Invariant distributions on the n-fold metaplectic covers of $GL(r, F)$, F p-adic*

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Dedicated to John Benedetto on his 60^{th} birthday.

Abstract

We describe the unitary and tempered dual of the n-fold metaplectic covers of $SL(2, F)$, where F is a p-adic field with p not dividing $2n$. We show that any tempered distribution on the n-fold metaplectic covers of $SL(2, F)$ or of $GL(r, F)$ (satisfying the assumptions of §1.1 below) may be expressed as a distributional integral over the tempered dual. We also show that any invariant distribution on the n-fold metaplectic covers of $SL(2, F)$ or of $GL(r, F)$ is supported on the tempered dual.

Contents

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1 Introduction

Since the days of Fourier, it has been known that any "nice" function on $\mathbb{R}$ has a Fourier transform,

$$f^\wedge(\pi) = \int_{\mathbb{R}} f(x)\pi(x)\ dx,$$

where $\pi \in \{e^{sx} \mid s \in \mathbb{C}\} = \mathbb{R}^\wedge$ denotes the dual space. π is unitary if and only if $s \in i\mathbb{R}$. Let $(\mathbb{R})^\wedge_u$ denote the unitary dual space and denote the Schwartz space by

$$\mathcal{C}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid |D^n f(y)| <<_{N,n,f} (1+|y|)^{-N}\}.$$

The image of the Schwartz space under unitary Fourier transform is

$$\mathcal{C}(\mathbb{R})^\wedge_u \cong \mathcal{C}(\mathbb{R})$$

This result enables us to define, for each tempered $T \in \mathcal{C}(\mathbb{R})'$, $T^\wedge \in \mathcal{C}(\mathbb{R})'$ by $T^\wedge(f^\wedge) = T(f)$. A compactly supported tempered distribution is given by integration against some distributional derivative $D^n u$, some $n \geq 0$ and some $u \in C_c(\mathbb{R})$. These last few facts are well-known results of L. Schwartz [Sch]. Thanks, to R. Paley and N. Wiener, the image of $C_c^\infty(\mathbb{R})$ under unitary Fourier transform has also been classified (in terms of the "the Paley-Wiener space," a space of complex-anaytic functions satisfying certain boundedness conditions).

We want analogs of these results for metaplectic covers of p-adic $SL(2)$, $GL(r)$. In fact, Schwartz' classification of the compactly supported distributions will be used to prove its own p-adic analog. However, the image of the Fourier transform shall only be determined here in the $SL(2)$ case.

Kazhdan [K] has shown that if G is a connected reductive p-adic group with compact center then any (not necessarily tempered) invariant distribution on G

is supported on the tempered dual. In Theorem 23 below, we prove this result in the case of n-fold metaplectic covers $\overline{G}$ of $SL(2, F)$ or of $GL(r, F)$, as in §1.1 below. We also express the Fourier transform of any tempered distribution of $\overline{G}$ as a distributional integral over the tempered dual. Both of these results require some understanding of the tempered dual of $\overline{G}$, which may be found either in §2 below (in the $SL(2)$ case) or [FK] (in the $GL(r)$ case).

Most of this paper is written in a general context in the belief that the arguments should be valid in a more general setting. The motivation for this paper is the application to constructing the Arthur invariant trace formula on the 2-fold metaplectic cover of $SL(2, \mathbb{A})$ (or the n-fold metaplectic covers of $GL(r, \mathbb{A})$ [Me]) using the assumptions (in local harmonic analysis) of [A1]. Though some of these local assumptions remain unproven at this time for the n-fold metaplectic covers of $GL(r, \mathbb{A})$, thanks to results here, it appears that all the assumptions in local harmonic analysis have been proven now to establish the Arthur invariant trace formula on the 2-fold metaplectic cover of $SL(2, \mathbb{A})$.

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1.1 Assumptions on $\overline{G}$

Throughout most of this paper we shall denote by G the group of F-rational points of a connected reductive algebraic group $\underline{G}$ over F. We denote by $\overline{G}$ a group which is a finite cyclic central topological extension,

$$1 \to \mu_n \to \overline{G} \to G \to 1,$$

where μ_n denotes the group of n^{th} roots of unity, F contains all n^{th} roots of unity,

- $\underline{G} = SL(2)$,
- p does not divide $2n$,

or

- $\underline{G} = GL(r)$,
- p does not divide n,
- n is relatively prime to all composite positive integers less than or equal to r.

We denote the above projection by $\rho : \overline{G} \to G$.

Let F be a p-adic field with uniformizer π_F, ring of integers $\mathcal{O}_F$, residual characteristic $p = char(\mathcal{O}_F/\pi_F\mathcal{O}_F)$, $q = |\mathcal{O}_F/\pi\mathcal{O}_F|$, and normalized valuation $|...|_F$. Let

$$N = \begin{cases} n, & n \text{ odd}, \\ n/2, & n \text{ even} \end{cases} \tag{1}$$

and let $\underline{N_0}$ denote the unipotent upper triangular subgroup of $\underline{G}$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL(2, F)$, let

$$x(g) = \begin{cases} c, & c \neq 0, \\ d, & c = 0, \end{cases}$$

and let $\beta = \beta_{n,F} : G \times G \to \mu_n$ be defined by

$$\begin{aligned} \beta(g_1, g_2) &= (x(g_1), x(g_2))_n \cdot (-x(g_1)^{-1}x(g_2), x(g_1g_2))_n \\ &= (\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)})_n, \end{aligned} \tag{2}$$

where $(..., ...)_n = (..., ...)_{n,F} : F^\times \times F^\times \to \mu_n$ denotes the Hilbert symbol [W]. This cocycle defines a cover $\overline{G}$ satisfying the properties above. Elements of $\overline{G}$ will be denoted by (g, ς), where $g \in G$, $\varsigma \in \mu_n$.

The cocycle for $GL(r, F)$ is described in [FK].

1.2 Basic notation

If H is any subset of G then denote $\overline{H} = \rho^{-1}(H)$. In particular, if G_r denotes the set of regular elements of G in the sense of [K], let $\overline{G}_r$ be the pull-back of G_r via the projection ρ.

