A note on gaps between zeroes of L-functions

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1984-85 (revised 1990, 1995)

Assuming the Riemann hypothesis, H. Montgomery [M] stated his well-known pair-correlation conjecture

$$\sum_{\substack{0<\gamma,\gamma'<T \\ \frac{2\pi a}{\log T}\leq\gamma-\gamma'\leq\frac{2\pi b}{\log T}}} 1 \sim \left(\int_a^b (1-(\frac{\sin(\pi x)}{\pi x})^2)dx + \delta(a,b)\right) N(T), \tag{0.1}$$

where

$$\delta(a,b) := \begin{cases} 1, & 0\in[a,b], \\ 0, & \text{otherwise}, \end{cases}$$

and $N(T)$ counts the zeros (according to multiplicity) $\beta + i\gamma$ with $0 \leq \gamma \leq T$ of the Riemann zeta function $\zeta(s)$. This conjecture is of considerable interest in prime number theory, as has been shown (for example) by D. Goldston [G], P. Gallagher and J. Mueller [GM], and R. Heath-Brown [H-B].

If $L(s)$ is an L-function satisfying axioms I-VI below then one might expect that the following analog of (0.1) holds:

$$\sum_{\substack{0<\gamma,\gamma'<T \\ \frac{2\pi a}{a_0(L)\log T}\leq\gamma-\gamma'\leq\frac{2\pi b}{a_0(L)\log T}}} 1 \sim \left(\int_a^b (1-(\frac{\sin(\pi x)}{\pi x})^2)dx + \delta(a,b)\right) N_L(T), \tag{0.2}$$

where $N_L(T)$ is the counting function for the zeros of $L(s)$. It is conceivable that such a statement might eventually be of some relavance to the study of Sato-Tate-type distribution questions of semi-simple conjugacy classes in L-groups associated to the unramified components of an automorphic representation. The main result of this note implies that, for the L-functions satisfying axioms I-VI (e.g., all Dirichlet L-functions), we have a lower bound

$$\lim_{\substack{a\to 0+\\ b\to\infty}} \sum_{\substack{0<\gamma,\gamma'<T\\ \frac{2\pi a}{a_0(L)\log T}\le\gamma-\gamma'\le\frac{2\pi b}{a_0(L)\log T}}} 1 \succeq \left(\int_0^\infty (1-(\frac{\sin\pi x}{\pi x})^2)dx+\delta(a,b)\right) N_L(T), \quad (0.3)$$

where $f(T) \succeq g(T)$ means $\lim_{T\to\infty} f(T)/g(T) \geq 1$ and the other unexplained notation is defined below. We do not assume a generalized Riemann hypothesis for this.

Let us index the zeros $\beta+i\gamma$ as $\beta_n+i\gamma_n$, ordered lexicographically and counted according to multiplicity. The asymptotic inequality (0.3) is a consequence of our main result which states that, for L-functions satisfying axioms I-VI below and for sufficiently large but fixed $r > r_0(L)$,

$$\frac{1}{N_L(T)}\#\{0<\gamma_n<T \mid \frac{\gamma_{n+r}-\gamma_n}{r}\frac{a_0(L)\log\gamma_n}{2\pi} > 1+c_1\frac{\sqrt{\log r}}{r}\} > e^{-c_2r^2}, \quad (0.4)$$

and

$$\frac{1}{N_L(T)}\#\{0<\gamma_n<T \mid \frac{\gamma_{n+r}-\gamma_n}{r}\frac{a_0(L)\log\gamma_n}{2\pi} < 1-c_1\frac{\sqrt{\log r}}{r}\} > e^{-c_2r^2}, \quad (0.5)$$

where c_i are constants depending only on L and $a_0(L)$ is defined below. Again, we do not assume a generalized Riemann hypothesis for this. For the Riemann zeta function, A. Fujii [F] stated such a result with $\frac{\sqrt{\log r}}{r}$ replaced by e^{-cr^2} and $e^{-c_2r^2}$ replaced by some (undetermined) $\delta_r > 0$. (However, his result holds for all $r \geq 1$.) Again for the Riemann zeta function, A. Selberg proved a slightly stronger result assuming the Riemann hypothesis (unpublished but see [J], Theorem 4.5, chapter 3).

