product algebra

$$\begin{aligned} V := \{xi + yj + zk \mid x, y, z \in \mathbb{Q},\ i \times i = j \times j = k \times k = 0, \\ i \times j = k,\ j \times k = i,\ k \times j = i,\ j \times i = -k,\ \text{etc.}\}. \end{aligned}$$

Define

$$G_2(F) := \mathrm{Aut}_F(C \otimes F), \tag{1.5}$$

where F is any field containing $\mathbb{Q}$. Since Der(C) $\cong$ Lie(Aut C), it follows that the Lie algebra $g_2 :=$ Lie G_2 may be identified with Der(C). For G_2, these are 2 real forms: one being the non-compact $\mathbb{Q}$–split from above, the other being a compact form associated to a division algebra (see[Jac, pp. 142-147] or [Tits]). We have

$$\dim_{\mathbb{Q}} C = 8,\ \dim_{\mathbb{Q}} g_2 = 14,$$

and there is a trace bilinear form which induces an embedding

$$G_2 = \mathrm{Aut}(C) \hookrightarrow SO(8).$$

Here SO(8) is regarded as the special orthogonal group associated to the trace form on C. Regarding G_2 as a subgroup of SO(8), we have the following

Triality theorem 1.6([Jac])**.** *Given any orthogonal map*

$$A_0 : C \to C,$$

there are orthogonal maps $A_1,\ A_2$ *such that, for all* $x,\ y \in C$*,*

$$A_0(xy) = A_1(x)A_2(y).$$

Moreover, the only other orthogonal maps satisfying this property are $-A_1,\ -A_2$*, so*

$$A_0 \longmapsto (A_1, A_2)$$

is well–defined, up to signs.

proof. We will verify that our form of triality is the global equivalent of Jacobson's form [Jac, p. 8]. Let

$$\begin{aligned} A_0 &= 1 + \delta B_0, \\ A_1 &= 1 + \delta B_1, \\ A_2 &= 1 + \delta B_2, \end{aligned}$$

in $SO(8, \mathbb{Q})$ (regarded as a group of orthogonal maps with respect to the trace form on C), where $\delta^2 = 0$ and $B_i \in so(8, \mathbb{Q})$. Thus $A_0,\ A_1,\ A_2$ may be regarded as lying in the tangent space to SO(8) at the identity (cf. [Bor, chapter 0]). Then $A_0(xy) = A_1(x)A_2(y)$ is equivalent to

$$B_0(xy) = B_1(x)y + xB_2(y).$$

This is Jacobson's expression.

Corollary 1.7. *The map*

$$\begin{aligned} SO(8, F) &\to PSO(8, F), \\ A_0 &\longmapsto \quad \pm A_1, \end{aligned}$$

defines a projective representation of SO(8), i.e., belongs to PSO(8). In particular, it induces an outer automorphism of degree 3 (the "triality" automorphism) corresponding to a rotation of the Dynkin diagram of $PSO(8, F)$*.*

Remarks. (1.8) If $A_0 \in G_2 = \mathrm{Aut}(C)$ then triality sends $A_0 \longmapsto A_0$. Conversely, if triality sends $A_0 \longmapsto A_0$ then $A_0 \in G_2$, so G_2 is the subgroup fixed by triality. In general, triality sends (up to signs)

$$A_0 \longmapsto A_1,\ A_1 \longmapsto A_2,\ A_2 \longmapsto A_0.$$

This way we can explicitly construct an outer automorphism of G_2.

(1.9) Triality can be described explicitly on the standard Borel subgroup B of PSO(8): triality cyclically permutes the "root groups" U_α, defined below, in the same way as it permutes the simple roots via its action on the Dynkin diagram; it also permutes the embeddings $PSL(2) \hookrightarrow PSO(8)$ associated to the simple roots. The images of the maximal split torus $T \subset PSL(2)$ embedded in $PSO(8)$ generate the maximal split torus in $PSO(8)$ and, of course, the simple root groups generate the unipotent radical of B. Putting these all together explicitly gives the action of triality on B.

Consequently, we know explicitly how triality acts on the characters of B, their induced representations, and in particular on the principal series representations of $PSO(8)$. Therefore, we will know how triality acts on local Langlands L-functions of principal series representations of $PSO(8)$.