Let $\mathcal{L}(G)$ denote the set of standard Levi subgroups of G (with respect to a given maximal split torus of G). We write A for the diagonal subgroup of G. Let $\mathcal{L}(\overline{G})$ denote the set of Levis in $\mathcal{L}(G)$ pulled back to $\overline{G}$ via ρ. We call these the standard Levi subgroups of $\overline{G}$. For each $M \in \mathcal{L}(G)$, let $X(M)$ denote the variety of unramified characters of M and let $X^{un}(M)$ denote the variety of unramified unitary characters of M. If $M = \overline{A} \in \mathcal{L}(\overline{G})$, let $X(M)$ denote the variety of unramified characters of A^n (which we may identify with a character of $\overline{A^n}$) and let $X^{un}(M)$ denote the variety of unramified unitary characters of A^n. Let $W = N_G(A)/A$ denote the Weyl group of A. When $G = SL(2, F)$, we sometimes identify W (as a *set*) with $\{1, w_0\}$ or sometimes (as a group) with $\{1, w_1\}$, where

$$w_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in G, \qquad w_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and when W is to act on $\overline{A}$ instead of A, we sometimes identify (using a slight abuse of notation) W (as a *set*) with $\{1, \overline{w_0}\}$, where $\overline{w_0} = (w_0, 1)$.

We shall often identify the unipotent radical

$$N = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & \dots & 1 \end{bmatrix} \right\} \subset G \tag{3}$$

with the subgroup $\{(n,1) \mid n \in N\} \subset \overline{G}$. Let $K_0 = \underline{G}(\mathcal{O}_F)$. It is known that if $(p, 2n) = 1$ then $\overline{K_0}$ splits (see [G] for $SL(2)$ and [FK] for $GL(r)$).

We call a function f of $\overline{G}$ (resp., of any subgroup H of $\overline{G}$) **genuine** if it satisfies

$$f(g, \varsigma) = \varsigma^{-1} \cdot f(g, 1), \tag{4}$$

for all $g \in G$ (resp., $(g, \varsigma) \in H$). Let $C_c^\infty(G)$ denote the space of smooth (i.e., locally constant and compactly supported) functions on G and let $C_c^\infty(\overline{G})$ denote the space of smooth genuine functions on $\overline{G}$.

Let $||g|| = \max(|g_{ij}|)$, where $g = [g_{ij}] \in G$, and let $\sigma(g) = \log ||g||$. For each compact open subgroup $K \subset\subset \overline{G}$, let

$$\mathcal{C}_K(\overline{G}) = \left\{ f \in C_c(\overline{G}//K) \mid \begin{array}{c} f \text{ genuine}, \\ |f(x)| <<_{r,f} \frac{\Xi(x)}{(1+\sigma(x))^r}, \ \forall \overline{x} = (x, \varsigma) \in \overline{G}, \\ \text{for each } r > 0 \end{array} \right\},$$

where $C_c(\overline{G}//K)$ denotes the space of compacty supported functions which are bi-K-invariant and where

$$\Xi(x) = \int_{K_0} \delta_B(xk)^{-1/2} dk.$$

Here, δ_B denotes the usual modulus function defined for $\overline{x} = (x, \zeta) \in \overline{G_r}$ by $\delta_B(\overline{x}) = |\det(Ad(x_d))_{\mathfrak{n}}|$, where x_d denotes a diagonalization of x in $\underline{G}(\overline{F})$, where $\overline{F}$ denotes a separable algebraic closure of F and the valuation $|...|$ has been extended to $\overline{F}$, and where $\mathfrak{n}$ denotes the Lie algebra of N in (3) (more precisely, the Lie algebra of $\underline{N}(\overline{F})$). We topologize $\mathcal{C}_K(\overline{G})$ via the semi-norms

$$v_k(f) = \sup_{x \in \overline{G}} |f(x)| \frac{(1+\sigma(x))^k}{\Xi(x)}.$$

Let

$$\mathcal{C}(\overline{G}) = \bigcup_K \mathcal{C}_K(\overline{G}),$$

where K runs over all compact open subgroups of $\overline{G}$. This is the **Schwartz space** of $\overline{G}$. Let S denote the collection of all seminorms on $\mathcal{C}(\overline{G})$ whose restriction to each $\mathcal{C}_K(\overline{G})$ is continuous. In the semi-norm topology induced by S, the Schwartz space is a complete locally convex topological vector space. Moreover, $\mathcal{C}(\overline{G}) \subset L^2(\overline{G})$ and $\mathcal{C}(\overline{G})$ is an algebra under convolution.

We call a representation π of $\overline{G}$ (resp., of any subgroup H of $\overline{G}$) **genuine** if it satisfies

$$\pi(g, \varsigma) = \varsigma \cdot \pi(g, 1), \tag{5}$$

for all $g \in G$ (resp., $g \in \rho(H)$). If π denotes an admissible representation of G then let Θ_π denote the character of π. Likewise, if π denotes an admissible genuine representation of $\overline{G}$ then let Θ_π denote the character of π (see [J2], §4, for details on how the results in §4.8 of [Sil] to metaplectic covers). We may regard Θ_π as either a locally integrable genuine function on $\overline{G}_r$, or as an invariant distribution, whichever is appropriate for the context, as in [HC1], [HC2].

For each Levi component M of $\overline{G}$, let

- $\Pi(M) = \Pi(M)_a$ denote the genuine admissible dual of M (the set of equivalence classes of genuine irreducible admissible representations[1] of M),
- $\Pi(M)_t$ denote the genuine tempered dual of M,
- $\Pi(M)_d$ denote the genuine discrete series dual of M (i.e., the set of equivalence classes of genuine irreducible representations of G whose matrix coefficients are L^2 mod center - see §2 of [HC4] in the case $n = 1$),
- $\Pi(M)_u$ denote the genuine unitary dual of M,
- $\Pi(M)_c$ denote the genuine supercuspidal dual of M.

Furthermore, let

- $R_t(\overline{G})_{\mathbb{Z}}$ denote the Grothendieck group of genuine, tempered, admissible representations of $\overline{G}$ and let $R_t(\overline{G}) = R_t(\overline{G})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$,
- $PW_t(\overline{G})$ denote the **tempered Paley-Wiener space**, i.e., the space of functionals on $R_t(\overline{G})$ of the form $\phi_f : \pi \to \Theta_\pi(f)$, for some $f \in \mathcal{C}(\overline{G})$. Similarly, let $PW(\overline{G})$ denote the **Paley-Wiener space**, i.e., the space of functionals on $R(\overline{G})$ of the form $\phi_f : \pi \to \Theta_\pi(f)$, for some $f \in C_c^\infty(\overline{G})$.

