Introduction to group homology and cohomology

David Joyner

11-15-2003

Contents

These are expanded lecture notes of a series of expository talks surveying basic aspects of group cohomology and homology.

First, some words of motivation.

1 Introduction

Let G be a group and A a G-module[1].

Let A^G denote the largest submodule of A on which G acts trivially. Let us begin by asking ourselves the following natural question.

Question: Suppose A is a submodule of a G-module B and x is an arbitrary G-fixed element of B/A. Is there an element b in B, also fixed by G, which maps onto x under the quotient map?

The answer to this question can be formulated in terms of group cohomology. (Yes, if $H^1(G, A) = 0$.) The details, given below, will help motivate the introduction of group cohomology.

Let A_G is the largest quotient module of A on which G acts trivially. Next, we ask ourselves the following analogous question.

Question: Suppose A is a submodule of a G-module B and b is an arbitrary element of B_G which maps to 0 under the natural map $B_G \to (B/A)_G$. Is there an element a in a_G which maps onto b under the inclusion map?

The answer to this question can be formulated in terms of group homology. (Yes, if $H_1(G, A) = 0$.) The details, given below, will help motivate the introduction of group homology.

Group cohomology arises as the right higher derived functor for $A \longmapsto A^G$. The **cohomology groups of G with coefficients in A** are defined by

[1]We call an abelian group A (written additively) which is a left $\mathbb{Z}[G]$-module a **G-module**.

$$H^n(G,A) = Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

(See §4 below for more details.) These groups were first introduced in 1943 by S. Eilenberg and S. MacLane [EM]. The functor $A \longmapsto A^G$ on the category of left G-modules is additive and left exact. This implies that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of G-modules then we have a **long exact sequence of cohomology**

$$\begin{array}{c} 0 \to A^G \to B^G \to C^G \to H^1(G,A) \to \\ H^1(G,B) \to H^1(G,C) \to H^2(G,A) \to \dots \end{array} \tag{1}$$

Similarly, group homology arises as the left higher derived functor for $A \longmapsto A_G$. The **homology groups of G with coefficients in A** are defined by

$$H_n(G,A) = Tor_n^{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

(See §5 below for more details.) The functor $A \longmapsto A_G$ on the category of left G-modules is additive and right exact. This implies that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of G-modules then we have a **long exact sequence of homology**

$$\begin{array}{c} \dots \to H_2(G,C) \to H_1(G,A) \to H_1(G,B) \to \\ H_1(G,C) \to A_G \to B_G \to C_G \to 0. \end{array} \tag{2}$$

Here we will define both cohomology $H^n(G,A)$ and homology $H_n(G,A)$ using projective resolutions and the higher derived functors Ext^n and Tor_n. We "compute" these when G is a finite cyclic group. We also give various functorial properties, such as corestriction, inflation, restriction, and transfer. Since some of these cohomology groups can be computed with the help of computer algebra systems, we also include some discussion of how to use computers to compute them. We include several applications to group theory.

A few questions (whose answers are unknown, as far as the author knows[2]) are scattered thoughout.

One can also define $H^1(G,A)$, $H^2(G,A)$, $\ldots$, by explicitly constructing cocycles and coboundaries. Similarly, one can also define $H_1(G,A)$, $H_2(G,A)$, $\ldots$, by explicitly constructing cycles and boundaries. For the proof that these constructions yield the same groups, see Rotman [R], chapter 10.

For the general outline, we follow §7 in chapter 10 of [R] on homology. For some details, we follow Brown [B], Serre [S] or Weiss [W].

For a recent expository account of this topic, see for example Adem [A]. Another good reference is Brown [B].

2 Differential groups

In this section cohomology and homology are viewed in the same framework. This "differential groups" idea was introduced by Cartan and Eilenberg [CE], chapter IV, and developed in R. Godement [G], chapitre 1, §2. However, we shall follow Weiss [W], chapter 1.

2.1 Definitions

A **differential group** is a pair (L,d), L an abelian group and $d : L \to L$ a homomorphism such that $d^2 = 0$. We call d a **differential operator**. The group

$$H(L) = Ker(d)/Im(d)$$

is the **derived group** of (L,d). If

$$L = \oplus_{n=-\infty}^{\infty} L_n$$

then we call L **graded**. Suppose d (more precisely, $d|_{L_n}$) satisfies, in addition, for some fixed $r \neq 0$,

$$d : L_n \to L_{n+r}, \qquad n \in \mathbb{Z}.$$

We say d is **compatible** with the grading provided $r = \pm 1$. In this case, we call (L,d,r) a **graded differential group**. As we shall see, the case $r = 1$

[2]Which isn't that much:-).

corresponds to cohomology and the the case $r = -1$ corresponds to homology. Indeed, if $r = 1$ then we call (L, d, r) a (differential) **group of cohomology type** and if $r = -1$ then we call (L, d, r) a **group of homology type**. Note that if $L = \oplus_{n=-\infty}^{\infty} L_n$ is a group of cohomology type then $L' = \oplus_{n=-\infty}^{\infty} L'_n$ is a group of homology type, where $L'_n = L_{-n}$, for all $n \in \mathbb{Z}$.

For the impatient: For *cohomology*, we shall eventually take $L = \oplus_n Hom_G(X_n, A)$, where the X_n form a chain complex (with $+1$ grading) determined by a certain type of resolution. The group $H(L)$ is an abbreviation for $\oplus_n Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$. For *homology*, we shall eventually take $L = \oplus_n \mathbb{Z} \otimes_{\mathbb{Z}[G]} X_n$, where the X_n form a chain complex (with -1 grading) determined by a certain type of resolution. The group $H(L)$ is an abbreviation for $\oplus_n Tor_n^{\mathbb{Z}[G]}(\mathbb{Z}, A)$.

Let $(L, d) = (L, d_L)$ and $(M, d) = (M, d_M)$ be differential groups[3]. A homomorphism $f : L \to M$ satisfying $d \circ f = f \circ d$ will be called **admissible**. For any $n \in \mathbb{Z}$, we define $nf : L \to M$ by $(nf)(x) = n \cdot f(x) = f(x) + \dots + f(x)$ (n times). If f is admissible then so is nf, for any $n \in \mathbb{Z}$. An admissible map f gives rise to a map of derived groups: define the map $f_* : H(L) \to H(M)$, by $f_*(x + dL) = f(x) + dM$, for all $x \in L$.

2.2 Properties

Let f be an admissible map as above.

1. The map $f_* : H(L) \to H(M)$ is a homomorphism.
2. If $f : L \to M$ and $g : L \to M$ are admissible, then so is $f + g$ and $(f + g)_* = f_* + g_*$.

[3] Of course, to be more precise, we should use different symbols for the differential operators of L and M. For notational simplicity, we use the same symbol and hope the context removes any ambiguity.

3. If $f : L \to M$ and $g : M \to N$ are admissible then so is $g \circ f : L \to N$ and $(g \circ f)_* = g_* \circ f_*$.

4. If

$$0 \to L \xrightarrow{i} M \xrightarrow{j} N \to 0 \tag{3}$$

is an exact sequence of differential groups with admissible maps i, j then there is a homomorphism $d_* : H(N) \to H(L)$ for which the following triangle is exact:

$$\begin{tikzcd} & H(L) \arrow[dd, "i_*"] \\ H(N) \arrow[ur, "d_*"] & \\ & H(M) \arrow[ul, "j_*"] \end{tikzcd} \tag{4}$$

This diagram[4] encodes both the long exact sequence of cohomology (1) and the long exact sequence of homology (2).

Here is the construction of d_*:

Recall $H(N) = Ker(d)/Im(d)$, so any $x \in H(N)$ is represented by an $n \in N$ with $dn = 0$. Since j is surjective, there is an $m \in M$ such that $j(m) = n$. Since j is admissible and the sequence is exact, $j(dm) = d(j(m)) = dn = 0$, so $dm \in Ker(j) = Im(i)$. Therefore, there is an $\ell \in L$ such that $dm = i(\ell)$. Define $d_*(x)$ to be the class of ℓ in $H(L)$, i.e., $d_*(x) = \ell + dL$.

Here's the verification that d_* is well-defined:

[4]This is a special case of Théorème 2.1.1 in [G].