1.3. Root systems for G_2 and $SO(8)$.

Since the symmetric group S_3 acts on the Dynkin diagram of $SO(8)$ by triality, it acts on its 24 roots. Of these, 6 roots remain fixed under this action, and 18 are permuted cyclically in groups of three. In particular, the roots corresponding to

$$\{E_{17}\}, \quad \{E_{16}\}, \quad \{E_{23}\},$$

are fixed and the triples of roots corresponding to

$$\{E_{14},\ E_{15},\ E_{26}\},\ \{E_{13},\ E_{24},\ E_{25}\},\ \{E_{12},\ E_{34},\ E_{35}\},$$

are each permuted by the action of S_3. More explicitly, let us represent a basis of $so(8)$ by

$$\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \epsilon_3 - \epsilon_4,\ \epsilon_3 + \epsilon_4,$$

where

$$\epsilon_i : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \longmapsto a_i, \tag{1.10}$$

in the notation of (1.1), $1 \le i \le 4$. Here E_{12} corresponds to $\epsilon_1 - \epsilon_2$, E_{13} to $\epsilon_1 - \epsilon_2 + \epsilon_2 - \epsilon_3 = \epsilon_1 + \epsilon_3$, etc. Since $\epsilon_2 - \epsilon_3$ is connected to the others (since its Killing form inner product with them is non–zero), it corresponds to the center of the Dynkin diagram

$$\begin{array}{ccc} & & \bullet\,\epsilon_1 - \epsilon_2 \\ \epsilon_3 - \epsilon_4\ \bullet & \bullet\ \epsilon_2 - \epsilon_3 & \\ & & \bullet\,\epsilon_3 + \epsilon_4 \end{array}$$

Therefore, under the action of S_3, the triality automorphism of so(8) induces a linear map ϕ on the roots such that

$$\phi(\epsilon_1-\epsilon_2)=\epsilon_3-\epsilon_4,\ \phi(\epsilon_3-\epsilon_4)=\epsilon_3+\epsilon_4,\ \phi(\epsilon_3+\epsilon_4)=\epsilon_1-\epsilon_2,\ \phi(\epsilon_2-\epsilon_3)=\epsilon_2-\epsilon_3. \tag{1.11}$$

Thus

$$\begin{aligned}\phi(\epsilon_1)&=\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4),\\ \phi(\epsilon_2)&=\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4),\\ \phi(\epsilon_3)&=\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4),\\ \phi(\epsilon_4)&=\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4).\end{aligned}$$

Call the induced automorphism on the Lie algebra so(8), ϕ^*. As we've already observed, the Lie subalgebra

$$\{X\in so(8)\mid \phi^*(X)=X\}$$

is a Lie algebra of type G_2, denoted g_2. Identifying its Lie group with a subgroup of $SO(8)$, we find that its maximal $\mathbb{Q}-$split torus is of the form

$$T':=\left\{\begin{pmatrix}A&0\\0&B\end{pmatrix},\ \mid A:=\begin{pmatrix}a_1&&&\\&a_2&&\\&&a_3&\\&&&1\end{pmatrix},\ B:=\begin{pmatrix}1&&&\\&a_3^{-1}&&\\&&a_2^{-1}&\\&&&a_1^{-1}\end{pmatrix}\right.$$
$$\left.\text{and } a_1=a_2a_3\right\}. \tag{1.12}$$

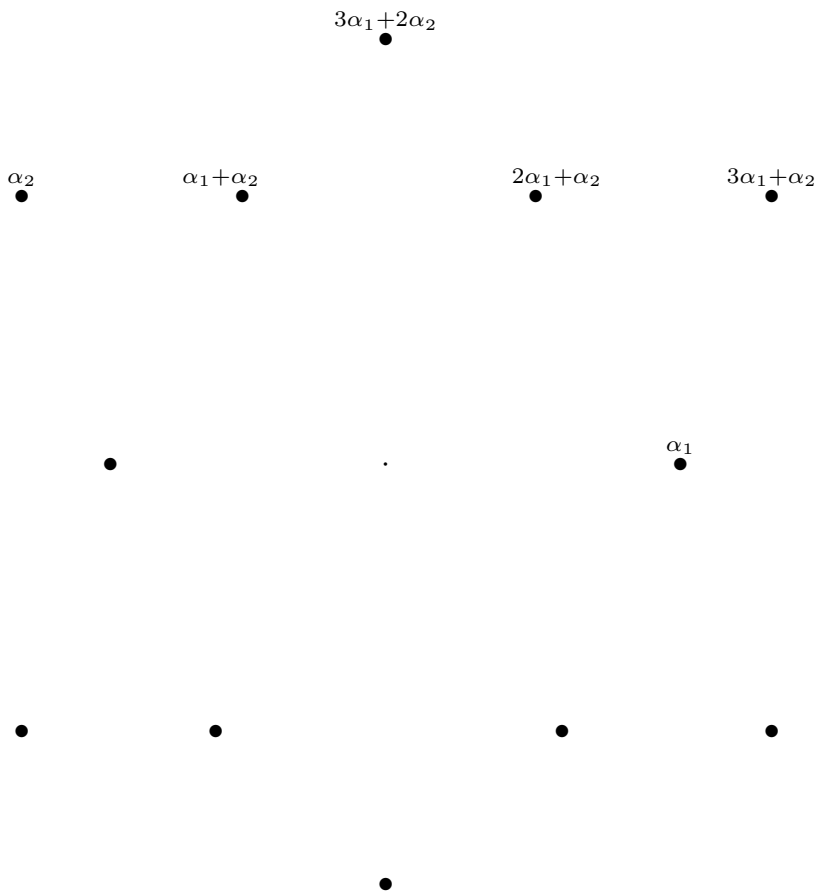
It will be convenient later to identify T' with the maximal split torus of SL(3) by means of the map