[1] From this point on, all representations will be assumed to be admissible unless otherwise stated.

2 Basic lemmas on orbital integrals

If $p < \infty$ then

$$F^\times = \pi^{\mathbb{Z}} \cdot \mu_{q-1} \cdot U_1,$$

a direct product. If $(p, 2n) = 1$ then $\mu_n \subset F^\times$ implies $q \equiv 1 \pmod{n}$. Recall

$$N = \begin{cases} n, & n \text{ odd}, \\ n/2, & n \text{ even}. \end{cases}$$

Lemma 1 *Suppose A is the diagonal subgroup of $SL(2, F)$.*

(a) If $(p, N) = 1$ then $C = \pi^{\mathbb{Z}} O_F^{\times N} = \pi^{\mathbb{Z}} \mu_{q-1}^N (1 + \pi O_F)$ is a maximal subgroup of $F^\times$ for which $\overline{C} \subset \overline{A}$ is abelian.

(b) If $(p, n) = 1$ then $C = \pi^{\mathbb{Z}} (1 + \pi O_F) \mu_{q-1}^N$ has index N in $F^\times$.

The (straightforward) verification of this fact is omitted (see [J2] or [J4]).

Let

$$\begin{aligned} D_{G/A}\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) &= \det\left(1 - Ad\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right)\right)_{\mathfrak{g}/\mathfrak{a}} \\ &= (1 - a^2)(1 - a^{-2}) = -(a - a^{-1})^2, \end{aligned}$$

where $\mathfrak{g}, \mathfrak{a}$ denote the Lie algebras of G, A, resp.. This map pulls back to $\overline{G}_r$ via ρ. For $t \in \overline{G}_r$, let $T = Cent(t, \overline{G})$, denote the centralizer[2]. Define the **orbital integral** of $f \in C_c^\infty(\overline{G})$ by

$$F_f^T(t) = |D(t)|^{1/2} \int_{T \backslash \overline{G}} f(x^{-1} t x) \frac{dx}{dt}. \tag{6}$$

(This exists as a simple consequence of a well-known result of Harish-Chandra [HC].) We define D as above by identifying T with A over the algebraic closure. If $a \in \overline{A}$ is regular, then define

$$F_f^{A^N}(a) = |D(a)|^{1/2} \int_{\overline{A^N} \backslash \overline{G}} f(x^{-1} a x) \frac{dx}{da}. \tag{7}$$

Lemma 2 *Let $\overline{G}$ be a cover of $SL(2, F)$ as in §1.1.*

(a) For $f \in C_c^\infty(\overline{G})$, $a \in \overline{A} - \overline{A^N}$, we have $F_f^{A^N}(a) = 0$.

[2]Note T need not be equal to the metaplectic cover of a centralizer of G. In other words, if $t = (x, 1)$ then in general $Cent(t, \overline{G}) \neq \overline{Cent(x, G)}$.

(b) *The map* $f \longmapsto F_f^{A^N}$ *defines a surjection* $C_c^\infty(\overline{G}) \to C_c^\infty(\overline{A^N})^W$*, where the action of* W *on* $\overline{A^N}$ *is as in §1.2.*

(c) *The map* $f \longmapsto F_f^{A^N}$ *defines a continuous surjection* $\mathcal{C}(\overline{G}) \to \mathcal{C}(\overline{A^N})^W$*.*

The relatively straightforward proofs of these results will be omitted. (However, one may find detailed proofs in [J2].)

For orbital integrals on $\overline{GL(r,F)}$, we refer to chapter 1 of [FK]. For example, the analog of part (b) above follows from §I.7-I.8 in [FK].

3 Unitary and tempered dual of $\overline{SL(2,F)}$

In this section, let $\overline{G}$ be a cover of $SL(2,F)$ as in §1.1. We shall need some facts about the tempered dual of $\overline{G}$ for the main result in the next section. In particular, we recall the classification of the unitary and tempered dual of $\overline{G}$ in order to state a theorem of "Paley-Weiner type" for the unitary and tempered Fourier transforms in §4.1 below. All the results of this section are essentially in the literature in one form or another but see [J4] for more details.

For $\overline{GL(r,F)}$, the necessary results on the tempered dual may be deduced from §19 of [FK].

It is remarked in [BD], §2.2 that the arguments of [BZ1], chapter 2 carry over to finite central extensions of reductive groups over a p-adic field (see also [KP], §1.2). The arguments of [BZ2], section 2 and the corresponding sections of [Ca] also carry over to finite central extensions of split reductive groups over a p-adic field. Such results reduce the determination of the unitary dual of $\overline{G}$ down to classifying the supercuspidal representations (done in [J5] when $gcd(p,n)=1$ and [J3] for any p,n) and the constituents of the induced representations, as indicated in the following fact.

Proposition 3 *(Jacquet [BZ1], section 3.19) If* $\pi \in \Pi(\overline{G})$ *then there is a Levi* $M \in \mathcal{L}(G)$ *and a supercuspidal* $\sigma \in \Pi(\overline{M})$ *such that* π *is a constituent of* $I_{\overline{M}}(\sigma)$*.*

3.1 Principal series

If $\overline{P} = \overline{M}N$ denotes a levi decomposition of a parabolic subgroup of $\overline{G}$ and $(\sigma, W) \in \Pi(\overline{M})_c$ (which we extend to $\overline{P} = \overline{M}N$ trivially), then define $I_M(\sigma) : \overline{G} \to Aut(V)$ to be the **unitarily induced representation**: the representation of $\overline{G}$ by right translation on

$$V \;=\; \{f:\overline{G}\to W \ \text{ genuine} \mid \begin{array}{c} (1)\ \ f(mg)=\delta_P(m)^{1/2}\sigma(m)f(g), \\ \forall g\in\overline{G},\ m\in\overline{M} \\ (2)\ \text{ for some open subgp } K\subset\subset\overline{G},\ f(gk)=f(g), \\ \forall k\in K,\ g\in\overline{G}\ \} \end{array}.$$

Here δ_P denotes the modulus function in §1.2.