We must show that if we defined instead $d_*(x) = \ell' + dL$, some $\ell' \in L$, then $\ell' - \ell \in dL$. Pull back the above $n \in N$ with $dn = 0$ to an $m \in M$ such that $j(m) = n$. As above, there is an $\ell \in L$ such that $dm = i(\ell)$. Represent $x \in H(N)$ by an $n' \in N$, so $x = n' + dN$ and $dn' = 0$. Pull back this n' to an $m' \in M$ such that $j(m') = n'$. As above, there is an $\ell' \in L$ such that $dm' = i(\ell')$. We know $n' - n \in dN$, so $n' - n = dn''$, some $n'' \in N$. Let $j(m'') = n''$, some $m'' \in M$, so $j(m' - m - dm'') = n' = n - j(dm'') = n' - n - dj(m'') = n' - n - dn'' = 0$. Since the sequence $L - M - N$ is exact, this implies there is an $\ell_0 \in L$ such that $i(\ell_0) = m' - m - dm''$. But $di(\ell_0) = i(d\ell_0) = dm' - dm = i(\ell') - i(\ell) = i(\ell' - \ell)$, so $\ell' - \ell \in dL$.

5. If $M = L \oplus N$ then $H(M) = H(L) \oplus H(N)$.

   proof: To avoid ambiguity, for the moment, let d_X denote the differential operator on X, where $X \in \{L, M, N\}$. In the notation of (3), j is projection and i is inclusion. Since both are admissible, we know that $d_M|_L = d_L$ and $d_M|_N = d_N$. Note that $H(X) \subset X$, for any differential group X, so $H(M) = H(M) \cap L \oplus H(M) \cap N \subset H(L) \oplus H(N)$. It follows from this that that $d_* = 0$. From the exactness of the triangle (4), it therefore follows that this inclusion is an equality.

   $\square$

6. Let L, L', M, M', N, N' be differential groups. If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{j} & N & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & L' & \xrightarrow{i'} & M' & \xrightarrow{j'} & N' & \longrightarrow & 0 \end{array} \qquad (5)$$

   is a commutative diagram of exact sequences with i, i', j, j', f, g, h all admissible then

$$\begin{array}{ccc} H(L) & \xrightarrow{i_*} & H(M) \\ {\scriptstyle f_*}\Big\downarrow & & {\scriptstyle g_*}\Big\downarrow \\ H(L') & \xrightarrow{i'_*} & H(M') \end{array}$$

commutes,

$$\begin{array}{ccc} H(M) & \xrightarrow{j_*} & H(N) \\ {\scriptstyle g_*}\Big\downarrow & & {\scriptstyle h_*}\Big\downarrow \\ H(M') & \xrightarrow{i'_*} & H(N') \end{array}$$

commutes, and

$$\begin{array}{ccc} H(N) & \xrightarrow{d_*} & H(L) \\ {\scriptstyle h_*}\Big\downarrow & & {\scriptstyle f_*}\Big\downarrow \\ H(N') & \xrightarrow{d_*} & H(L') \end{array}$$

commutes.

This is a case of Theorem 1.1.3 in [W] and of Théorème 2.1.1 in [G]..

The proofs that the first two squares commute are similar, so we only verify one and leave the other to the reader. By assumption, (5) commutes and all the maps are admissible. Representing $x \in H(M)$ by $x = m + dM$, we have

$$\begin{aligned} h_*j_*(x) &= h_*(j(m) + dN) = hj(m) + dN' = gi'(m) + dN' \\ &= g_*(i'(m) + dM') = g_*i'_*(m + dM) = g_*i'_*(x), \end{aligned}$$

as desired.

The proof that the last square commutes is a little different than this, so we prove this too. Represent $x \in H(N)$ by $x = n + dN$ with $dn = 0$ and recall that $d_*(x) = \ell + dL$, where $dm = i(\ell)$, $\ell \in L$, where $j(m) = n$, for $m \in M$. We have

$$f_*d_*(x) = f_*(\ell + dL) = f(\ell) + dL'.$$

On the other hand,

$$d_*h_*(x) = d_*(h(n) + dN') = \ell' + dL',$$

for some $\ell' \in L'$. Since $h(n) \in N'$, by the commutativity of (5) and the definition of d_*, $\ell' \in L'$ is an element such that $i'(\ell') = gi(\ell)$. Since i' is injective, this condition on ℓ' determines it uniquely mod dL'. By the commutativity of (5), we may take $\ell' = f(\ell)$.

7. Let L, L', M, M', N, N' be differential graded groups with grading $+1$ (i.e., of "cohomology type"). Suppose that we have a commutative diagram, with all maps admissible and all rows exact as in (5). Then the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H_{n-1}(N) & \xrightarrow{d_*} & H_n(L) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(N) & \xrightarrow{d_*} & H_{n+1}(L) & \longrightarrow & \cdots \\
 & & {\scriptstyle h_*}\downarrow & & {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle h_*}\downarrow & & {\scriptstyle f_*}\downarrow & & \\
\cdots & \longrightarrow & H_{n-1}(N') & \xrightarrow{d_*} & H_n(L') & \xrightarrow{i'_*} & H_n(M') & \xrightarrow{j'_*} & H_n(N') & \xrightarrow{d_*} & H_{n+1}(L') & \longrightarrow & \cdots
\end{array}$$

This is Proposition 1.1.4 in [W]. As pointed out there, it is an immediate consequence of the properties, 1-6 above.

Compare this with Proposition 10.69 in [R].

8. Let L, L', M, M', N, N' be differential graded groups with grading -1 (i.e., of "homology type"). Suppose that we have a commutative diagram, with all maps admissible and all rows exact, as in (5). Then the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H_{n+1}(N) & \xrightarrow{d_*} & H_n(L) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(N) & \xrightarrow{d_*} & H_{n-1}(L) & \longrightarrow & \cdots \\
 & & {\scriptstyle h_*}\downarrow & & {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle h_*}\downarrow & & {\scriptstyle f_*}\downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(N') & \xrightarrow{d_*} & H_n(L') & \xrightarrow{i'_*} & H_n(M') & \xrightarrow{j'_*} & H_n(N') & \xrightarrow{d_*} & H_{n-1}(L') & \longrightarrow & \cdots
\end{array}$$

This is the analog of the previous property and is proven similarly.

Again, this is similar to Proposition 10.58 in [R].

9. Let (L, d) be a differential graded group with grading r. If $d_n = d|_{L_n}$ then $d_{n+r} \circ d_n = 0$ and

$$\cdots \to L_{n-r} \overset{d_{n-r}}{\to} L_n \overset{d_n}{\to} L_{n+r} \overset{d_n}{\to} L_{n+2r} \to \ldots \tag{6}$$

is exact.

10. If $\{L_n \mid n \in \mathbb{Z}\}$ is a sequence of abelian groups with homomorphisms d_n satisfying (6) then (L, d) is a differential group, where $L = \oplus_n L_n$ and $d = \oplus_n d_n$.

2.3 Homology and cohomology

When $r = 1$, we call L_n the **group of n-cochains**, $Z_n = L_n \cap Ker(d_n)$ the group of **n-cocycles**, and $B_n = L_n \cap d_{n-1}(L_{n-1})$ the group of **n-coboundaries**. We call $H_n(L) = Z_n/B_n$ the n^{th} **cohomology group**. When $r = -1$, we call L_n the **group of n-chains**, $Z_n = L_n \cap Ker(d_n)$ the group of **n-cycles**, and $B_n = L_n \cap d_{n+1}(L_{n+1})$ the group of **n-boundaries**. We call $H_n(L) = Z_n/B_n$ the n^{th} **homology group**.

3 Complexes

We introduce complexes in order to define explicit differential groups which will then be used to construct group (co)homology.

3.1 Definitions

Let R be a non-commutative ring, for example $R = \mathbb{Z}[G]$.

We shall define a "finite free, acyclic, augmented chain complex" of left R-modules.