$$\begin{pmatrix}A&0\\0&B\end{pmatrix}\in T'\longmapsto \mathrm{diag}(a_2,a_3,a_1^{-1}). \tag{1.13}$$

1.4. The Chevalley basis.

The embedding of g_2 into s0(8) gives a convenient framework for doing global calculations with explicit matrices. We now write down the Chevalley basis of G_2. The notation

for the root system is given by the following diagram:



the root system for G_2

$$X_{\alpha_1} := \begin{pmatrix} 0 & 1 & & & & & & \\ & 0 & & & & & & \\ & & 0 & 1 & 1 & & & \\ & & & 0 & 0 & -1 & & \\ & & & & 0 & -1 & & \\ & & & & & 0 & & \\ & & & & & & 0 & -1 \\ & & & & & & & 0 \end{pmatrix}, \quad h_{\alpha_1} := \begin{pmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & 2 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & -2 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{pmatrix}.$$

(Note that $X_{\alpha_1} = E_{12} + E_{34} + E_{35}$.)

$$X_{\alpha_2} := \begin{pmatrix} 0 & & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & -1 & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad h_{\alpha_2} := \begin{pmatrix} 0 & & & & & & & \\ & 1 & & & & & & \\ & & -1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & 0 \end{pmatrix}.$$

(Note that $X_{\alpha_2} = E_{23}$.) For $\alpha = \alpha_1 + \alpha_2$:

$$X_{\alpha} := \begin{pmatrix} 0 & & -1 & & & & & \\ & 0 & & 1 & 1 & & & \\ & & 0 & & & & & \\ & & & 0 & & & -1 & \\ & & & & 0 & & -1 & \\ & & & & & 0 & & 1 \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad h_{\alpha} := \begin{pmatrix} 1 & & & & & & & \\ & 2 & & & & & & \\ & & -1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & -2 & \\ & & & & & & & -1 \end{pmatrix}.$$

(Note that $X_{\alpha_1+\alpha_2} = E_{13} + E_{24} + E_{25}$.) For $\alpha = 2\alpha_1 + \alpha_2$:

$$X_{\alpha} := \begin{pmatrix} 0 & & & 1 & 1 & & & \\ & 0 & & & & 1 & & \\ & & 0 & & & & -1 & \\ & & & 0 & & & & -1 \\ & & & & 0 & & & -1 \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad h_{\alpha} := \begin{pmatrix} 2 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -2 \end{pmatrix}.$$

(Note that $X_{2\alpha_1+\alpha_2} = E_{14} + E_{15} + E_{26}$.) For $\alpha = 3\alpha_1 + \alpha_2$:

$$X_{\alpha} := \begin{pmatrix} 0 & & & & & 1 & & \\ & 0 & & & & 0 & & \\ & & 0 & & & 0 & 0 & -1 \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad h_{\alpha} := \begin{pmatrix} 1 & & & & & & & \\ & 0 & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & -1 \end{pmatrix}.$$

(Note that $X_{3\alpha_1+\alpha_2} = E_{16}$.) For $\alpha = 3\alpha_1 + 2\alpha_2$:

$$X_\alpha := \begin{pmatrix} 0 & & & & & & 1 & \\ & 0 & & & & & 0 & -1 \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad h_\alpha := \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & -1 & \\ & & & & & & & 1 \end{pmatrix}.$$

(Note that $X_{3\alpha_1+2\alpha_2} = E_{17}$.)