Now let us state the axioms on which our results depend.

(I) The Dirichlet series $L(s)$ converges absolutely in some half-plane $Re(s) > 1$ and may be meromorphically continued to the entire plane, except possibly for a simple pole at $s = 1$.

(II) Aside from "trivial" zeros on the negative real axis, all zeros of $L(s)$ lie in the "critical strip" $0 \leq Re(s) \leq 1$, and

$$N_{tr}(T) := \#\{\beta+i\gamma \mid L(\beta+i\gamma)=0,\ -T<\beta<0\} \sim A_L\cdot T, \quad (0.6)$$

and

$$
\begin{aligned}
N_L(T) \quad &:= \#\{\rho = \beta + i\gamma \mid L(\beta + i\gamma) = 0,\ 0 < \gamma < T\} \\
&\sim \tfrac{a_0(L)}{2\pi} \cdot T \log \tfrac{T}{2\pi} + \tfrac{a_1(L)}{2\pi} \cdot T + a_2(L) + S_L(T) + O_L(1/T),
\end{aligned}
\tag{0.7}
$$

as $T \to \infty$. Here

$$S_L(T) := \frac{1}{\pi} arg L(1/2 + it),$$

where arg is taken in the same way as the Riemann zeta function (as the variation starting from $+\infty$, where the argument is zero, ...).

(III) There is an Euler product

$$L(s) = \prod_p L_p(s), \quad Re(s) > 1,$$

where $L_p(s)$ is a rational function of p^{-s}, such that

$$\frac{L'}{L}(s) = -\sum_{n \geq 2} a_n \Lambda(n) n^{-s}, \quad Re(s) > 1,$$

where $a_n \in \mathbb{C}$ are constants satisfying

$$|a_p| < C, \text{ for all primes } p,$$

for some $C = C(L)$, and $\Lambda(n)$ is the von Mangoldt function.

(IV) $L(s)$ satisfies the growth conditions:

$$\frac{L'}{L}(s) = \sum_{|s-\rho| \leq 2} \frac{1}{s - \rho} + O_L(\log T), \quad -1 < Re(s) < 2; \tag{0.8}$$

for some $\sigma_n \to \infty$,

$$\frac{L'}{L}(-\sigma_n + it) << _L \sigma_n \log T, \quad |t| < T; \tag{0.9}$$

for all $0 \leq \sigma \leq 1$,

$$L(\sigma + it) << _L T^{c(L)}, \quad |t| < T, \tag{0.10}$$

for some constant $c(L) > 0$ with $T > 1$.

(V) The Rankin convolution of L with itself, $\tilde{L}(s)$, satisfies axioms (I-IV), where

$$\tilde{L}(s) := \sum_n |c_n|^2 n^{-s},$$

if

$$L(s) = \sum_n c_n n^{-s}, \quad Re(s) > 1.$$

(VI) Let $T^{1/2+\epsilon} \leq H \leq T$, $x := T^{\epsilon/100k}$, and $0 < h = h(T) < 1$. For any positive integer $k \leq (\log\log T)^{1/10}$ we have

$$\begin{aligned} \tfrac{1}{H}\textstyle\int_T^{T+H}(S_L(t+h)-S_L(t))^{2k}dt &= \tfrac{(2k)!}{(2\pi)^{2k}k!}(r(\tilde{L})\log(3+h\log T))^k \\ &+O_L((Ak)^{2k}(\log(3+h\log T))^{k-1/2}), \end{aligned}$$

where $r(\tilde{L})$ denotes the residue of $\tilde{L}(s)$ at $s=1$.

1. Preliminary lemma

Our result relies on a distribution theorem for $S_L(t+h) - S_L(t)$, $h = c/\log t$, similar to the one given by A. Ghosh [Gh].