A **complex** (or chain complex or R-complex with a negative grading) is a sequence of maps

$$\cdots \to X_{n+1} \stackrel{\partial_{n+1}}{\to} X_n \stackrel{\partial_n}{\to} X_{n-1} \stackrel{\partial_{n-1}}{\to} X_{n-2} \to \ldots \tag{7}$$

for which $\partial_n \partial_{n+1} = 0$, for all n. If each X_n is a free R-module with a finite basis over R (so is $\cong R^k$, for some k) then the complex is called **finite free**. If this sequence is exact then it is called an **acyclic complex**. The complex is **augmented** if there is a surjective R-module homomorphism $\epsilon : X_0 \to \mathbb{Z}$ and an injective R-module homomorphism $\mu : \mathbb{Z} \to X_{-1}$ such that $\partial_0 = \mu \circ \epsilon$, where (as usual) $\mathbb{Z}$ is regarded as a trivial R-module.

The **standard diagram** for such an R-complex is

$$
\begin{tikzcd}
\dots \arrow[r] & X_2 \arrow[r, "\partial_2"] & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\partial_0"] \arrow[d, "\epsilon"'] & X_{-1} \arrow[r, "\partial_{-1}"] & X_{-2} \arrow[r] & \dots \\
& & & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[u, "\mu"'] & & \\
& & & 0 & 0 \arrow[u] & &
\end{tikzcd}
$$

Such an acyclic augmented complex can be broken up into the **positive part**

$$\dots \to X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

and the **negative part**

$$0 \to \mathbb{Z} \xrightarrow{\mu} X_{-1} \xrightarrow{\partial_{-1}} X_{-2} \xrightarrow{\partial_{-2}} X_{-3} \to \dots \ .$$

Conversely, given a positive part and a negative part, they can be combined into a standard diagram by taking $\partial_0 = \mu \circ \epsilon$.

If X is any left R-module, let $X^* = Hom_R(X, \mathbb{Z})$ be the dual R-module, where $\mathbb{Z}$ is regarded as a trivial R-module. Associated to any $f \in Hom_R(X, Y)$ is the pull-back $f^* \in Hom_R(Y^*, X^*)$. (If $y^* \in Y^*$ then define $f^*(y^*)$ to be $y^* \circ f : X \to \mathbb{Z}$.) Since "dualizing" reverses the direction of the maps, if you dualize the entire complex with a -1 grading, you will get a complex with a $+1$ grading. This is the **dual complex**.

When $R = \mathbb{Z}[G]$ then we call a finite free, acyclic, augmented chain complex of left R-modules, a **G-resolution**.

3.2 Constructions

Let $R = \mathbb{Z}[G]$.

3.2.1 Bar resolution

This section follows §1.3 in [W].

Define a symbol $[.]$ and call it the **empty cell**. Let $X_0 = R[.]$, so X_0 is a finite free (left) R-module whose basis has only 1 element. For $n > 0$, let $g_1, \dots, g_n \in G$ and define an **n-cell** to be the symbol $[g_1, \dots, g_n]$. Let

$$X_n = \oplus_{(g_1,\ldots,g_n)\in G^n} R[g_1, \ldots, g_n],$$

where the sum runs over all ordered n-tuples in G^n.

Define the differential operators d_n and the augmentation ϵ, as G-module maps, by

$$\begin{aligned}
\epsilon(g[.]) &= 1, \qquad g \in G \\
d_1([g]) &= g[.] - [.], \\
d_2([g_1, g_2]) &= g_1[g_2] - [g_1 g_2] + [g_1], \\
&\vdots \\
d_n([g_1, \ldots, g_n]) &= g_1[g_2, \ldots, g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n] \\
&\quad + (-1)^n [g_1, \ldots, g_{n-1}],
\end{aligned}$$

for $n \geq 1$. Note that the condition $\epsilon(g[.]) = 1$ for all $g \in G$ is equivalent to saying $\epsilon([.]) = 1$. This is because ϵ is a G-module homomorphism and $\mathbb{Z}$ is a trivial G-module, so $\epsilon(g[.]) = g\epsilon([.]) = g \cdot 1 = 1$, where the (trivial) G-action on $\mathbb{Z}$ is denoted by a $\cdot$.

The X_n are finite free G-modules, with the set of all n-cells serving as a basis.

Proposition 1 *With these definitions, the sequence*

$$\cdots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

is a free G-resolution.

Sometimes this resolution is called the **bar resolution**. There are two other resolutions we shall consider. One is the closely related "homogeneous resolution" and the other is the "normalized bar resolution".

This simple-looking proposition is not so simple to prove. First, we shall show it is a complex, i.e., $d^2 = 0$. Then, and this is the most non-trivial part of the proof, we show that the sequence is exact.

First, we need some definitions and a lemma.

Let $f : L \to M$ and $g : L \to M$ be $+1$-graded admissible maps. We say f is **homotopic** to g if there is a homomorphism $D : L \to M$, called a **homotopy**, such that

- $D_n = D|_{L_n} : L_n \to M_{n+1}$,
- $f - g = Dd + dD$.

If $L = M$ and the identity map $1 : L \to L$ is homotopic to the zero map $0 : L \to L$ then the homotopy is called a **contracting homotopy for** L.

Lemma 2 *If L has a contracting homotopy then $H(L) = 0$.*

proof: Represent $x \in H(L)$ by $\ell \in L$ with $d\ell = 0$. But $\ell = 1(\ell) - 0(\ell) = dD(\ell) + Dd(\ell) = dD(\ell)$. Since $D : L \to L$, this shows $\ell \in dL$, so $x = 0$ in $H(L)$. $\square$

Next, we construct a contracting homotopy for the complex X_* in Proposition 1 with differential operator d. Actually, we shall *temporarily* let $X_{-1} = \mathbb{Z}$, $X_{-n} = 0$ and $d_{-n} = 0$ for $n > 1$, so that that the complex is infinite in both directions. We must define $D : X \to X$ such that

- $D_{-1} = D|_{\mathbb{Z}} : \mathbb{Z} \to X_0$,
- $D_n = D|_{X_n} : X_n \to X_{n+1}$,
- $\epsilon D_{-1} = 1$ on $\mathbb{Z}$,
- $d_1 D_0 + D_{-1}\epsilon = 1$ on X_0,
- $d_{n+1} D_n + D_{n-1} d_n = 1$ in X_n, for $n \geq 1$.

Define

$$\begin{aligned} D_{-n} &= 0, \qquad n > 1, \\ D_{-1}(1) &= [.], \\ D_0(g[.]) &= [g], \\ D_n(g[g_1, \ldots, g_n]) &= [g, g_1, \ldots, g_n], \qquad n > 0, \end{aligned}$$

and extend to a $\mathbb{Z}$-basis linearly.

Now we must verify the desired properties.

By definition, for $m \in \mathbb{Z}$, $\epsilon D_{-1}(m) = \epsilon(m[.]) = m\epsilon([.]) = m$. Therefore, ϵD_{-1} is the identity map on $\mathbb{Z}$.

Similarly,

$$\begin{aligned}(d_1 D_0 + D_{-1}\epsilon)(g[.]) &= d_1([g]) + D_{-1}(1)\\ = g[.] - [.] + D_{-1}(1) &= g[.] - [.] + [.] = g[.].\end{aligned}$$

For the last property, we compute

$$\begin{aligned} d_{n+1}D_n(g[g_1,\dots,g_n]) &= d_{n+1}([g,g_1,\dots,g_n])\\ &= g[g_1,\dots,g_n] - [gg_1,\dots,g_n]\\ &\quad + \sum_{i=1}^{n-1}(-1)^{i-1}[g,g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n]\\ &\quad + (-1)^{n+1}[g,g_1,\dots,g_{n-1}],\end{aligned}$$

and

$$\begin{aligned} &D_{n-1}d_n(g[g_1,\dots,g_n])\\ &\qquad = D_{n-1}(gd_n([g_1,\dots,g_n]))\\ &\qquad = D_{n-1}(gg_1[g_2,\dots,g_n]\\ &\qquad\quad + \sum_{i=1}^{n-1}(-1)^i g[g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n]\\ &\qquad\quad + (-1)^n g[g_1,\dots,g_{n-1}])\\ &\qquad = [gg_1,g_2,\dots,g_n]\\ &\qquad\quad + \sum_{i=1}^{n-1}(-1)^i[g,g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n]\\ &\qquad\quad + (-1)^n[g,g_1,\dots,g_{n-1}].\end{aligned}$$

All the terms but one cancels, verifying that $d_{n+1}D_n + D_{n-1}d_n = 1$ in X_n, for $n \geq 1$.