There is a canonical embedding $G_2 \hookrightarrow SO(8)$, given by the natural faithful representation associated to the Cayley algebra. In fact, each one–parameter unipotent subgroup U_α associated to a short root α of G_2 is contained in a three dimensional unipotent subgroup of SO(8) generated by three (uniquely determined) roots of D_4 which are permuted by S_3. The possibilities for this embedding of unipotents can be tabulated as follows:

$$\begin{aligned} &U_{\alpha_1} \subset G_2 \longleftrightarrow U := \exp[E_{12}F + E_{34}F + E_{35}F], \\ &U_{2\alpha_1+\alpha_2} \subset G_2 \longleftrightarrow U := \exp[E_{14}F + E_{15}F + E_{26}F], \\ &U_{\alpha_1+\alpha_2} \subset G_2 \longleftrightarrow U := \exp[E_{13}F + E_{24}F + E_{25}F]. \end{aligned} \tag{1.14}$$

For the long roots of G_2, we have

$$\begin{aligned} &U_{\alpha_2} \subset G_2 \longleftrightarrow U := \exp[E_{23}F], \\ &U_{3\alpha_1+2\alpha_2} \subset G_2 \longleftrightarrow U := \exp[E_{17}F], \\ &U_{3\alpha_1+\alpha_2} \subset G_2 \longleftrightarrow U := \exp[E_{16}F]. \end{aligned} \tag{1.15}$$

In other words, under the action of S_3 on the roots of D_4, the six stable roots correspond to the long roots of G_2 and the other 18, permuted in orbits of 3, correspond to the short roots of G_2.

§2. The Casselman–Shalika formula for $\mathbf{G}_2$

From section 1, we may identify the maximal $\mathbb{Q}$–split torus in G_2 with

$$T := \left\{ \tau := \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \mid t_1t_2t_3 = 1 \right\}. \tag{2.1}$$

The simple roots are

$$\text{(short) } \alpha_1 : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \longmapsto t_2,$$

$$\text{(long) } \alpha_2 : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \longmapsto t_1t_2^{-1}.$$

The fundamental weights are

$$\lambda_1 := 2\alpha_1 + \alpha_2,\ \lambda_2 := 3\alpha_1 + 2\alpha_2. \tag{2.2}$$

The reflections corresponding to α_1 and α_2 are

$$w_1 : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \longmapsto \begin{pmatrix} t_1^{-1} & & \\ & t_2^{-1} & \\ & & t_3^{-1} \end{pmatrix},$$
$$w_2 : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \longmapsto \begin{pmatrix} t_2 & & \\ & t_1 & \\ & & t_3 \end{pmatrix},$$

These permutation matrices $w_1,\ w_2$ of course generate the Weyl group D_{12} (the dihedral group of order 12) of G_2. The action of the Weyl group on the roots is given by

$$(w\alpha)(\tau) := \alpha(w^{-1}\tau).$$

Let a_α, for α simple, denote the Casselman–Shalika element of the maximal split torus $T'(F)$, where F denotes a fixed non–archimedian field with uniformizer π [CS, p. 217, p. 219]. Using the Chevalley basis $\{h_\alpha\}$ of section one, by definition we obtain

$$a_{\alpha_1} = \begin{pmatrix} \pi^{-1} & & \\ & \pi^2 & \\ & & \pi^{-1} \end{pmatrix},$$
$$a_{\alpha_2} = \begin{pmatrix} \pi & & \\ & \pi^{-1} & \\ & & 1 \end{pmatrix},$$
$$a_{\alpha_1+\alpha_2} = \begin{pmatrix} \pi^2 & & \\ & \pi^{-1} & \\ & & \pi^{-1} \end{pmatrix},$$
$$a_{2\alpha_1+\alpha_2} = \begin{pmatrix} \pi & & \\ & \pi & \\ & & \pi^{-2} \end{pmatrix},$$
$$a_{3\alpha_1+\alpha_2} = \begin{pmatrix} 1 & & \\ & \pi & \\ & & \pi^{-1} \end{pmatrix},$$
$$a_{3\alpha_1+2\alpha_2} = \begin{pmatrix} \pi & & \\ & 1 & \\ & & \pi^{-1} \end{pmatrix}.$$

Let χ denote a character on $T'(F)$ given by

$$\chi(\tau) = \chi\Big(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}\Big) = z_1^{\nu(t_1)} z_2^{\nu(t_2)} z_3^{\nu(t_3)}, \tag{2.3}$$

where $z_i \in \mathbb{C}^\times$ are fixed and $\nu : F^\times \to \mathbb{Z}$ is the normalized valuation, so $\nu(\pi) = 1$. The Weyl group acts on χ by

$$(w\chi)(\tau) := \chi(w^{-1}\tau).$$

It will be convenient to write an arbitrary element of $T'(F)$ in the form

$$\tau = \begin{pmatrix} \pi^{k_1+k_2} & & \\ & \pi^{k_1} & \\ & & \pi^{-2k_1-k_2} \end{pmatrix},$$

so that, for example, if $w = 1 \in D_{12}$ then

$$(w\chi)(\tau) = z_1^{k_1+k_2} z_2^{k_1} z_3^{-2k_1-k_2}.$$

To apply the Casselman–Shalika formula, let us now tabulate, for each $w \in D_{12}$, the neccessary data.