Lemma 1.1. *Assume $L(s)$ satisfies axioms I-VI above. Let $0 < h = h(T)$ be bounded as $T \to \infty$ and suppose $h \log T > C_0(L)$ is sufficiently large. Let*

$$M_T = \log\log(h\log T)/\log\log\log(h\log T),$$

and let $b(r) := \pi^{-r-1}\Gamma(\frac{r+1}{2})$. Then for any real number r, $1 \leq r \leq (\log M_T)^{1/4}$, we have

$$\frac{1}{T}\int_0^T \left|\frac{S(t+h)-S(t)}{\sqrt{r(\tilde{L})\log(h\log T)}}\right|^r dt = b(r)(1+O_L((\log M_T)^{-1/2})),$$

as $T \to \infty$.

proof For $h << 1$, we let

$$B_h(t) := |S_L(t+h) - S_L(t)|/\sqrt{r(\tilde{L})\log(h\log T)},$$

and recall that $2^{-r}\Gamma(r+1)/\Gamma(r/2+1) = \pi^{-1/2}\Gamma(r/2+1/2)$. Thus, by axiom VI,

$$\frac{1}{T}\int_0^T B_h(t)^{2k}dt = b(2k) + O((A_L k)^{4k}(\log(3+h\log T))^{-1/2}),$$

for integers k, such that $1 \leq k \leq M_T$.

Choose N be an integer with $N < (\log M_T)^{3/4}$ and write, by the Taylor expansion,

$$(\sin u)^4 = \frac{1}{8}\sum_{j=2}^{N} b_j u^{2j} + O(\frac{(4u)^{2N+2}}{(2N+2)!}),$$

where $b_j := \frac{(-4)^{j+1}}{(2j)!}(4^{j-1}-1)$. As in A. Ghosh [Gh], §2, we obtain for $r \leq \sqrt{\log M_T}$,

$$\begin{aligned}
&\tfrac{1}{T}\textstyle\int_0^T B_h(t)^r dt - b(r)\\
&<<_L b(2m)X^{-1-\theta} + (A_L M)^{4M}(\log(3+h\log T))^{-1/2}X^{-1-\theta}e^{X^2}\\
&+\tfrac{16^N X^{2N+1-\theta}}{N(2N+2)!}[b(2M+2) + (A_L M)^{4M}(\log(3+h\log T))^{-1/2}],
\end{aligned}$$

where $r = 2m+1+\theta$, $0 < \theta \leq 2$, $M = m+N$, and N is as above. Taking $r \leq (\log M_T)^{3/4}$, $N = (\log M_T)^{3/4}$, and $X = (\log M_T)^{1/4}$, we obtain the estimate desired. □

Theorem 1.2. *Let*

$$\mu_T(x, B_h) := \frac{1}{T} meas\{0 \leq t \leq T \mid B_h(t) \leq x\}$$

and let

$$\mu(x) := \frac{2}{\sqrt{\pi}}\int_0^{\pi x} e^{-t^2}dt.$$

If $0 << h << 1$ and if $h\log T$ is sufficiently large then

$$|\mu_T(x, B_h) - \mu(x)| << (\log\log M_T)^{1/2}.$$

In other words, for each $c > 0$, we have

$$\begin{aligned}
&\mathrm{meas}\{0 \leq t \leq T \mid |S(t+h) - S(t)| < \tfrac{c}{\pi}\sqrt{r(\tilde{L})\log(3+h\log T)}\}\\
&= T\left(\tfrac{2}{\sqrt{\pi}}\textstyle\int_0^{\pi x} e^{-t^2}dt + O(\tfrac{1}{\sqrt{\log M_T}})\right).
\end{aligned}$$

proof Let N denote the integral part of $M_T^{1/8}$ and consider

$$\begin{aligned}\Phi_T(y) &:= \tfrac{1}{T}\int_0^T e^{iyB_h(t)}dt \\ &= \sum_{j=0}^{N-1}\tfrac{(iy)^j}{j!}\tfrac{1}{T}\int_0^T B_h(t)^j dt + O(\tfrac{|y|^N}{N!}\tfrac{1}{T}\int_0^T B_h(t)^N dt) \\ &= \sum_{j=0}^{N-1}\tfrac{(iy)^j}{j!}b(j) + O(\tfrac{|y|^N}{N!}b(N)) + O(M_T^{-1/4}\sum_1^{N-1}\tfrac{|y|^j}{j!}b(j)).\end{aligned}$$

