# Principal Ideals and Associate Rings

Dennis Spellman
Saint Joseph's University
Philadelphia, PA 19124

Georgia M. Benkart
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706

Anthony M. Gaglione *
W. David Joyner
Mark E. Kidwell
Mark D. Meyerson
William P. Wardlaw*
United States Naval Academy
Annapolis, MD 21402

**Abstract**

A commutative ring $A$ with 1 is **associate** provided whenever two elements $a$ and $b$ generate the same principal ideal there is a unit $u$ such that $ua = b$. The main results proved here are:

- Every commutative Noetherian ring with 1 is a subdirect product of rings which have the property that all their unital subrings are associate.

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- Every commutative ring embeds into an associate ring.
- Every commutative unital algebraically closed or principal ideal ring is associate.
- The direct sum of associate rings is associate.

One novel feature here is that we prove these results using model theory. For comparison and completeness, the main result is also sketched using a localization argument.

# 1 Ring Theoretic Preliminaries

A **ring** here shall always have an associative multiplication together with a multiplicative identity $1 \neq 0$. Subrings are required to contain the identity 1 of the over ring in question and ring homomorphisms are required to respect unities. The **unrestricted direct sum** of an indexed family $\{A_i\}_{i \in I}$ of rings is the set of all choice functions

$$\bigoplus_{i \in I} A_i = \left\{ a \mid a : I \to \bigcup_{i \in I} A_i \text{ with } a(i) \in A_i, \text{for all } i \in I \right\} \tag{1}$$

with operations defined componentwise. When we refer to the direct sum of rings, we shall always mean the unrestricted direct sum given in (1), especially in the case where I is infinite. Following McCoy [Mc] a subring $A$ of $\bigoplus_{i \in I} A_i$ is a **subdirect sum** provided for each fixed $j \in I$, projection onto the j-th coordinate remains an epimorphism with image $A_j$ when restricted to $A$.

If $\mathcal{X}$ is a nonempty class of rings closed under isomorphism, then we call the ring $A$ **residually-$\mathcal{X}$** provided to every nonzero element $a \neq 0$ in $A$ there is a ring $B_a \in \mathcal{X}$ and an epimorphism $\phi_a : A \to B_a$ which does not annihilate $a$ (see [C], page 101). An immediate consequence of the treatment in McCoy (see [Mc], Theorem 3.6, page 52) is

**Lemma 1** *Let $\mathcal{X}$ be a nonempty class of rings closed under isomorphism. The ring $A$ is residually-$\mathcal{X}$ if and only if $A$ is embeddable as a subdirect sum of an indexed family $\{A_i\}_{i \in I}$ of rings such that $A_i \in \mathcal{X}$ for all $i \in I$.*

It is easy to see that

**Lemma 2** *Let $\mathcal{X}$ be a nonempty class of rings closed under isomorphism. The ring $A$ is residually-$\mathcal{X}$ if and only if $\bigcap K = \{0\}$, where the intersection is taken over all two-sided ideals $K$ such that $A/K \in \mathcal{X}$.*

A ring $A$ is called the **direct union** of a family $\mathcal{F} = \{A_\alpha\}_{\alpha \in \Lambda}$ of subrings of $A$ if and only if $\mathcal{F}$ satisfies the following two properties:

(i) For each $a \in A$ there is at least one $\alpha \in \Lambda$ such that $a \in A_\alpha$, and

(ii) Whenever $\alpha, \beta \in \Lambda$ there is at least one $\gamma \in \Lambda$ such that $A_\alpha \subseteq A_\gamma$ and $A_\beta \subseteq A_\gamma$.

Note that (i) is equivalent to $A = \bigcup_{\alpha \in \Lambda} A_\alpha$. We call $A$ **finitely generated** provided that, as an associative $\mathbb{Z}$-algebra with 1, it has at least one finite set of generators. It is easy to see that every ring is the direct union of its finitely generated subrings (see [G], Lemma 3, page 130). Furthermore, a finitely generated commutative ring is a homomorphic image of a polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ (see [L], page 60) and so, as an immediate consequence of the Hilbert Basis Theorem, must be Noetherian. It follows that every commutative ring is the direct union of a family of Noetherian subrings.