If $w = w_1$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{2k_1+k_2} & & \\ & \pi^{-k_1} & \\ & & \pi^{-k_1-k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{2k_1+k_2} z_2^{-k_1} z_3^{-k_1-k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_1\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1 z_2^{-2} z_3.$$

If $w = w_2$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{k_1} & & \\ & \pi^{k_1+k_2} & \\ & & \pi^{-2k_1-k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{k_1} z_2^{k_1+k_2} z_3^{-2k_1-k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-1} z_2.$$

If $w = w_1 w_2$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{-k_1} & & \\ & \pi^{2k_1+k_2} & \\ & & \pi^{-k_1-k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{-k_1} z_2^{2k_1+k_2} z_3^{-k_1-k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_2, \alpha_1 + \alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-3} z_2^2 z_3.$$

If $w = w_2 w_1$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{2k_1+k_2} & & \\ & \pi^{-k_1-k_2} & \\ & & \pi^{-k_1} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{2k_1+k_2} z_2^{-k_1-k_2} z_3^{-k_1}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_1, 3\alpha_1 + \alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1 z_2^{-3} z_3^2.$$

If $w = w_1 w_2 w_1$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{k_1+k_2} & & \\ & \pi^{-2k_1-k_2} & \\ & & \pi^{k_1} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{k_1+k_2} z_2^{-2k_1-k_2} z_3^{-k_1}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_2^{-4} z_3^4.$$

If $w = w_2 w_1 w_2$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{-k_1-k_2} & & \\ & \pi^{2k_1+k_2} & \\ & & \pi^{-k_1} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{-k_1-k_2} z_2^{2k_1+k_2} z_3^{-k_1}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-5} z_3^5.$$

If $w = w_1 w_2 w_1 w_2$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{-2k_1-k_2} & & \\ & \pi^{k_1+k_2} & \\ & & \pi^{k_1} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{-2k_1-k_2} z_2^{k_1+k_2} z_3^{k_1}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-5} z_2 z_3^4.$$

If $w = w_2 w_1 w_2 w_1$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{k_1} & & \\ & \pi^{-2k_1-k_2} & \\ & & \pi^{k_1+k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{k_1} z_2^{-2k_1-k_2} z_3^{k_1+k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_1, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-1} z_2^{-4} z_3^{5}.$$

If $w = w_1w_2w_1w_2w_1$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{-k_1} & & \\ & \pi^{-k_1-k_2} & \\ & & \pi^{2k_1+k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{-k_1} z_2^{-k_1-k_2} z_3^{2k_1+k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-3} z_2^{-3} z_3^{6}.$$

If $w = w_2w_1w_2w_1w_2$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{-2k_1-k_2} & & \\ & \pi^{k_1} & \\ & & \pi^{k_1+k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{-2k_1-k_2} z_2^{k_1} z_3^{k_1+k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-5} z_3^{5}.$$

If $w = w_1w_2w_1w_2w_1w_2$ then

$$w^{-1}\tau = \begin{pmatrix} \pi^{-k_1-k_2} & & \\ & \pi^{-k_1} & \\ & & \pi^{2k_1+k_2} \end{pmatrix},$$

so $(w\chi)(\tau) = z_1^{-k_1-k_2} z_2^{-k_1} z_3^{2k_1+k_2}$, and $\{\alpha > 0 \mid w\alpha < 0\} = \{$ all roots $\}$, so