In general, if ρ is a representation of a group $H\subset G$ and $x\in N_G(H)$ then we let ρ^x be the representation defined by $\rho^x(h)=\rho(x^{-1}hx)$, for $h\in H$.

Let $\chi,\chi'\in\Pi(\overline{A})$ and let $\overline{w}=(w,1)$, for $w\in W$. If $\chi^{\overline{w}}\neq\chi$ for all $w\in W-\{1\}$ then we call χ **regular**. We say that χ,χ' are W**-conjugate** if $\chi'=\chi^{\overline{w}}$ for some $w\in W$. The following result was proven for metaplectic covers of $GL(r,F)$ in [KP], however it holds in the present case as well.

Lemma 4 *([BZ2], Corollary 2.13, Theorem 2.9(b)) Let $\chi,\chi'\in\Pi(\overline{A})$. If χ is regular then*

$$dim\ Hom_{\overline{G}}(I_{\overline{A}}(\chi),I_{\overline{A}}(\chi'))\leq 1,$$

with equality if and only if χ',χ are W-conjugate.

In other words, distinct W-conjugacy classes of $\chi\in\Pi(\overline{A})$ yield inequivalent representations.

Suppose that $\pi\in\Pi(\overline{G})_u$. We call π a (unitary) **principal series** representation if $\pi=I_{\overline{A}}(\chi)$ for some $\chi\in\Pi(\overline{A})_u$. These representations are tempered. In case $I_{\overline{A}}(\chi)$ is reducible and $\chi\in\Pi(\overline{A})_u$, we call the irreducible constituents **reducible principal series** (or, more precisely, **reducible principal series constituents**).

Let $\chi\in\Pi(\overline{A})$. The induced representation $I_{\overline{A}}(\chi)$ is in general not irreducible. However, we do have the following result.

Proposition 5 *(Moen [Mo2])*

- *(a) If n is even and $gcd(p,n)=1$ then $I_{\overline{A}}(\chi)$ is irreducible and unitary for all $\chi\in\Pi(\overline{A})_u$.*
- *(b) If n is odd and $gcd(p,n)=1$ then $I_{\overline{A}}(\chi)$ is irreducible and unitary for all $\chi\in\Pi(\overline{A})_u$ such that (a) $\chi=1$ or (b) $\chi^{\overline{w_0}}\neq\chi$ where $\overline{w_0}=(w_0,1)$. If $\chi^{\overline{w_0}}=\chi$ and $\chi\neq 1$ then $I_{\overline{A}}(\chi)$ is reducible and has two irreducible constituents.*

In fact, C. Moen [Mo1] explicitly computes the intertwining operators as matrices using the Kirillov model when $gcd(p,n)=1$.

Proposition 6 *$I_{\overline{A}}(\chi)$ is irreducible and unitary for all $\chi \in \Pi(\overline{A})_u$ such that $\chi^{\overline{w_0}} \neq \chi$ where $\overline{w_0} = (w_0, 1)$.*

The above result has a direct proof, based on Bruhat theory, but it can also be deduced from results in [FK].

3.2 Complementary series

In this subsection, we shall briefly review some of the results of Ariturk [Ar] and use some results of Flicker and Kazhdan [FK] to generalize them to the n-fold cover ([Ar] assumed $n = 3$ and $p > 3$). In case $n = 2$, these results were essentially known to Gelbart-Sally [GS].

We call an irreducible unitary representation π a **complementary series** representation if $\pi = I_{\overline{A}}(\chi)$ for some $\chi \in \Pi(\overline{A}) - \Pi(\overline{A})_u$. These representations are not tempered.

Let $\mu \in \Pi(\overline{C})$, $\chi = \chi_\mu = Ind_{\overline{C}}^{\overline{A}}\mu \in \Pi(\overline{A})$. If $\mu(x) = \mu_0(x)|x|^s$, for some character μ_0 of finite order and some $s \in \mathbb{C}$ then we write $s = s(\mu) = s(\chi)$.

Let $K(\mu)$ denote the space of locally constant functions $f : F \times \overline{A} \to \mathbb{C}$ such that

(i) $f(x, a_1a_2) = \mu(a_1)f(x, a_2)$, $a_1 \in \overline{C}$, $a_2 \in \overline{A}$,

(ii) $|x|\chi(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, 1)f(x, a)$ is constant for $|x|$ large.

Let $R \subset \overline{A}$ denote a complete set of representatives of $\overline{A}/\overline{C}$, and let r denote the cardinality of R. The elements $f \in K(\mu)$ may be identified with the r-tuple $(f(x, a))_{a \in R}$.

Let $V(\mu)$ denote the space of all locally constant functions $\varphi : \overline{G} \times \overline{A} \to \mathbb{C}$ such that

(i) $\varphi(g, a_1a_2) = \mu(a_1)\varphi(g, a_2)$, $a_1 \in \overline{C}$, $a_2 \in \overline{A}$,

(ii) $\varphi(a_1ng, a_2) = \delta(a_1)\varphi(g, a_2a_1)$, where $a_1 \in \overline{A}$, $a_2 \in \overline{A}$, $n \in N_0$. Here δ denotes the usual modulus function as defined in §1.2 above. For $\varphi \in V(\mu)$ and $w \in W$, define the map $T = T_w$ by

$$T\varphi(g, a) = \int_F \varphi(\overline{w} \cdot (\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, 1) \cdot g, \overline{w}a\overline{w}^{-1})dx, \quad Re(s(\mu)) > 0,$$

where $\overline{w} = (w, 1)$.

Lemma 7 *(Ariturk) T intertwines $I_{\overline{A}}(\mu)$ and $I_{\overline{A}}(\mu^{\overline{w}})$.*

This lemma does not require us to assume $gcd(p,n)=1$.

Let $L(\overline{G},\overline{B})$ denote the space of all locally constant functions φ on $\overline{G}$ such that

$$\varphi(\left(\begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix}, \varsigma\right)\cdot g) = |a|^2\varphi(g).$$

For $\varphi_1 \in V(\mu)$, $\varphi_2 \in V(\mu^w)$, the function

$$g \longmapsto \int_{\overline{A}/\overline{C}} \varphi_1(g,a)\varphi_2(g,a)da$$

belongs to $L(\overline{G},\overline{B})$. Therefore,

$$\begin{aligned} <\ \varphi_1,\varphi_2 >&= \int_{\overline{B}\backslash\overline{G}} \int_{\overline{A}/\overline{C}} \varphi_1(g,a)\varphi_2(g,a)dadg \\ &= \int_F \int_{\overline{A}/\overline{C}} \varphi_1(\overline{w}^{-1}\cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a)\varphi_2(\overline{w}^{-1}\cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a)dad^\times x \end{aligned}$$

gives a non-degenerate bilinear form on $V(\mu)\times V(\mu^{\overline{w}})$.