Now we show $d^2 = 0$. One verifies $d_1 d_2 = 0$ directly (which is left to the reader). Multiply $d_k D_{k-1} + D_{k-2}d_{k-1} = 1$ on the right by d_k and $d_{k+1}D_k + D_{k-1}d_k = 1$ on the left by d_k:

$$d_k D_{k-1} d_k + D_{k-2} d_{k-1} d_k = d_k = d_k d_{k+1} D_k + d_k D_{k-1} d_k.$$

Cancelling like terms, the induction hypothesis $d_{k-1}d_k = 0$ implies $d_k d_{k+1} = 0$. This shows $d^2 = 0$ and hence that the sequence in Proposition 1 is exact. This completes the proof of Proposition 1. □

The above complex can be "dualized" in the sense of §3.1. This dualized complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{\mu} X_{-1} \xrightarrow{d_{-1}} X_{-2} \xrightarrow{d_{-2}} X_{-3} \to \dots .$$

The **standard G-resolution** is obtained by splicing these together.

3.2.2 Normalized bar resolution

Define the **normalized cells** by

$$[g_1, ..., g_n]^* = \begin{cases} [g_1, ..., g_n], & \text{if all } g_i \neq 1, \\ 0, & \text{if some } g_i = 1. \end{cases}$$

Let $X_0 = R[.]$ and

$$X_n = \oplus_{(g_1,...,g_n)\in G^n} R[g_1, \dots, g_n]^*, \qquad n \geq 1,$$

where the sum runs over all ordered n-tuples in G^n. Define the differential operators d_n and the augmentation map exactly as for the bar resolution.

Proposition 3 *With these definitions, the sequence*

$$\cdots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

is a free G-resolution.

Sometimes this resolution is called the **normalized bar resolution**.
proof: See Theorem 10.117 in [R]. □

3.2.3 Homogeneous resolution

Let $X_0 = R$, so X_0 is a finite free (left) R-module whose basis has only 1 element. For $n > 0$, let X_n denote the $\mathbb{Z}$-module generated by all $(n+1)$-tuples $(g_0, \dots, g_n)$. Make X_i into a G-module by defining the action by $g : X_n \to X_n$ by

$$g : (g_0, ..., g_n) \longmapsto (gg_0, \dots, gg_n), \qquad g \in G.$$

Define the differential operators ∂_n and the augmentation ϵ, as G-module maps, by

$$\epsilon(g) = 1,$$
$$\partial_n(g_0, \ldots, g_n) = \sum_{i=0}^{n-1} (-1)^i (g_0, \ldots, g_{i-1}, \hat{g}_i, g_{i+1}, \ldots, g_n),$$

for $n \geq 1$.

Proposition 4 *With these definitions, the sequence*

$$\cdots \to X_2 \stackrel{\partial_2}{\to} X_1 \stackrel{\partial_1}{\to} X_0 \stackrel{\epsilon}{\to} \mathbb{Z} \to 0,$$

is a G-resolution.

Sometimes this resolution is called the **homogeneous resolution**.

Of the three resolutions presented here, this one is the most straightforward to deal with.

proof: See Lemma 10.114, Proposition 10.115, and Proposition 10.116 in [R]. $\Box$

4 Definition of $H^n(G, A)$

For convenience, we briefly recall the definition of Ext^n. Let A be a left R-module, where $R = \mathbb{Z}[G]$, and let (X_i) be a G-resolution of $\mathbb{Z}$. We define

$$\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) = ker(d^*_{n+1})/im(d^*_n),$$

where

$$d^*_n : Hom(X_{n-1}, A) \to Hom(X_n, A),$$

is defined by sending $f : X_{n-1} \to A$ to $fd_n : X_n \to A$. It is known that this is, up to isomorphism, independent of the resolution choosen. Recall $\text{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ is the right-derived functors of the right-exact functor $A \longmapsto A^G = Hom_G(\mathbb{Z}, A)$ from the category of G-modules to the category of abelian groups. We define

$$H^n(G,A) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},A), \tag{8}$$

When we wish to emphasize the dependence on the resolution choosen, we write $H^n(G,A,X_*)$.

For example, let X_* denote the bar resolution in §3.2.1 above. Call $C^n = C^n(G,A) = Hom_G(X_n,A)$ the **group of n-cochains of G in A**, $Z^n = Z^n(G,A) = C^n \cap Ker(\partial)$ the group of n-**cocycles**, and $B^n = B^n(G,A) = \partial(C^{n-1})$ the group of n-**coboundaries**. We call $H^n(G,A) = Z^n/B^n$ the n^{th} **cohomology group of G in A**. This is an abelian group.

We call also define the cohomology group using some other resolution, the normalized bar resolution or the homogeneous resolution for example. If we wish to express the dependence on the resolution X_* used, we write $H^n(G,A,X_*)$. Later we shall see that, up to isomorphism, this abelian group is independent of the resolution.

The group $H^2(G,\mathbb{Z})$ (or $H^2(G,\mathbb{C})$) is sometimes called the **Schur multiplier** of G.

We say that the group G has **cohomological dimension** n, written $cd(G) = n$, if $H^{n+1}(H,A) = 0$ for all G-modules A and all subgroups H of G, but $H^n(H,A) \neq 0$ for some such A and H.

Remark 1

- *If $cd(G) < \infty$ then G is torsion-free*[5].
- *If G is a free abelian group of finite rank then $cd(G) = rank(G)$.*
- *If $cd(G) = 1$ then G is free. This is a result of Stallings and Swan (see for example [R], page 885).*

4.1 Computations

We briefly discuss computer programs which compute cohomology and some examples of known computations.

4.1.1 Computer computations of cohomology

GAP [Gap], MAGMA [MAGMA], and Macaulay 2 [Mac] can compute some cohomology groups.

[5] This follows from the fact that if G is a cyclic group then $H^n(G,\mathbb{Z}) \neq 0$, discussed below.

For example, GAP will compute the Schur multiplier $H^2(G, \mathbb{C})$ using the `AbelianInvariantsMultiplier` command. To find $H^2(A_5, \mathbb{C})$, where A_5 is the alternating group on 5 letters, type

```
gap> A5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> AbelianInvariantsMultiplier(A5);
[ 2 ]
```

So, $H^2(A_5, \mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

GAP will do other cohomology computations (in low dimension) as well (see §37.22 of the GAP manual or D. Holt's GAP package **cohomo**, [Gap]).

Example 5 `> G:=CyclicGroup(5);`

```
> M := TrivialModule(G, Integers());
> X := CohomologyModule(G, M);
> CohomologyGroup(X, 1);
Full Quotient RSpace of degree 0 over Integer Ring
Column moduli:
[ ]
> CohomologyGroup(X, 2);
Full Quotient RSpace of degree 1 over Integer Ring
Column moduli:
[ 5 ]
>
> G := Alt(5);
> M := PermutationModule(G, GF(7));
> X := CohomologyModule(G, M);
> CohomologyGroup(X, 1);
Full Vector space of degree 0 over GF(7)
> CohomologyGroup(X, 2);
Full Vector space of degree 0 over GF(7)
```

We also refer to the MAGMA manual [MAGMA] and to the paper [EK].

For Macaulay, we refer to the Macaulay 2 manual [Mac].

4.1.2 Examples

Some example computations.

1. $H^0(G, A) = A^G$.

   This is by definition.

2. Let L/K denote a Galois extension with finite Galois group G. We have $H^1(G, L^\times) = 1$. This is often called Hilbert's Theorem 90.

   See Theorem 1.5.4 in [W] or Proposition 2 in §X.1 of [S].

3. Let G be a finite cyclic group and A a trivial torsion-free G-module. Then $H^1(G, A) = 0$.

   This is a consequence of properties given in the next section.

4. If G is a finite cyclic group of order m and A is a trivial G-module then

$$H^2(G, A) = A/mA$$

   This is a consequence of properties given below.

   For example, $H^2(\mathbb{F}_q^\times, \mathbb{C}) = 0$.

5. If $|G| = m$, $rA = 0$ and $gcd(r, m) = 1$, then $H^n(G, A) = 0$, for all $n \geq 1$.

   This is Corollary 3.1.7 in [W].