$$\prod_{w\alpha<0} \chi(a_\alpha)^{-1} = z_1^{-4} z_2^{-2} z_3^{6}.$$

Motivated by this data and the Casselman–Shalika formula, we are led to define

$$\begin{aligned} A'(k_1,k_2) := {} & z_1^{k_1+k_2} z_2^{k_1} z_3^{-2k_1-k_2} - z_1^{2k_1+k_2+1} z_2^{-k_1-2} z_3^{-k_1-k_2+1} \\ & + z_1^{-k_1-3} z_2^{2k_1+k_2+2} z_3^{-k_1-k_2+1} - z_1^{k_1+k_2} z_2^{-2k_1-k_2-4} z_3^{k_1+4} \\ & + z_1^{-2k_1-k_2-5} z_2^{k_1+k_2+1} z_3^{k_1+4} - z_1^{-k_1-3} z_2^{-k_1-k_2-3} z_3^{2k_1+k_2+6} \\ & + z_1^{-k_1-k_2-4} z_2^{-k_1-2} z_3^{2k_1+k_2+6} - z_1^{-2k_1-k_2-5} z_2^{k_1} z_3^{k_1+k_2+5} \\ & + z_1^{k_1-1} z_2^{-2k_1-k_2-4} z_3^{k_1+k_2+5} - z_1^{-k_1-k_2-4} z_2^{2k_1+k_2+2} z_3^{-k_1+2} \\ & + z_1^{2k_1+k_2+1} z_2^{-k_1-k_2-3} z_3^{-k_1+2} - z_1^{k_1-1} z_2^{k_1+k_2+1} z_3^{-2k_1-k_2}. \end{aligned} \tag{2.4}$$

Let

$$S(k_1,k_2) := \frac{A'(k_1,k_2)}{A'(0,0)}, \tag{2.5}$$

let ψ denote a fixed non–trivial character of the unipotent radical $U(F)$ of a minimal parabolic $B(F)$ of $G_2(F)$, and let $W_\chi(\tau)$ denote the G_2-Whittaker function [CS, §5]. The **Casselman–Shalika formula** says that

$$W_\chi(\tau) = \delta(\tau)^{1/2} S(k_1,k_2), \tag{2.6}$$

where $\tau \in T'(F)$, and δ is the modulus function. It can be checked, using the data given above, that if

$$A(k_1,k_2) := z_1^2 z_2 z_3^{-3} A'(k_1,k_2)$$

then

$$A(k_1,k_2) = \mathcal{A}(\chi)(\tau), \tag{2.7}$$

where

$$\mathcal{A}(\chi)(\tau) := \sum_{w\in D_{12}} (\text{sgn } w) w\chi(\tau).$$

This should be compared with the expressions occurring in the Weyl character formula. Indeed, if λ_1 and λ_2 are the fundamental weights of G_2 then the **Weyl character formula** states that

$$\text{tr } r_\lambda(\tau) = \frac{\mathcal{A}((\delta+\lambda)(\tau))}{\mathcal{A}(\delta(\tau))}, \tag{2.8}$$

where r_λ denotes the finite–dimensional representation of G_2 with highest weight $\lambda = k_1\lambda_1 + k_2\lambda_2,\ k_i \geq 0$.

§3. **The calculation**

We may embed a copy of $SL(3,\mathbb{C})$ into ${}^L{G_2}^o(\mathbb{C}) \cong G_2(\mathbb{C})$ by identifying one of the simple positive roots, say β_1, of $SL(3)$ with the short root α_1 of G_2 and the other simple positive root, β_2, of SL(3) with $\alpha_1+\alpha_2$. The Weyl group D_{12} acts on the embedded maximal split torus

$$T' \subset SL(3,\mathbb{C})) \hookrightarrow {}^L{G_2}^o(\mathbb{C})$$

consistently with the embedding $S_3 \hookrightarrow D_{12}$. Let

$$\mathcal{A} := \sum_{w\in D_{12}} (\text{sgn } w) w : \mathbb{C}[t_1,t_2,t_3,t_1^{-1},t_2^{-1},t_3^{-1}] \to \mathbb{C}[t_1,t_2,t_3,t_1^{-1},t_2^{-1},t_3^{-1}], \tag{3.1}$$

denote the **anti–symmetrizer**. It acts via identifying τ with the ordered triple (t_1,t_2,t_3); for example, if $w : \text{diag}(t_1,t_2,t_3) \longmapsto \text{diag}(t_3^{-1},t_2^{-1},t_1^{-1})$ then $w : t_1^a t_2^b t_3^c \longmapsto t_1^{-c} t_2^{-b} t_3^{-a}$.