Lemma 8 *(Ariturk) $I_{\overline{A}}(\mu^{\overline{w}})$ is the contragredient of $I_{\overline{A}}(\mu)$.*

This lemma does not require us to assume $gcd(p,n)=1$.

For $f \in K(\mu)$, define the Fourier transform of f by

$$f^\wedge(x,a) = \int_{F^\times} f(y,a)\psi(xy)dy,$$

where ψ is a fixed additive character of F.

Lemma 9 *(Ariturk) Assume $gcd(p,n)=1$. For φ_1, $\varphi_2 \in V(\mu)$, $\mu(x)=|x|^s$, we have*

$$<\varphi_1, T\varphi_2> = \int_F \int_{\overline{A}/\overline{C}} f_1^\wedge(x,a)\overline{(Jf_2^\wedge)(x,a)}dadx,$$

where $J=J_\mu$ is a linear transformation on $K(\mu)^\wedge$ and

$$f_i(x,a) = \varphi_i(\overline{w}^{-1}\cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a), \quad i=1,2.$$

We may identify the map $J=J_\mu$ defined in the above lemma with an $r\times r$ matrix which we still denote by J.

Lemma 10 *(Langlands, Ariturk) Assume* $gcd(p, 2n) = 1$. *If* $0 \leq Re(s(\mu)) \leq 1/n$ *and* $|Im(s(\mu))| \leq \pi/n \ln(q)$ *then the image of* J_μ *is an irreducible representation of* $\overline{G}$.

Proposition 11 *(Flicker-Kazhdan) If* $0 < s(\mu) < 1/n$ *then* $I_{\overline{A}}(\mu)$ *is a unitarizable representation of* $\overline{G}$.

Corollary 12 *If* $0 < s < 1/n$ *and* $\mu(x) = |x|^s$ *then* $< \varphi_1, T\varphi_2 >$ *is a positive definite form. In particular,* $I_{\overline{A}}(\mu)$ *is unitary in this range. If, in addition,* $gcd(p, 2n) = 1$ *then* J_μ *is a positive definite matrix.*

Remark 1 *In the case* $n = 2$, *this follows from [GS], Proposition 1 following the argument of [Ar]. See also [Mo1] if* $p > 2$ *and [G], §5.4. In the case* $n = 3$, $p > 3$, *this was proven in [Ar].*

This corollary shows that some of the results of Ariturk [Ar] generalize to the n*-fold cover case without any condition on the prime* p.

3.3 The special and the "trash" representations

For the origin of the term "trash" representation, see [G].

As a consequence of the above-mentioned facts, we have the following result.

Proposition 13 *Let* $s = 1/n$ *and* $\mu(x) = |x|^s$.

(a) *The irreducible subrepresentation of* $I_{\overline{A}}(\mu)$ *(if* $gcd(p, 2n) = 1$, *the kernel of* J_μ*) is the "special" representation* π_{sp}. *It is tempered and square-integrable (hence unitary). If* $gcd(p, n) = 1$ *then it also contains an Iwahori fixed vector.*

(b) *If* $n > 1$ *then the irreducible quotient of* $I_{\overline{A}}(\mu)$ *(if* $gcd(p, 2n) = 1$, *image of* J_μ*) is an infinite-dimensional, non-tempered representation* π_{nt}. *If* $gcd(p, n) = 1$ *then it is also spherical.*

Remark 2 *In the case* $n = 3$ *and* $p > 3$, *this proposition follows from [Ar]. In case* $n = 2$, *most of the statements are proven in [GS].*

Proposition 14 *(Kazhdan-Patterson [KP])* [3] π_{nt} *is unitary.*

Remark 3 *This was known earlier in the cases* $n = 2$ *([GS], Theorem 2) and* $n = 3$, $p \neq 3$ *([Ar], Theorem 5.4).*

[3] This was originally only a conjecture in [J4]. An anonomous referee of [J4] pointed out that it followed from [KP].

3.4 Classification

We summarize the above results.

Theorem 15 *Let $\overline{G}$ be as in §1.1 above. If $\pi \in \Pi(\overline{G})_u$ then one of the following holds.*

- *(Principal series) There is a $\chi \in \Pi(\overline{A})_u$ such that ($\chi^{\overline{w_0}} \neq \chi$ and) $\pi = I_{\overline{A}}(\chi)$, as in §2.2.*
- *(Complementary series) There is a $\chi \in \Pi(\overline{A}) - \Pi(\overline{A})_u$ such that $\pi = I_{\overline{A}}(\chi)$, as in §2.3.*
- *("Reducible principal series") There is a $\chi \in \Pi(\overline{A})_u$ such that ($\chi^{\overline{w_0}} = \chi$ and) π is either a subrepresentation or a quotient of $I_{\overline{A}}(\chi)$, as in §2.2.*
- *π is a "special" or "trash" representation as in §2.4.*
- *π is a supercuspidal representation as in [J5].*

Theorem 16 *Let $\overline{G}$ be as in §1.1 above. If $\pi \in \Pi(\overline{G})_t$ then one of the following holds.*

- *(Principal series) There is a $\chi \in \Pi(\overline{A})_u$ such that ($\chi^{\overline{w_0}} \neq \chi$ and) $\pi = I_{\overline{A}}(\chi)$, as in §2.2.*
- *("Reducible principal series") There is a $\chi \in \Pi(\overline{A})_u$ such that ($\chi^{\overline{w_0}} = \chi$ and) π is either a subrepresentation or a quotient of $I_{\overline{A}}(\chi)$, as in §2.2.*
- *π is a "special" representation as in §2.4.*
- *π is a supercuspidal representation as in [J5].*

4 Invariant distributions

We classify the image of $C_c^\infty(\overline{G})$ and of $\mathcal{C}(\overline{G})$ under the "scalar-valued Fourier transform" or "trace map", $f \longmapsto \operatorname{tr}\pi(f)$, where $\overline{G}$ is as in §1.1 above. We prove that all invariant distributions on $\overline{G}$ are supported on tempered characters, where $\overline{G}$ is either a cover of $SL(2,F)$ or a cover of $GL(r,F)$ as in §1.1 Finally, we show that, for $\overline{G}$ as in §1.1, we can write any invariant tempered distribution D on $\overline{G}$ as an integral on the tempered dual.