   For example, $H^1(A_5, \mathbb{Z}/7\mathbb{Z}) = 0$.

5 Definition of $H_n(G, A)$

We say A is **projective** if the functor $B \longmapsto Hom_G(A, B)$ (from the category of G-modules to the category of abelian groups) is exact. Recall, if

$$P_{\mathbb{Z}} = \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{9}$$

is a projective resolution of $\mathbb{Z}$ then

$$\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A) = ker(d_n \otimes 1_A)/im(d_{n+1} \otimes 1_A).$$

It is known that this is, up to isomorphism, independent of the resolution choosen. Recall $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, A)$ are the right-derived functors of the right-exact functor $A \longmapsto A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ from the category of G-modules to the category of abelian groups. We define

$$H_n(G, A) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A), \tag{10}$$

When we wish to emphasize the dependence on the resolution, we write $H_n(G, A, P_{\mathbb{Z}})$.

5.1 Computations

We briefly discuss computer programs which compute cohomology and some examples of known computations.

5.1.1 Computer computations of homology

MAGMA has some limited ability to construct complexes, chain maps, and homology groups of a complex but I've not figured out the syntax to give an example. Both MAGMA and GAP allow one to compute the commutator subgroup, so $H_i(G, \mathbb{Z})$ can be computed for $i = 1, 2$, when G is a finitely presented group (see the next subsection).

For Macaulay, we refer to the Macaulay 2 manual [Mac].

5.1.2 Examples

Some example computations.

1. If A is a G-module then $Tor_0^{\mathbb{Z}[G]}(\mathbb{Z}, A) = H_0(G, A) = A_G \cong A/DA$.

   proof: We need some lemmas.

   Let $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ be the augmentation map. This is a ring homomorphism (but not a G-module homomorphism). Let $D = ker(\epsilon)$ denote its kernel, the **augmentation ideal**. This is a G-module.

   Lemma 6 *As an abelian group, D is free abelian generated by $G - 1 = \{g - 1 \mid g \in G\}$.*

   We write this as $D = \mathbb{Z}\langle G - 1\rangle$.

   proof of lemma: If $d \in D$ then $d = \sum_{g\in G} m_g g$, where $m_g \in \mathbb{Z}$ and $\sum_{g\in G} m_g = 0$. Thus, $d = \sum_{g\in G} m_g(g-1)$, so $D \subset \mathbb{Z}\langle G-1\rangle$. To show D is free: If $\sum_{g\in G} m_g(g-1) = 0$ then $\sum_{g\in G} m_g g - \sum_{g\in G} m_g = 0$ in $\mathbb{Z}[G]$. But $\mathbb{Z}[G]$ is a free abelian group with basis G, so $m_g = 0$ for all $g \in G$. $\square$

   Lemma 7 *$\mathbb{Z} \otimes_{\mathbb{Z}[G]} A = A/DA$, where DA is generated by elements of the form $ga - a$, $g \in G$ and $a \in A$.*

Recall A_G denotes the largest quotient of A on which G acts trivially[6].

proof of lemma: Consider the G-module map, $A \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$, given by $a \longmapsto 1 \otimes a$. Since $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ is a trivial G-module, it must factor through A_G. The previous lemma implies $A_G \cong A/DA$. (In fact, the quotient map $q : A \to A_G$ satisfies $q(ga - a) = 0$ for all $g \in G$ and $a \in A$, so $DA \subset ker(q)$. By maximality of A_G, $DA = ker(q)$. QED) So, we have maps $A \to A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$. By the definition of tensor products, the map $\mathbb{Z} \times A \to A_G$, $1 \times a \longmapsto 1 \cdot aDA$, corresponds to a map $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A \to A_G$ for which the composition $A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \to A_G$ is the identity. This forces $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$. $\Box$

See also # 11 in §6.

2. If G is a finite group then $H_0(G, \mathbb{Z}) = \mathbb{Z}$.

   This is a special case of the example above (taking $A = \mathbb{Z}$, as a trivial G-module).

3. $H_1(G, \mathbb{Z}) \cong G/[G, G]$, where $[G, G]$ is the commutator subgroup of G.

   This is Proposition 10.110 in [R], §10.7.

   proof: First, we **claim**: $D/D^2 \cong G/[G, G]$, where D is as in Lemma 6. To prove this, define $\theta : G \to D/D^2$ by $g \longmapsto (g-1) + D^2$. Since $gh - 1 - (g-1) - (h-1) = (g-1)(h-1)$, it follows that $\theta(gh) = \theta(g)\theta(h)$, so θ is a homomorphism. Since D/D^2 is abelian and $G/[G, G]$ is the maximal abelian quotient of G, we must have $ker(\theta) \subset [G, G]$. Therefore, θ factors through $\theta' : G/[G, G] \to D/D^2$, $g[G, G] \longmapsto (g-1) + D^2$. Now, we construct an inverse. Define $\tau : D \to G/[G, G]$ by $g - 1 \longmapsto g[G, G]$. Since $\tau(g - 1 + h - 1) = g[G, G] \cdot h[G, G] = gh[G, G]$, it is not hard to see that this is a homomorphism. We would be essentially done (with the construction of the inverse of θ', hence the proof of the claim) if we knew $D^2 \subset ker(\tau)$. (The inverse would be the composition of the quotient $D/D^2 \to D/ker(\tau)$ with the map induced from τ, $D/ker(\tau) \to G/[G, G]$.) This follows from the fact that any $x \in D^2$ can be written as $x = (\sum_g m_g(g-1))(\sum_h m'_h(h-1)) = (\sum_{g,h} m_g m'_h (g-1)(h-1))$, so $\tau(x) = \prod_{g,h}(ghg^{-1}h^{-1})^{m_g m'_h}[G, G] = [G, G]$. QED (claim)

   Next, we show $H_1(G, \mathbb{Z}) \cong D/D^2$. From the short exact sequence

$$0 \to D \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

   we obtain the long exact sequence of homology

$$\begin{array}{c} \cdots \to H_1(G, D) \to H_1(G, \mathbb{Z}[G]) \to \\ H_1(G, \mathbb{Z}) \xrightarrow{\partial} H_0(G, D) \xrightarrow{f} H_0(G, \mathbb{Z}[G]) \xrightarrow{\epsilon_*} H_0(G, \mathbb{Z}) \to 0. \end{array} \tag{11}$$

[6] Implicit in the words "largest quotient" is a universal property which we leave to the reader for formulate precisely.

Since $\mathbb{Z}[G]$ is a free $\mathbb{Z}[G]$-module, $H_1(G, \mathbb{Z}[G]) = 0$. Therefore ∂ is injective. By item # 1 above (i.e., $H_0(G, A) \cong A/DA \cong A_G$, we have $H_0(G, \mathbb{Z}) \cong \mathbb{Z}_G = \mathbb{Z}$ and $H_0(G, \mathbb{Z}[G]) \cong \mathbb{Z}[G]/D \cong \mathbb{Z}$. By (11), ϵ_* is surjective. Combining the last two statements, we find $\mathbb{Z}/ker(\epsilon_*) \cong \mathbb{Z}$.This forces ϵ_* to be injective. This, and (11), together imply f must be 0. Since this forces ∂ to be an isomorphism, we are done. $\square$

4. Let $G = F/R$ be a presentation of G, where F is a free group and R is a normal subgroup of relations. **Hopf's formula** states: $H_2(G, \mathbb{Z}) \cong (F \cap R)/[F, R]$, where $[F, R]$ is the commutator subgroup of G.

   See [R], §10.7.

   The group $H_2(G, \mathbb{Z})$ is sometimes called the **Schur multiplier** of G.

6 Basic properties of $H^n(G, A)$, $H_n(G, A)$

Let R be a (possibly non-commutative) ring and A be an R-module. We say A is **injective** if the functor $B \longmapsto Hom_G(B, A)$ (from the category of G-modules to the category of abelian groups) is exact. (Recall A is projective if the functor $B \longmapsto Hom_G(A, B)$ is exact.) We say A is **co-induced** if it has the form $Hom_{\mathbb{Z}}(R, B)$ for some abelian group B. We say A is **relatively injective** if it is a direct factor of a co-induced R-module. We say A is **relatively projective** if

$$\begin{array}{rrl} \pi: & \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \rightarrow & A, \\ & x \otimes a \longmapsto & xa, \end{array}$$

maps a direct factor of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ isomorphically onto A. These are the G-modules A which are isomorphic to a direct factor of the induced module $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$. When G is finite, the notions of relatively injective and relatively projective coincide[7].