The fundamental weights of G_2 are

$$\lambda_1 := 2\alpha_1 + \alpha_2, \quad \lambda_2 := 3\alpha_1 + 2\alpha_2,$$

and half the sum of the positive roots of G_2 is

$$\rho := \frac{1}{2}\sum_{\alpha>0} \alpha = 5\alpha_1 + 3\alpha_2 = 2\beta_1 + 3\beta_2. \tag{3.2}$$

On the other hand, half the sum of the positive roots of SL(3) is, with the above embedding into G_2, λ_1, i.e.,

$$\lambda_1 = 2\alpha_1 + \alpha_2 = \beta_1 + \beta_2 =: \rho_{SL(3)}.$$

Therefore,

$$\begin{aligned} \rho(\tau) &= \beta_1(\tau)^2\beta_2(\tau)^3 = t_1^2 t_2 t_3^{-3}, \\ \lambda_1(\tau) &= t_1 t_3^{-1}. \end{aligned}$$

Let $r_{\lambda_2} = r_{\text{Cayley}}$ denote the irreducible 7-dimensional representation of G_2, so

$$r_{\lambda_2} : \tau \longmapsto \begin{pmatrix} t_1t_2^{-1} & & & & & & \\ & t_1t_3^{-1} & & & & & \\ & & t_2t_3^{-1} & & & & \\ & & & 1 & & & \\ & & & & t_1^{-1}t_2 & & \\ & & & & & t_1^{-1}t_3 & \\ & & & & & & t_2^{-1}t_3 \end{pmatrix}. \tag{3.3}$$

Observe that, as a function of τ, $L_x(\tau) := \det(1 - r_{\lambda_2}(\tau)x) \in \mathbb{C}[t_1, \ldots, t_3^{-1}]$ is invariant under the action of the Weyl group D_{12}. Therefore,

$$\mathcal{A}(L_x(\tau)t_1^a t_2^b t_3^c) = L_x(\tau)\mathcal{A}(t_1^a t_2^b t_3^c), \tag{3.4}$$

and this identity extends, by linearity, to all of $\mathbb{C}[t_1, \ldots, t_3^{-1}]$. Consequently, we have

$$\begin{aligned} &(1 - t_1t_2^{-1}x)(1 - t_2t_3^{-1}x)(1 - t_3t_1^{-1}x)(1 - t_2t_1^{-1}x)\times \\ &\qquad \times (1 - t_3t_2^{-1}x)(1 - t_1t_3^{-1}x)S(t_1^2t_2t_3^{-3}(1 - t_1t_3^{-1}x)^{-1}) \\ &\quad = S(t_1^2t_2t_3^{-3}(1 - t_1t_2^{-1}x)(1 - t_2t_3^{-1}x)(1 - t_3t_1^{-1}x)(1 - t_2t_1^{-1}x)(1 - t_3t_2^{-1}x)). \end{aligned} \tag{3.5}$$

Furthermore, it is easy to see that

$$S(t_1^2t_2t_3^{-3}(1 - t_1t_3^{-1}x)^{-1}) = \sum_{k=0}^{\infty} S(0,k)x^k, \tag{3.6}$$

Using (3.5-3.6), we want to express the local Langlands L-function on G_2 associated to r_{λ_2} at an unramified non–archimedean place,

$$
\begin{aligned}
L(s,\pi,r_{\lambda_2}) &:= \det(1-r_{\lambda_2}(\tau)x)^{-1} \\
&= (1-x)^{-1}(1-t_1t_2^{-1}x)^{-1}(1-t_2t_3^{-1}x)^{-1}(1-t_3t_1^{-1}x)^{-1}\times \\
&\quad \times (1-t_2t_1^{-1}x)^{-1}(1-t_3t_2^{-1}x)^{-1}(1-t_1t_3^{-1}x)^{-1}
\end{aligned}
\tag{3.7}
$$

as an integral of the associated Whittaker function, where $x = Nv^{-s}$ and $r_{\lambda_2}(\tau) \in {}^LG_2(\mathbb{C})^o$ determines the conjugacy class associated to the unramified representation π of $G_2(F)$ via the L–group correspondence. To this end, we will prove

Lemma 3.8.

$$
\frac{\mathcal{A}(t_1^2t_2t_3^{-3}(1-t_1t_2^{-1}x)(1-t_2t_3^{-1}x)(1-t_3t_1^{-1}x)(1-t_2t_1^{-1}x)(1-t_3t_2^{-1}x))}{\mathcal{A}(t_1^2t_2t_3^{-3})} = 1+x.
$$

proof. We must calculate the effect of $\mathcal{A}$ on

$$
t_1^2t_2t_3^{-3}(1-t_1t_2^{-1}x)(1-t_2t_3^{-1}x)(1-t_3t_1^{-1}x)(1-t_2t_1^{-1}x)(1-t_3t_2^{-1}x),
$$

which we write, for convenience, as

$$
t_1^2t_2t_3^{-3}(1-t_1t_2^{-1}u)(1-t_2t_3^{-1}u)(1-t_3t_1^{-1}u)(1-t_2t_1^{-1}v)(1-t_3t_2^{-1}v),
$$

with u = v = x. Representing $t_1^at_2^bt_3^c$ by (a,b,c), the following table describes the terms in the expansion of this expression (note that we must always have $a+b+c=0$):