4.1 Tempered Paley-Wiener theorem

In this section, $\overline{G}$ is a cover of $SL(2,F)$ as in §1.1. Let $C \subset A$ be a maximal subgroup of A for which $\overline{C} \subset \overline{A}$ is abelian.

Next we classify the image of the Fourier transforms of a "generic" *unitary* principal series representation

$$\pi = I_{\overline{A}}(\chi), \quad \chi = Ind_{\overline{C}}^{\overline{A}}\mu \in \Pi(\overline{A}), \quad \mu \in \Pi(\overline{C}),$$

on $C_c^\infty(\overline{G})$. Note that both the Weyl group W and $H = \overline{A}/\overline{C}$ act on $C_c(\Pi(\overline{C}))$ by conjugation and $\phi_f \in C_c(\Pi(\overline{C}))$. Let $C_c(\Pi(\overline{C}))^H$ denote the subspace of H-invariant functions and let $C_c(\Pi(\overline{C}))^{WH}$ denote the subspace of functions which are both H-invariant and W-invariant.

Lemma 17 *In the notation above, for each $f \in C_c^\infty(\overline{G})$, $\phi_f : \mu \longmapsto \mathrm{tr}\, I_{\overline{A}}(\chi)(f)$ belongs to $C_c(\Pi(\overline{C}))^{WH}$.*

proof: The map

$$\begin{array}{rcl} f & \longmapsto & f^H, \\ f^H(\chi) & = & \frac{1}{|H|}\sum_{h\in H} f(\chi^h), \end{array}$$

defines a surjection $C_c(\Pi(\overline{C})) \to C_c(\Pi(\overline{C}))^H$. The H-invariance of $I_{\overline{A}}(\chi)$ implies $\phi_f \in C_c(\Pi(\overline{C}))^H$, for $f \in C_c^\infty(\overline{G})$. The W-invariance follows from Lemma 4. □

Recall the Fourier transform with respect to the principal series,

$$\phi_f(\mu) = \Theta_\pi(f), \quad \pi = I_{\overline{A}}(\chi), \quad \chi = Ind_{\overline{C}}^{\overline{A}}\mu \in \Pi(\overline{A}), \quad \mu \in \Pi(\overline{C}).$$

When μ is unitary we call this the Fourier transform with respect to the *unitary principal series*. If the restriction of μ to an $diag(x, x^{-1}) \in A^n$ is of the form $|x|^s$ then we write $\phi_f(\mu) = \phi_f(s)$. When μ is of this form and s is real, we call this the Fourier transform with respect to the *complementary series*.

Proposition 18 *For $f \in C_c^\infty(\overline{G})$, the image $C_c^\infty(\overline{G})^\wedge_{ps}$ of the Fourier transform $f \longmapsto \phi_f$ with respect to the unitary principal series, is given by*

$$C_c^\infty(\overline{G})^\wedge_{ps} = \{h \in C_c(\Pi(\overline{C})_u)^{WH} \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } \Pi(\overline{C})_u \end{array} \}.$$

The image $C_c^\infty(\overline{G})^\wedge_{cs}$ of the Fourier transform $f \longmapsto \phi_f$ with respect to the complementary series, is given by

$$C_c^\infty(\overline{G})^\wedge_{cs} \cong \{h \in C_c^\infty(\Pi(\overline{C}))^{WH} \text{ restricted to } 0 < s < 1/n, \text{ a polynomial in } q^s \}.$$

This follows from character formulas for induced representations and from results on p-adic Mellin transforms in [Ta], pp. 43-44.

Analogous to Proposition 18 above, we have the following result.

Proposition 19 *For $f \in \mathcal{C}(\overline{G})$, the image $\mathcal{C}(\overline{G})^{\wedge}_{ps}$ of the Fourier transform $f \longmapsto \phi_f$ with respect to the unitary principal series, is given by*

$$\mathcal{C}(\overline{G})^{\wedge}_{ps} = C_c^{\infty}(\Pi(\overline{C})_u)^{WH}.$$

4.2 The Fourier transform

Let

$$J = \mathrm{span}\{f - f^g \mid f \in C_c^{\infty}(\overline{G}) \ , g \in \overline{G}\},$$

and recall $PW_t(\overline{G})$ denotes the tempered Paley-Wiener space.

Proposition 20 *Let $\overline{G}$ be as in §1.1. The kernel of the trace map*

$$\wedge : C_c^{\infty}(\overline{G}) \to PW_t(\overline{G}),$$

defined by $f^{\wedge}(\pi) = \Theta_{\pi}(f)$, is J.

Remark 4 *When $n = 1$, the result above is a special case of [K], Theorem 0. Moreover, in this case Arthur ([A2], page 175) showed (in particular) that the trace map $\mathcal{C}(\overline{G}) \to PW_t(\overline{G})$ is a continuous (in a natural topology on $PW_t(\overline{G})$) linear transformation. We note that his proof of this part of his main result easily extends to the case $n > 1$.*

proof: For $\overline{G}$ as in §1.1, Vignéras (see Proposition 3.2 and §2.3 in [Vig]) showed that the kernel of the orbital integral map

$$\Phi : C_c^{\infty}(\overline{G}_r) \to C_c^{\infty}(\overline{G}_r)^{\overline{G}}$$

is J. Let K denote the kernel of the trace map $\wedge : C_c^{\infty}(\overline{G}) \to PW_t(\overline{G})$. The Weyl integration formula implies $J \subset K$. Theorem 19.2 of Flicker and Kazhdan [FK] [4] implies $K \subset J$, if $\overline{G}$ is a metaplectic cover of $GL(r, F)$. If $\overline{G}$ is a cover of $SL(2, F)$ as in §1.1 above then probably the argument is analogous - hence perhaps could

[4]This section of [FK] uses the global trace formula, hence requires the assumption that n is relatively prime to all composite positive integers less than or equal to r and to the residual characteristic over F.

be left at that - but we give an alternative (and much simpler) argument below in this special case.