Since

$$b(j) = \pi^{-j-1/2}\Gamma(\frac{j+1}{2}) = 2\pi^{-j-1/2}\int_0^\infty t^j e^{-t^2}dt,$$

this is (as in [Gh], p. 101)

$$\begin{aligned}&\tfrac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\sum_{j=0}^{N-1}\tfrac{(iyt/\pi)^j}{j!}dt + O(M_T^{-1/4}\sum_1^{N-1}\tfrac{|y|^j}{j!}b(j)) + O(\tfrac{|y|^N}{N!}b(N)) \\ &\tfrac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}e^{iyt/\pi}dt + O(M_T^{-1/4}\sum_1^{N-1}\tfrac{|y|^j}{j!}b(j)) + O(\tfrac{|y|^N}{N!}b(N)),\end{aligned}$$

so

$$\Phi_T(y) - \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}e^{iyt/\pi}dt << M_T^{-1/4}\sum_1^{N-1}\frac{|y|^j}{j!}b(j) + \frac{|y|^N}{N!}b(N).$$

As in [Gh], pp. 100-102, the Berry-Essen theorem implies

$$\mu_T(x, B_h) - \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}e^{iyt/\pi}dt << (\log M_T)^{-1/2},$$

as desired. □

Now let us prove (0.4) and (0.5). Let $h = \frac{2\pi C}{T}$, with C sufficiently large. By the above lemma,

$$\text{meas}\{0 \le t \le T \mid |S(t+h) - S(t)| > \frac{1}{20}\sqrt{r(\tilde{L})\log C}\} > \frac{1}{30}e^{-C^2}T,$$

for $T > T_0(C, L)$. This implies that for a proportion of at least $\frac{1}{5}e^{-C^2}$ of the $t \in (0, T)$, we have

$$|N_L(t + \frac{2\pi C}{a_o(L)\log T}) - N_L(T) - C| > \frac{1}{20}\sqrt{r(\tilde{L})\log C},$$

for C sufficiently large. Since the average of $N_L(t + \frac{2\pi C}{a_o(L)\log T}) - N_L(T)$ is C, we may suppose without loss of generality that our positive proportion in fact satisfies

$$N_L(t + \frac{2\pi C}{a_o(L)\log T}) - N_L(T) > C + \frac{1}{20}\sqrt{r(\tilde{L})\log C}.$$

Let $C = r - \frac{1}{35}\sqrt{\log r}$, so that

$$N_L(t + \frac{2\pi r - \frac{1}{35}\sqrt{\log r}}{a_o(L)\log T}) - N_L(T) > r,$$

for the above proportion of $t \in (0, T)$. This implies (0.4). This actually implies (0.5) since the average gap $\gamma_{n+r} - \gamma_n$ is $\frac{2\pi r}{a_0(L)\log \gamma_n}$. $\square$

Theorem 1.3. *Fix ϵ, $0 < \epsilon < 1/10$ arbitarily small. For $C > C_0(\epsilon, L)$, $r_0(L) < r < (1-3\epsilon)C$ fixed, and $T > T_0(\epsilon, C, L)$, we have*

$$\frac{1}{N_L(T)}\#\{0 < \gamma_n < T \mid \gamma_{n+r} - \gamma_n < \frac{2\pi C}{a_0(L)\log \gamma_n}\} > 1 - O(\frac{1}{\sqrt{\log\log\log\log C}}).$$