	1	$-v$	$-v$	v^2
1	$\boxed{(2,1,-3)}$	(2,2,−4)	(2,0,−2)	(2,1,−3)
$-u$	(3,0,−3)	(3,1,−4)	(3,1,−2)	(3,0,−3)
$-u$	$\boxed{(1,1,-2)}$	(1,2,−3)	(1,0,−1)	(1,1,−2)
$-u$	(1,2,−3)	(1,3,−4)	(1,1,−2)	(1,2,−3)
u^2	(2,0,−2)	(2,1,−3)	(2,−1,−1)	(2,0,−2)
u^2	(2,1,−3)	(2,2,−4)	(2,0,−2)	(2,1,−3)
u^2	(0,2,−2)	(0,3,−3)	(0,1,−1)	(0,2,−2)
$-u^3$	(1,1,−2)	(1,2,−3)	(1,0,−1)	(1,1,−2).

After applying $\mathcal{A}$ to the resulting expansion, one can verify that only the boxed terms survive, from which the claim follows.□

From this claim and the definition of r_{λ_2}, we obtain

$$
\det(1-r_{\lambda_2}(\tau)x)^{-1} = (1-u^2)^{-1}\sum_{k=0}^{\infty} S(0,k)x^k. \tag{3.9}
$$

Let

$$d(y) := \begin{pmatrix} y & & & & & & & \\ & y & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & y^{-1} & \\ & & & & & & & y^{-1}, \end{pmatrix}$$

and let $\delta = \delta_{G_2}$ denote the modulus function for G_2. The above computation along with the Casselman–Shalika formula shows that

Lemma 3.10. *If π is an unramified principal series representation then*

$$L(s, \pi, r_{\lambda_2}) = \zeta_v(2s) \int_{F^\times} W_\chi(d(y))|y|^s \delta(d(y))^{-1/2} \, d^\times y,$$

where ζ_v is the local factor of the Dedekind zeta function at the place v.

Let K denote a number field with adele ring $\mathbb{A}_K = \mathbb{A}$ and let ϕ denote a generic cusp form of $G_2(\mathbb{A})$. The above lemma implies

$$L(s, \pi_\phi, r_{\lambda_2}) \sim \zeta(2s) \int_{\mathbb{A}^\times} W_\phi(d(y))|y|^s \delta(d(y))^{-1/2} \, d^\times y, \tag{3.11}$$

where $\sim$ signifies that both sides are equal up to a finite number of factors, where π_ϕ denotes the global representation associated to ϕ, and where

(3.12)

$$W_\phi(g) := \int\int\int\int\int\int_{(\mathbb{A}/K)^6} \phi(u_1(x_1)u_2(x_2)u_3(x_3)u_4(x_4)u_5(x_5)u_6(x_6)g) \times \\ \times \psi(-x_1 - x_2) \, dx_1 dx_2 dx_3 dx_4 dx_5 dx_6.$$

Here $u_1, \ldots, u_6$ are the local coordinates in the one–dimensional unipotent groups associated to the positive roots:

$$U_{\alpha_1} = u_1(x), \; U_{\alpha_2} = u_2(x), \; U_{\alpha_1+\alpha_2} = u_3(x), \\ U_{2\alpha_1+\alpha_2} = u_4(x), \; U_{3\alpha_1+\alpha_2} = u_5(x), \; U_{3\alpha_1+2\alpha_2} = u_6(x).$$

Theorem 3.13. *The Langlands L–function can be expressed as the Mellin transform of its associated cusp form. More precisely,*

$$L(s, \pi_\phi, r_{\lambda_2}) \sim \xi(2s) \int_{\mathbb{A}^\times/K^\times} \int\int\int\int\int_{(\mathbb{A}/K)^5} \phi(u_1(x_1)u_3(x_3)u_4(x_4)u_5(x_5)u_6(x_6)d(y)) \times \\ \times \psi(-x_1)|y|^s \delta(d(y))^{-1/2} \, dx_1 dx_3 dx_4 dx_5 dx_6 \, d^\times y.$$

This follows by using commutation relations and then applying the Poisson sumation formula. The factor $\xi(2s)$ which occurs here suggests that this integral may be transformed into a Rankin-Selberg convolution, but thus far we've been unable to find such a representation.

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Added 4-2004: See the global integral of David Ginzburg described in the paper "On the standard L-function for G_2," Duke J. Math. 69(1993)315-333. Ginzburg's global integral "unfolds" to our local integral.