Fourier transforms of Harish-Chandra transforms and character formulae for induces representations (implicit in §2.1 above but see also explicit formulas in [J4] or [J2], for example) show that if $f \in K$ then $F_f^{AN}(a) = 0$. It remains to show that if $f \in K$ then $F_f^T(t) = 0$, where $T = Cent(t, \overline{G})$, is the centralizer of a regular elliptic element (see (6) above). The desired $F_f^T(t) = 0$ follows from (3.43) in [J1]. $\square$

Let V' denote the dual (consisting of linear functionals) of the complex vector space V. If V is in addition a $\overline{G}$-module, let $(V')^{\overline{G}}$ denote the subspace of $\overline{G}$-invariant linear functionals.

Lemma 21 *Let $\overline{G}$ be as in §1.1. The canonical map*

$$(C_c^\infty(\overline{G})/J)' \to (C_c^\infty(\overline{G})')^{\overline{G}}$$

is an isomorphism.

proof: For $f \in C_c^\infty(\overline{G})$, let $f \bmod J$ denotes its class in $C_c^\infty(\overline{G})/J$. First, note that the canonical map

$$\begin{array}{c} (C_c^\infty(\overline{G})/J)' \to (C_c^\infty(\overline{G})')^{\overline{G}} \\ D \longmapsto D^* \\ (f \bmod J \longmapsto D(f \bmod J)) \longmapsto (f \longmapsto D(f \bmod J)). \end{array}$$

is injective by definition.

To see that this is surjective, let $D \in (C_c^\infty(\overline{G})')^{\overline{G}}$. We must show that there is a $D_0 \in (C_c^\infty(\overline{G})/J)'$ such that $D = D_0^*$. Let

$$D_0(f \bmod J) = D(f), \qquad f \in C_c^\infty(\overline{G}).$$

We want to show that D_0 is a well-defined distribution, i.e., that if $f, f' \in C_c^\infty(\overline{G})$ and $f \bmod J = f' \bmod J$ then $D(f) = D(f')$. By definition of J, $f \bmod J = f' \bmod J$ implies $f' = f + \sum_{i \in I} c_i(f_i - f_i^{g_i})$, for some finite set I and some $c_i \in \mathbb{C}$, $f_i \in C_c^\infty(\overline{G})$, $g_i \in \overline{G}$. Since D is invariant, from linearity it follows that $D(f) = D(f')$, as desired. Therefore, the canonical map is surjective. $\square$

From these two results, we conclude the following important fact.

Theorem 22 *Let $\overline{G}$ be as in §1.1 above. The trace map*

$$\wedge : C_c^\infty(\overline{G}) \to PW_t(\overline{G})$$

factors through the canonical map

$$(C_c^\infty(\overline{G})/J)' \to (C_c^\infty(\overline{G})')^{\overline{G}}.$$

In other words, each invariant distribution is supported on tempered characters.

Remark 5 *(1) This is one of the assumptions needed to extend [A1] to the metaplectic covers of* $\overline{SL(2,F)}$, $\overline{GL(r,F)}$.

(2) In the case of $\overline{GL(r,F)}$, *this is the remark following Theorem 19.2 in [FK].*

This allows us to define, for each invariant distribution D on G, the **Fourier transform** $D^\wedge$ on $PW_t(\overline{G})$ by

$$D(f) = D^\wedge(f^\wedge), \qquad f \in C_c^\infty(\overline{G}). \tag{8}$$

By the results of §2 above and of [FK], the tempered dual has both a continuous part and a discrete part. The continuous part of $PW_t(\overline{G})$ decomposes into a vector space sum of smooth functions on compact real tori. It is noted for later reference that if $D^\wedge$ induced, by restriction, a distribution on each of these spaces of smooth functions then D must be tempered.

4.3 The Fourier transform as an integral over $\Pi(\overline{G})_t$

Note that any function m on the tempered dual $\Pi(\overline{G})_t$ extends by linearity to the Grothendieck group $R(\overline{G})_t$ (defined in [K] in the algebraic case; in the metaplectic case the definition is similar).

In the case $n = 1$, let $d\omega$ denote the canonical measure on the discrete dual of G as in §2 of [HC4]. The discrete dual has the structure of the disjoint union of compact real manifolds $\mathcal{O}$. We use Corollary 4.5.11 and Theorem 4.6.1 in [Sil], we extend this measure to $\Pi(G)_t$, which is parameterized by (a dense subset of) the discrete dual.

In case $G = GL(r,F)$, we use the correspondence between $\Pi(G)_t$ and $\Pi(\overline{G})_t$ proven in §19 of [FK] to pull these parameters and measures on $\Pi(G)_t$ back to $\Pi(\overline{G})_t$. In case $G = SL(2,F)$, we use the correspondence between $\Pi(G)_t$ and $\Pi(\overline{G})_t$ proven above to pull these parameters and measures on $\Pi(G)_t$ back to $\Pi(\overline{G})_t$. Let $d\mu$ denote the measure on the tempered dual $\Pi(\overline{G})_t$ corresponding to $d\omega$. Let $m(\pi)d\mu(\pi)$ denote a distribution on the tempered Paley-Wiener space $PW_t(\overline{G})$ such that

1. $m(\pi)$ is supported on finitely many orbits $\mathcal{O} = \mathcal{O}_\sigma$, for $\sigma \in \Pi(M)_d$ and some Levi M of $\overline{G}$,

2. there is a continuous function h on $\Pi(\overline{G})_t$ such that on each orbit $\mathcal{O} = \mathcal{O}_\sigma$, with σ, M as above, such that as distributions on $C_c^\infty(\mathcal{O})$, we have
$$m(I_M(\omega\sigma)) = \frac{\partial^I}{\partial\omega^I} h(\omega),$$
where $I = (i_1, ..., i_r)$ denotes a multi-index, r being the real dimension of $\mathcal{O}$, and $\frac{\partial^I}{\partial\omega^I}$ denotes partial differentiation on the real manifold $\mathcal{O}$.

We call such a distribution a **distribution of finite type** on $\Pi(\overline{G})_t$. A distribution satisfying (2) but not (1) will be called a **distribution of quasi-finite type** on $\Pi(\overline{G})_t$. The maximum of the integers $|I| = i_1 + ... + i_r$, where I runs over all multi-indices occurring in (2), is called the **order** of the distribution.