Finally a commutative ring $A$ is **algebraically closed** provided given any commutative extension $E \supseteq A$ it is the case that any finite system

$$p_i(X_1, \ldots, X_n) = 0, \quad i = 1, \ldots, k \qquad (p_i \in A[X_1, \ldots, X_n], \quad 1 \le i \le k)$$

of equations which has a solution $(X_1, \ldots, X_n) = (e_1, \ldots, e_n) \in E^n$ must already have a solution $(X_1, \ldots, X_n) = (a_1, \ldots, a_n) \in A^n$ (see [H], page 47). It follows from Theorem 3.2.1 of Hodges (see [H], page 48), that every commutative ring can be embedded in an algebraically closed commutative ring.

# 2 Model Theoretic Preliminaries

Let $L_{Ring}$ be the first-order language whose only relation symbol is $=$, always to be interpreted as the identity relation, and whose only function and constant symbols are: two binary operation symbols $+$ and $\cdot$, a unary operations symbol $-$, and two constant symbols 0 and 1. (For a more complete description of such a language see [BS] or [CK].) Observe that the axioms for a commutative ring can be formalized within $L_{Ring}$. Recall that a **sentence** of $L_{Ring}$ is a formula of $L_{Ring}$ containing no free occurrences of any variable.

We define two increasing chains of sets of formulas of $L_{Ring}$ ,

$$\Sigma_0 \subseteq \Sigma_1 \text{ and } \Pi_0 \subseteq \Pi_1 \subseteq \Pi_2$$

as follows.

A formula is in $\Pi_0$ if and only if it is logically equivalent to a quantifier free formula of $L_{Ring}$. A formula is in $\Sigma_0$ if and only if it is in $\Pi_0$. A formula is in $\Pi_1$ if either it is in $\Pi_0$ or is logically equivalent to a formula of the form $\forall \bar{x} \phi(\bar{x})$, where $\bar{x}$ is a tuple of variables and $\phi(\bar{x})$ is in $\Sigma_0$. A formula is in $\Sigma_1$ if either it is in $\Sigma_0$ or is logically equivalent to a formula of the form $\exists \bar{x} \phi(\bar{x})$, where $\bar{x}$ is a tuple of variables and $\phi(\bar{x})$ is in $\Pi_0$. A formula is in $\Pi_2$ if either it is in $\Pi_1$ or is logically equivalent to a formula of the form $\forall \bar{x} \phi(\bar{x})$, where $\bar{x}$ is a tuple of variables and $\phi(\bar{x})$ is in $\Sigma_1$. A formula which is simultaneously a $\Pi_n$-formula and a sentence of $L_{Ring}$ is a **$\Pi_n$-sentence** of $L_{Ring}$. Similarly, a formula which is simultaneously a $\Sigma_n$-formula and a sentence of $L_{Ring}$ is a **$\Sigma_n$-sentence** of $L_{Ring}$.

A **universal sentence** is one of the form $\forall \bar{x} \phi(\bar{x})$, where $\bar{x}$ is a tuple of variables and $\phi(\bar{x})$ is a quantifier free formula of $L_{Ring}$. An **existential sentence** is one of the form $\exists \bar{x} \phi(\bar{x})$ with similar provisos. A **universal-existential sentence** of $L_{Ring}$ is one of the form $\forall \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})$ with similar provisos.

Allowing vacuous quantifications and regarding each of $\forall x \phi$ and $\exists x \phi$ as logically equivalent to $\phi$ if the variable $x$ does not occur in the formula $\phi$ , we may assert that a sentence of $L_{Ring}$ is $\Pi_1$ if and only if it is logically equivalent to a universal sentence of $L_{Ring}$, a sentence of $L_{Ring}$ is $\Sigma_1$ if and only if it is logically equivalent to an existential sentence of $L_{Ring}$, and that a sentence of $L_{Ring}$ is $\Pi_2$ if and only if it is logically equivalent to a universal-existential sentence of $L_{Ring}$.