proof We have

$$\gamma_{n+r} - \gamma_n < \frac{2\pi C}{a_0(L)\log \gamma_n}$$

for a positive proportion δ_r of $\gamma_n's$ if and only if

$$N_L(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}) - N_L(\gamma_n) > r$$

for the same positive proportion δ_r of $\gamma_n's$. By axiom (II), we have

$$N_L(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}) - N_L(\gamma_n) - C = S_L(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}) - S_L(\gamma_n) + O_L(\frac{C}{\log \gamma_n}),$$

so to prove the theorem we need only show that

$$S_L(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}) - S_L(\gamma_n) > r - C + O_L(\frac{C}{\log \gamma_n})$$

occurs for a proportion of at least $1 - O(\frac{1}{\sqrt{\log\log\log\log C}})$ of the $\gamma_n's$. This follows easily from Theorem 1.2. $\square$

Corollary 1.4. *As $C \to \infty$, we have*

$$\lim_{T\to\infty} \frac{1}{N_L(T)}\#\{0 < \gamma, \gamma' < T \mid 0 < \gamma - \gamma' < \frac{2\pi C}{a_0(L)\log T}\} \succeq C.$$

proof By Theorem 1.3,

$$\begin{aligned}
&\tfrac{1}{N_L(T)}\#\{0<\gamma,\gamma'<T \mid 0<\gamma-\gamma'<\tfrac{2\pi C}{a_0(L)\log T}\}\\
&\geq \textstyle\sum_{r=r_0(L)}^{(1-\epsilon)C} \tfrac{1}{N_L(T)}\#\{0<\gamma_n<T \mid 0<\gamma_{n+r}-\gamma_n<\tfrac{2\pi C}{a_0(L)\log T}\}\\
&\geq (1-\epsilon-\tfrac{r_0(L)}{C})C(1-O(\tfrac{1}{\sqrt{\log\log\log\log C}}))\\
&\geq (1-2\epsilon)C,
\end{aligned}$$

for sufficiently large C. $\square$

2. Appendix: Errata to [J]

There is a gap in [J], Ch. 4, as was brought to my attention by D. Hejhal. To prove the results in [J], pp. 117-127, in addition to those axioms listed above, one must assume the following:
Missing Axiom: We have, for $H > T^{1/2+\epsilon}$, and $\sigma \geq 1/2$,

$$N_L(\sigma, T+H) - N_L(\sigma, T) <<_{L,\epsilon} \left(\frac{\sqrt{H}}{T}\right)^{\frac{1-2\sigma}{4}} H \log H, \tag{2.1}$$

where

$$N_L(\sigma, T) := \#\{\rho = \beta + i\gamma \mid L(\beta+i\gamma) = 0,\ 0<\gamma<T,\ \beta \geq \sigma\}.$$

First, this "missing axiom" is a well-known result of A. Selberg [S1] in the case $L(s) = \zeta(s)$. (It is also seems to be known for Dirichlet L-series, L-functions of quadratic fields and Langlands standard L-functions for $GL(2)$ [S2]. Selberg has other results assuming this axiom and another axiom vaguely related to our axiom (V).))

Second, axiom (VI) above is a well-known result of A. Selberg [S1] in the case $L(s) = \zeta(s)$ and is mentioned in the remark to [J], Proposition 4.2 (on page 158, $S(t+h) - S(t)$ should be $S_L(t+h) - S_L(t)$; in the remark there, $\log k$ should be $\log x$). Thus the results of this paper and [J] are theorems in case $L(s)$ is a Dirichlet L-function. It seems reasonable to expect that they should also hold for Langlands L-functions associated to tempered generic automorphic representations of $GL(n)$.

Lastly, the result (0.4-0.5) was listed as Conjecture 4.4 in [J], p. 84. There is a typo there: on page 84, line 4, $C\log r/\sqrt{r}$ should be $\frac{1}{50}\frac{\sqrt{\log r}}{r}$, and on page 84, line 9, $C\log r/r$ should be $C\sqrt{\log r}/r$.

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