Theorem 23 *Let $\overline{G}$ be as in §1.1 above. If D is an invariant tempered distribution on $C_c^\infty(\overline{G})$ then there is a distribution of quasi-finite type $m(\pi)d\mu(\pi)$ on the tempered Paley-Wiener space $PW_t(\overline{G})$ such that*
$$D(f) = \int_{\Pi(\overline{G})_t} \Theta_\pi(f) m(\pi) d\mu(\pi), \qquad f \in C_c^\infty(\overline{G}).$$
This formula extends continuously to all of $\mathcal{C}(G)$.

Remark 6 *If we replace $\mathcal{C}(\overline{G})$ by $\mathcal{C}_K(\overline{G})$ in the second part of the above theorem then we can replace quasi-finite by finite.*

proof: First, we know from Theorem 22 that D is supported on the tempered dual.

We claim that the tempered dual is contained in the unitary dual. In the $SL(2)$ case, see [J2] for the detailed case-by-case proof using the classification of the irreducible admissible representations of $\overline{G}$. In the $GL(r)$ case, see §§16-17 of Flicker-Kazhdan [FK] [5]. Therefore D arises from a distribution $D^\wedge$ on $\Pi(\overline{G})_t$. The above theorem is now an immediate consequence of equation (8), and the classification of L. Schwartz ([Sch], ch. III, Th. XXI) which we state in the present notation as follows.

Lemma 24 *(Schwartz) Let $\mathcal{O}$ be an orbit as above. If $T \in C_c^\infty(\mathcal{O})'$ then there is a continuous function h on $\mathcal{O}$ and a multi-index $I = (i_1, ..., i_m)$, $i_j \geq 0$ such that $T = \frac{\partial^I}{dx^I} h$ (as distributions), where $x = (x_1, ..., x_m)$ is a coordinate on $\mathcal{O}$.*

This completes the proof of the Theorem. □

[5] These sections of [FK] do *not* use the global trace formula, hence only requires the assumption that n is relatively prime to the residual characteristic of F.

4.4 Some corollaries

By Theorem 22, the Fourier transform of each invariant distribution is support on the tempered dual.

Corollary 25 *Let $\overline{G}$ be as in §1.1 above. If D is tempered then the Fourier transform $D^\wedge$ may be expressed in the form*

$$D^\wedge(h) = \int_{\Pi(\overline{G})_t} h(\pi)m(\pi)d\mu(\pi),$$

for all $h \in PW_t(\overline{G})$, where $m(\pi)d\mu(\pi)$ is a quasi-finite distribution.

It is natural to ask for a more explicit characterization of the admissible distributions [HC2]. The result below uses the above corollary to basically reduce the question of admissibility down to the behaviour of the distribution near the singular set.

Corollary 26 *Assume $\overline{G} = G$ $(n = 1)$. If D is an tempered then it is admissible on the regular set. In other words, if we identify $C_c^\infty(G_r)$ with the following subspace of $C_c^\infty(G)$,*

$$C_c^\infty(G_r) = \{f : G \to \mathbb{C} \mid f|_{G-G_r} = 0,\ f \in C_c^\infty(G_r)\},$$

then $D|_{C_c^\infty(G_r)}$ is admissible.

Remark 7 *This result cannot be extended from G_r to all of G since, for example, the distribution $f \longmapsto f(1)$ is not admissible on $\overline{G}$. However, as a distribution on $C_c^\infty(G_r)$, it is zero, hence admissible on G_r.*

proof: By hypothesis,

$$D(f) = D^\wedge(f^\wedge) = \int_{\Pi(G)_t} f^\wedge(\pi)m(\pi)d\mu(\pi),$$

where m is of quasi-finite type. For a fixed $x \in G$ regular, we may regard the character $\Theta_\pi(x)$, as a function of $\pi \in \Pi(G)_t$, as an element of $PW_t(G)$. Therefore,

$$F_D : x \longmapsto \int_{\Pi(G)_t} \Theta_\pi(x)m(\pi)d\mu(\pi),$$

defines a locally constant function on the regular set of G. From this it follows, by Fubini's theorem, that D is representable by a locally constant function on G_r: for any $f \in C_c^\infty(G_r)$, we have

$$D(f) = \int_G \int_{\Pi(G)_t} \Theta_\pi(x) m(\pi) d\mu(\pi) f(x) dx.$$

Let $\gamma \in G$ be regular, $U \subset G_r$ be an open set, and $K \subset G$ be a compact open subgroup such that $\gamma K \subset U$. Since characters are admissible, by [HC2], for any irreducible complex representation ρ of K with character χ_ρ we have $(\Theta_\pi * \chi_\rho^0)(x) = 0$, for all $x \in G$ regular. Here χ_ρ^0 denotes the function on G which is the extension by 0 of χ_ρ on K. We want to show that the same condition holds for D. The above formula and Fubini's theorem give,

$$\begin{aligned}(D * \chi_\rho^0)(f) &= \textstyle\int_G \int_{\Pi(G)_t} \Theta_\pi(x) m(\pi) d\mu(\pi) (f * \chi_\rho^0)(x) dx \\ &= \textstyle\int_K \int_G \int_{\Pi(G)_t} \Theta_\pi(x) m(\pi) d\mu(\pi) f(xk^{-1}) \chi_\rho^0(k) dx dk \\ &= \textstyle\int_K \int_G \int_{\Pi(G)_t} \Theta_\pi(xk) \chi_{\tilde\rho}^0(k^{-1}) m(\pi) d\mu(\pi) f(x) dx dk \\ &= \textstyle\int_G \int_{\Pi(G)_t} (\Theta_\pi * \chi_{\tilde\rho}^0)(x) m(\pi) d\mu(\pi) f(x) dx \\ &= 0,\end{aligned}$$

where $\tilde\rho$ denotes the contragredient. It follows that D is admissible on the regular set. □

Example 27 *Clearly $f \longmapsto f(1)$ is a tempered distribution. Theorem 23 implies that there is a quasi-finite m such that*

$$f(1) = \int_{\Pi(\overline{G})_t} \Theta_\pi(f) m(\pi) d\mu(\pi), \qquad f \in C_c^\infty(\overline{G}).$$

This is a weak case of Harish-Chandra's Plancherel theorem.

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