1. The definition of $H^n(G, A)$ does not depend on the G-resolution X_* of $\mathbb{Z}$ used.

2. If A is an projective $\mathbb{Z}[G]$-module then $H^n(G, A) = 0$, for all $n \geq 1$.

   This follows immediately from the definitions.

[7] These notions were introduced by Hochschild [Ho].

3. If A is an injective $\mathbb{Z}[G]$-module then $H_n(G,A)=0$, for all $n\geq 1$.

   See also [S], §VII.2.

4. If A is a relatively injective $\mathbb{Z}[G]$-module then $H^n(G,A)=0$, for all $n\geq 1$.

   This is Proposition 1 in [S], §VII.2.

5. If A is a relatively projective $\mathbb{Z}[G]$-module then $H^n(G,A)=0$, for all $n\geq 1$.

   This is Proposition 2 in [S], §VII.4.

6. If $A=A'\oplus A''$ then $H^n(G,A)=H^n(G,A')\oplus H^n(G,A'')$, for all $n\geq 0$. More generally, if I is any indexing family and $A=\oplus_{i\in I}A_i$ then $H^n(G,A)=\oplus_{i\in I}H^n(G,A_i)$, for all $n\geq 0$.

   This follows from Proposition 10.81 in §10.6 of Rotman [R].

7. If

   $$0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$$

   is an exact sequence of G-modules then we have a long exact sequence of cohomology (1). See [S], §VII.2, and properties of the *ext* functor [R], §10.6.

8. $A\longmapsto H^n(G,A)$ is the higher right derived functor associated to $A\longmapsto A^G=Hom_G(A,\mathbb{Z})$ from the category of G-modules to the category of abelian groups.

   This is by definition. See [S], §VII.2, or [R], §10.7.

9. If

   $$0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$$

   is an exact sequence of G-modules then we have a long exact sequence of homology (2). In the case of a finite group, see [S], §VIII.1. In general, see [S], §VII.4, and properties of the Tor functor in [R], §10.6.

10. $A \longmapsto H_n(G, A)$ is the higher left derived functor associated to $A \longmapsto A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ on the category of G-modules.

    This is by definition. See [S], §VII.4, or [R], §10.7.

11. If G is a finite cyclic group then

$$\begin{aligned} H_0(G, A) &= A_G, \\ H_{2n-1}(G, A) &= A^G/NA, \\ H_{2n}(G, A) &= \ker N/DA, \end{aligned}$$

for all $n \geq 1$.

To prove this, we need a lemma.

Lemma 8 *Let $G = \langle g \rangle$ be acyclic group of order k. Let $M = g - 1$ and $N = 1 + g + g^2 + ... + g^{k-1}$. Then*

$$\cdots \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{M} \mathbb{Z}[G] \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{M} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

is a free G-resolution.

proof of lemma: It is clearly free. Since $MN = NM = (g-1)(1 + g + g^2 + ... + g^{k-1}) = g^k - 1 = 0$, it is a complex. It remains to prove exactness. Since $ker(\epsilon) = D = im(M)$, by Lemma 6, this stage is exact.

To show $ker(M) = im(N)$, let $x = \sum_{j=0}^{k-1} m_j g^j \in ker(M)$. Since $(g-1)x = 0$, we must have $m_0 = m_1 = ... = m_{k-1}$. This forces $x = m_0 N \in im(N)$. Thus $ker(M) \subset im(N)$. Clearly $MN = 0$ implies $im(N) \subset ker(M)$, so $ker(M) = im(N)$.

To show $ker(N) = im(M)$, let $x = \sum_{j=0}^{k-1} m_j g^j \in ker(N)$. Since $Nx = 0$, we have $0 = \epsilon(Nx) = \epsilon(N)\epsilon(x) = k\epsilon(x)$, so $\sum_{j=0}^{k-1} m_j = 0$. Observe that

$$\begin{aligned} x &= m_0 \cdot 1 + m_1 g + m_2 g^2 + ... + m_{k-1} g^{k-1} \\ &= (m_0 - m_0 g) + (m_0 + m_1) g + m_2 g^2 + ... + m_{k-1} g^{k-1} \\ &= (m_0 - m_0 g) + (m_0 + m_1) g - (m_0 + m_1) g^2 \\ &\quad + (m_0 + m_1 + m_2) g^2 - (m_0 + m_1 + m_2) g^3 + ... \\ &\quad + (m_0 + .. + m_{k-1}) g^{k-1} - (m_0 + .. + m_{k-1}) g^k. \end{aligned}$$

where the last two terms are actually 0. This implies $x = -M(m_0 + (m_0 + m_1)g + (m_0 + m_1 + m_2)g^2 + ... + (m_0 + .. + m_{k-1})g^{k-1} \in im(M)$. Thus $ker(N) \subset im(M)$. Clearly $NM = 0$ implies $im(M) \subset ker(N)$, so $ker(N) = im(M)$.

This proves exactness at every stage.□

Now we can prove the claimed property. By property 1 in §5.1.2, it suffices to assume $n > 0$. Tensor the complex in Lemma 8 on the right with A:

$$\begin{aligned}\cdots \to \quad & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \overset{N_*}{\to} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \overset{M_*}{\to} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \overset{N_*}{\to} \\ & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \overset{M_*}{\to} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \overset{\epsilon}{\to} \mathbb{Z} \otimes \mathbb{Z}[G] A \to 0,\end{aligned}$$

where the new maps are distinguished from the old maps by adding an asterisk. By definition, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \cong A$, and by property 1 in §5.1.2, $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A \cong A/DA$. The above sequence becomes

$$\cdots \to A \overset{N_*}{\to} A \overset{M_*}{\to} A \overset{N_*}{\to} A \overset{M_*}{\to} A \overset{\epsilon}{\to} A/DA \to 0.$$

This implies, by definition of Tor,

$$Tor_{2n-1}^{\mathbb{Z}[G]}(\mathbb{Z}, A) = ker(M_*)/im(N_*) = A^G/NA,$$

and

$$Tor_{2n}^{\mathbb{Z}[G]}(\mathbb{Z}, A) = ker(N_*)/im(M_*) = A[N]/DA.$$

See also [S], §VIII.4.1 and the Corollary in §VIII.4.

12. The group $H^2(G, A)$ classifies group extensions of A by G.

    This is Theorem 5.1.2 in [W]. See also §10.2 in [R].

13. If G is a finite group of order $m = |G|$ then $mH^n(G, A) = 0$, for all $n \geq 1$.

    This is Proposition 10.119 in [R].

14. If G is a finite group and A is a finitely-generated G-module then $H^n(G, A)$ is finite, for all $n \geq 1$.

    This is Proposition 3.1.9 in [W] and Corollary 10.120 in [R].

15. The group $H^1(G, A)$ constructed using resolutions is the same as the group constructed using 1-cocycles. The group $H^2(G, A)$ constructed using resolutions is the same as the group constructed using 2-cocycles.

    This is Corollary 10.118 in [R].

16. If G is a finite cyclic group then

$$H^0(G,A) = A^G,$$
$$H^{2n-1}(G,A) = \ker N/DA,$$
$$H^{2n}(G,A) = A^G/NA,$$

for all $n \geq 1$. Here $N : A \to A$ is the norm map $Na = \sum_{g\in G} ga$ and DA is the augmentation ideal defined above (generated by elements of the form $ga - a$).

proof: The case $n = 0$: By definition, $H^0(G,A) = Ext^0_{\mathbb{Z}[G]}(\mathbb{Z}, A) = Hom_G(\mathbb{Z}, A)$. Define $\tau : Hom_G(\mathbb{Z}, A) \to A^G$ by sending $f \longmapsto f(1)$. It is easy to see that this is well-defined and, in fact, injective. For each $a \in A^G$, define $f = f_a \in Hom_G(\mathbb{Z}, A)$ by $f(m) = ma$. This shows τ is surjective as well, so case $n = 0$ is proven.

Case $n > 0$: Applying the functor $Hom_G(*, A)$ to the G-resolution in Lemma 8 to get

$$\cdots \leftarrow \quad Hom_G(\mathbb{Z}[G], A) \overset{N_*}{\leftarrow} Hom_G(\mathbb{Z}[G], A) \overset{M_*}{\leftarrow} Hom_G(\mathbb{Z}[G], A) \overset{\epsilon_*}{\leftarrow} Hom_G(\mathbb{Z}, A) \leftarrow 0.$$

It is known that $Hom_G(\mathbb{Z}[G], A) \cong A$ (see Proposition 8.85 on page 583 of [R]). It follows that

$$\cdots \leftarrow \quad A \overset{N_*}{\leftarrow} A \overset{M_*}{\leftarrow} A \overset{\epsilon_*}{\leftarrow} A^G \leftarrow 0.$$

By definition of Ext, for $n > 0$ we have

$$Ext^{2n}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = ker(M_*)/im(N_*) = A^G/NA,$$

and

$$Ext^{2n-1}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = ker(N_*)/im(M_*) = ker(N)/(g-1)A,$$

where g is a generator of G as in Lemma 8. $\square$

See also [S], §VIII.4.1 and the Corollary in §VIII.4.

17. If G is a finite cyclic group of order m and A is a *trivial* G-module then

$$H^0(G,A) = A^G,$$
$$H^{2n-1}(G,A) \cong A[m],$$
$$H^{2n}(G,A) \cong A/mA,$$

for all $n \geq 1$.

This is a consequence of the previous property.

7 Functorial properties

In this section, we investigate some of the ways in which $H^n(G, A)$ depends on G.

One way to construct all these in a common framework is to introduce the notion of a "homomorphism of pairs". Let G, H be groups. Let A be a G-module and B an H-module. If $\alpha : H \to G$ is a homomorphism of groups and $\beta : A \to B$ is a homomorphism of H-modules (using α to regard B as an H-module) then we call (α, β) a **homomorphism of pairs**, written

$$(\alpha, \beta) : (G, A) \to (H, B).$$

Let $G \subset H$ be groups and A an H-module (so, by restriction, a G-module). We say a map

$$f_{G,H} : H^n(G, A) \to H^n(H, A),$$

is **transitive** if $f_{G_2,G_3} f_{G_1,G_2} = f_{G_1,G_2}$, for all subgroups $G_1 \subset G_2 \subset G_3$.

Let X_* be a G-resolution and X'_* a H-resolution, each with a -1 grading. Associated to a homomorphism of groups $\alpha : H \to G$ is a sequence of H-homomorphisms

$$A_n : X'_n \to X_n, \tag{12}$$

$n \geq 0$, such that $d_{n+1} A_{n+1} = A_n d'_{n+1}$ and $\epsilon A_0 = \epsilon'$.

Theorem 9 *1. If $(\alpha, \beta) : (G, A) \to (G', A')$ and $(\alpha', \beta') : (G', A') \to (G'', A'')$ are homomorphisms of pairs then so is $(\alpha' \circ \alpha, \beta' \circ \beta) : (G, A) \to (G'', A'')$.*

2. Suppose $(\alpha, \beta) : (G, A) \to (G', A')$ is homomorphism of pairs, X_* is a G-resolution, and X'_* is a G'-resolution (each infinite in both directions, with a -1 grading). Let $H^n(G, A, X_*)$ denote the derived groups associated to the differential groups $Hom_G(X_*, A)$ with $+1$ grading. There is a homomorphism

$$(\alpha, \beta)_{X_*, X'_*} : H^n(G, A, X_*) \to H^n(G', A', X'_*)$$

satisfying the following properties.

(a) If $G = G'$, $A = A'$, $X = X'$, $\alpha = 1$ and $\beta = 1$ then $(1, 1)_{X_*, X'_*} = 1$.

(b) If $(\alpha', \beta') : (G', A') \to (G'', A'')$ is homomorphism of pairs, X''_* is a G''-resolution then

$$(\alpha' \circ \alpha, \beta' \circ \beta)_{X_*, X''_*} = (\alpha', \beta')_{X'_*, X''_*} \circ (\alpha, \beta)_{X_*, X'_*}.$$

(c) If $(\alpha, \gamma) : (G, A) \to (G', A')$ is homomorphism of pairs then

$$(\alpha, \beta + \gamma)_{X_*, X'_*} = (\alpha, \beta)_{X_*, X'_*} + (\alpha, \gamma)_{X_*, X'_*}.$$

Remark 2 *For an analogous result for homology, see §§III.8 in Brown [B].*

proof: We sketch the proof, following Weiss, [W], Theorem 2.1.8, pp 52-53.

(1): This is "obvious".

(2): Let $(\alpha, \beta) : (G, A) \to (G', A')$ be a homomorphism of pairs. Using (12), we have an associated chain map

$$\alpha^* : Hom_G(X_*, A) \to Hom_{G'}(X'_*, A')$$

of differential groups (Brown §III.8 in [B]). The homomorphism of cohomology groups induced by α^* is denoted

$$\alpha^*_{n, X_*, X'_*} : H^n(G, A, X_*) \to H^n(G', A', X'_*).$$

Properties (a)-(c) follow from §2.2 and the corresponding properties of α^*. $\square$

As the cohomology groups are independent of the resolution used, the map $(\alpha, \beta)_{X_*, X'_*} : H^n(G, A, X_*) \to H^n(G', A', X'_*)$ is sometimes simply denoted by

$$(\alpha, \beta)_* : H^n(G, A) \to H^n(G', A'). \tag{13}$$

7.1 Restriction

Let $X_* = X_*(G)$ denote the bar resolution.

If H is a subgroup of G then the cycles on G, $C^n(G, A) = Hom_G(X_n(G), A)$, can be restricted to H: $C^n(H, A) = Hom_H(X_n(H), A)$. The restriction map $C^n(G, A) \to C^n(H, A)$ leads to a map of cohomology classes:

$$Res : H^n(G, A) \to H^n(H, A).$$

In this case, the homomorphism of pairs is given by the inclusion map $\alpha : H \to G$ and the identity map $\beta : A \to A$. The map Res is the induced map defined by (13). By the properties of this induced map, we see that $Res_{H,G}$ is transitive: if $G \subset G' \subset G''$ then[8]

$$Res_{G',G} \circ Res_{G'',G'} = Res_{G'',G}.$$

A particularly nice feature of the restriction map is the following fact.

Theorem 10 *If G is a finite group and G_p is a p-Sylow subgroup and if $H^n(G,A)_p$ is the p-primary component of $H^n(G,A)$ then*

(a) there is a canonical isomorphism $H^n(G,A) \cong \oplus_p H^n(G,A)_p$, and

(b) $Res : H^n(G,A) \to H^n(G_p,A)$ restricted to $H^n(G,A)_p$ (identified with a subgroup of $H^n(G,A)$ via (a)) is injective.

proof: See Weiss, [W], Theorem 3.1.15. □

If H is a subgroup of finite index in G then there is an analogous restriction map in group homology (see for example Brown [B], §III.9).

Example 11 *Let $m = dk$ be a product with $d > 1$, $k > 1$, and let $G = \mathbb{Z}/m\mathbb{Z}$, $H = \mathbb{Z}/d\mathbb{Z}$. We can and do identify H with $(m/d)\mathbb{Z}/m\mathbb{Z}$. By §6, we have*

$$H^{2n}(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}, \quad H^{2n+1}(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}) = 0,$$

for $n > 0$. The map

$$Res : H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \to H^{2n}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z})$$

is simply the multiplication by m/d map:

$$Res : \mathbb{Z}/m\mathbb{Z} \stackrel{m/d}{\to} (m/d)\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

[8]There is an analog of the restriction for homology which also satisfies this transitive property (Proposition 9.5 in Brown [B]).

7.2 Inflation

Let X_* denote the bar resolution of G. Recall

$$X_n = \oplus_{(g_1,\dots,g_n)\in G^n} R[g_1,\dots,g_n],$$

where the sum runs over all ordered n-tuples in G^n. If H is a subgroup of G, let X_*^H denote the complex defined by

$$X_n^H = \oplus_{(g_1H,\dots,g_nH)\in (G/H)^n} R[g_1H,\dots,g_nH].$$

This is a resolution, and we have a chain map defined on n-cells by $[g_1,\dots,g_n]\longmapsto [g_1H,\dots,g_nH]$.

Suppose that H is a normal subgroup of G and A is a G-module. We may view A^H as a G/H-module. In this case, the homomorphism of pairs is given by the quotient map $\alpha : G\to G/H$ and the inclusion map $\beta : A^H\to A$. The **inflation** map Inf is the induced map defined by (13), denoted

$$Inf : H^n(G/H,A^H)\to H^n(G,A).$$

The **inflation-restriction sequence in dimension** n is

$$0\to H^n(G/H,A^H)\stackrel{Inf}{\to} H^n(G,A)\stackrel{Res}{\to} H^n(H,A).$$

For a proof, see Weiss, [W], §3.4.

There an analog of this inflation-restriction sequence for homology.

Example 12 *Let $H\subset G$ be as in Example 11 above. We have $G/H\cong \mathbb{Z}/(m/d)\mathbb{Z}$. The map*

$$Inf : H^{2n}((\mathbb{Z}/m\mathbb{Z})/(\mathbb{Z}/d\mathbb{Z}),\mathbb{Z})\to H^{2n}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$$

is simply the mod m map:

$$Inf : \mathbb{Z}/(m/d)\mathbb{Z}\stackrel{\mathrm{mod}\, m}{\to}\mathbb{Z}/m\mathbb{Z}.$$

Note that, by Example 11, the composition map

$$Res\circ Inf : H^{2n}((\mathbb{Z}/m\mathbb{Z})/(\mathbb{Z}/d\mathbb{Z}),\mathbb{Z})\to H^{2n}(\mathbb{Z}/d\mathbb{Z},\mathbb{Z})$$

is the mod m map followed by the multiplication by m/d map. The kernel of $\mathbb{Z}/m\mathbb{Z}\stackrel{m/d}{\to}(m/d)\mathbb{Z}/m\mathbb{Z}\cong\mathbb{Z}/d\mathbb{Z}$ is $\cong(\mathbb{Z}/m\mathbb{Z})/(\mathbb{Z}/d\mathbb{Z})$, so $Res\circ Inf=0$ map, as the inflation-restriction sequence predicts.

7.3 Transfer

This map is also called **corestriction**.

Let H be a subgroup of G of finite index and let $G = \cup_{i=1}^{m} g_i H$, where $S = \{g_i \mid 1 \leq i \leq m\}$ denotes a complete set of coset representatives.

For any G-module A, define the **trace** $T = T_{G/H} : A \to A$ by

$$T(a) = \sum_{s \in S} s(a), \qquad a \in A.$$

Properties of the trace:

1. T is independent of the set S of representatives choosen (so T only depends on G/H),
2. T is transitive: if $G \subset G' \subset G''$ then
$$T_{G'/G} \circ T_{G''/G'} = T_{G''/G}.$$
3. If $H = 1$ then $T : A \to A^G$ is a G-module homomorphism.
4. $T_{G/H}$ induces a homomorphism $T^*_{G/H} : Hom_H(A, B) \to Hom_G(A, B)$.

Let X_* denote the bar resolution of G. Then X_* is also a resolution of $\mathbb{Z}$ as H-modules. The trace map induces chain maps $T^*_{G/H,n} : Hom_H(X_n, A) \to Hom_G(X_n, A)$ (as in the proof of Theorem 9). This induces a map on cohomology denoted

$$Cor : H^n(H, A) \to H^n(G, A).$$

This is the **corestriction** or **transfer map** on cohomology.

There an analog of this for homology (Brown [B], §II.9).

Remark 3 *According to Brown §III.9 in [B], this does not arise from a homomorphism of pairs.*

This transfer map is also transitive. Moreover, it commutes with restriction and inflation (see Propositions 2.4.5 and 2.4.6 in Weiss [W] for the precise statements).

Lemma 13 *If H has index m in G then*

$$Cor \circ Res : H^n(G, A) \to H^n(G, A)$$

is the multiplication by m map.

proof: See Corollary 2.4.9 on page 76, in §2.4 of Weiss [W]. $\square$

There is an analog of this property for homology (see §§III.8-III.9 in Brown [B]).

Example 14 *Let $H \subset G$ be as in Example 11 above. The map*

$$Cor : H^{2n}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}) \to H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$$

is simply the multiplication by m/d map:

$$Inf : \mathbb{Z}/d\mathbb{Z} \stackrel{m/d}{\to} \mathbb{Z}/m\mathbb{Z}.$$

Note that, by Example 11, the composition map

$$Cor \circ Res : H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \to H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$$

is the multiplication by m/d map followed by the mod m map. So, $Cor \circ Res = m/d$, as the above lemma predicts.

7.4 Shapiro's lemma

Let H be a subgroup of G and let A be an H-module. Let

$$Ind_H^G(A) = \{f : G \to A \mid f(hg) = h \cdot f(g), \quad h \in H, \ g \in G\}.$$

This is the G-module A **induced from H to G**. The action of G on $Ind_H^G(A)$ is by right-translation: $g : f(x) \longmapsto f(xg)$.

In this case, the homomorphism of pairs is given by the inclusion map $\alpha : H \to G$ and the map $\beta : Ind_H^G(A) \to A$ defined by $\beta(f) = f(1)$. The map

$$(\alpha, \beta)_* : H^n(G, Ind_H^G(A)) \to H^n(H, A)$$

is the induced map defined by (13).

Theorem 15 *(Shapiro's lemma) This map $(\alpha, \beta)_*$ is an isomorphism.*

proof: See Weiss, [W], §3.7.14. □

There is an analog of Shapiro's lemma for homology (Proposition 9.5 in Brown [B]).

Acknowledgements: I thank M. Mazur and J. Feldvoss for correspondence which improved the content of these notes.

References

[A] A. Adem, "Recent devolopments in the cohomology of finite groups," Notices AMS, vol 44(1997)806-812. Available online at `http://www.ams.org/notices/199707/199707-toc.html`

[AM] A. Adem, R. Milman, **Cohomology of finite groups**, Springer-Verlag, 1994.

[B] K. Brown, **Cohomology of groups**, Springer-Verlag, 1982.

[CE] E. Cartan and S. Eilenberg, **Homological algebra**, Princeton Univ. Press, 1956.

[EM] S. Eilenberg, S. MacLane, "Relations between homology and homotopy groups," Proc. Nat. Acad. Sci. U. S. A. 29 (1943). 155–158.

[EK] G. Ellis and I. Kholodna, "Computing second cohomology of finite groups with trivial coefficients," Homology, Homotopy and Applications, vol 1 (1999) 163-168 `http://www.bangor.ac.uk/hha/volumes/1999/volume1-1.htm`

[E] L. Evens, **The cohomology of groups**, Oxford Univ. Press, 1991.

[Gap] The GAP Group, **GAP – Groups, Algorithms, and Programming, Version 4.3**, 2000 (`http://www.gap-system.org`).

[G] R. Godement, **Topologie algébrique et théorie des faisceaux**, Hermann, 1958.

[Ho] G. Hochschild, "Relative homological algebra," Trans. Amer. Math. Soc. 82 (1956), 246–269.

[K] G. Karpilovsky, **The Schur multiplier**, Oxford Univ. Press, 1987.

[L] S. Lang, **Topics in cohomology of groups**, Springer Lecture Notes in Mathematics, vol 1625, 1996.

[Mac] Macaulay 2,
`http://www.math.uiuc.edu/Macaulay2/`, especially
`http://www.math.uiuc.edu/Macaulay2/Manual/0027.html`

[MAGMA] W. Bosma, J. Cannon, C. Playoust, "The MAGMA algebra system, I: The user language," J. Symb. Comp., 24(1997)235-265.
(See also the MAGMA homepage at
`http://www.maths.usyd.edu.au:8000/u/magma/`)

[R] J. Rotman, **Advanced modern algebra**, Prentice Hall, 2002.

[S] J.-P. Serre, **Local fields**, Springer-Verlag, 1979.

[Sh] S. Shatz, **Profinite groups, arithmetic, and geometry**, Princeton Univ. Press, 1972.

[W] E. Weiss, **Cohomology of groups**, Academic Press, 1969.

Index