A set T of sentences of a language is called a **theory** in the language. Henceforth we tacitly assume all sets of sentences of $L_{Ring}$ under consideration contain the ring axioms. A **model** of a theory $T$ of $L_{ring}$ is a ring satisfying all the sentences of $T$ (see [H], page 9). Let $T$ be a theory of $L_{Ring}$. $T$ is said to be **inconsistent** if at least one **contradiction**, i.e., a formula of the form $\psi \wedge \sim \psi$, is a formal consequence of $T$ ; otherwise, $T$ is said to be **consistent**. The Gödel-Henkin Completeness Theorem asserts that $T$ is consistent if and only if $T$ has a model (see [BS], page 102, or [CK]). In the event that $T$ is a consistent theory in $L_{Ring}$ containing the ring axioms, we may form the **model class** $\mathbb{M}(T)$ consisting of all rings satisfying all of the sentences $T$ and be assured that $\mathbb{M}(T)$ is not empty. It will be convenient for us to summarize here some of the preservation theorems of classical model theory. (See [CK] and [G]).

**Lemma 3** *Let $T$ be a consistent set of sentences of $L_{Ring}$. Then*

1. *$\mathbb{M}(T)$ is closed under the formation of subrings if and only if there is a consistent set $S$ of $\Pi_1$-sentences of $L_{Ring}$ such that $\mathbb{M}(T) = \mathbb{M}(S)$.*

2. *$\mathbb{M}(T)$ is closed under the formation of extensions if and only if there is a consistent set $S$ of $\Sigma_1$-sentences of $L_{Ring}$ such that $\mathbb{M}(T) = \mathbb{M}(S)$.* [1]

3. *$\mathbb{M}(T)$ is closed under the formation of direct unions if and only if there is a consistent set $S$ of $\Pi_2$-sentences of $L_{Ring}$ such that $\mathbb{M}(T) = \mathbb{M}(S)$.*

Lemma 3 may be paraphrased by asserting that a model class is closed under

1. Subrings if and only if it has a set of universal axioms;

2. Extensions if and only if it has a set of existential axioms;

3. Direct unions if and only if it has a set of universal-existential axioms.

Recall that $\Pi_1 \subseteq \Pi_2$ (or that universal sentences are considered degenerate cases of universal-existential sentences). We see that every class axiomatized by a consistent set of $\Pi_1$-sentences (or universal sentences, if one prefers) of $L_{Ring}$ is closed under the formation of direct unions.

Given a consistent theory $T$ in $L_{Ring}$ we let $T_\forall$ be the set of all $\Pi_1$-sentences of $L_{Ring}$ which are formal consequences of $T$. $T_\forall$ is surely consistent since every model $A$ of $T$ must also be a model of $T_\forall$. Furthermore, $\mathbb{M}(T_\forall)$ is closed under the formation of subrings since it has a set of $\Pi_1$-axioms. Thus, every subring of a model of $T$ is a model of $T_\forall$. The converse is Corollary 3.1.2. of Hodges (see [H], page 37), which we here record as a lemma.

[1] This is to be interpreted in the present context as asserting that other than the ring axioms the remaining sentences in $S$ (if any) are all in $\Sigma_1$. Moreover by extensions, of course, we mean ring extensions. (One direction of this requires Corollary 4, page 273, of [G].)

**Lemma 4** *: If $T$ is a consistent set of sentences of $L_{Ring}$ then $\mathbb{M}(T_\forall)$ consists of precisely the subrings of models of $T$.*

# 3 Associate and Superassociate Rings

**Definition 5** *Let $A$ be a commutative ring with identity. $A$ is* **associate** *provided whenever $a, b \in A$ and $Aa = Ab$ there is a unit $u \in A$ such that $ua = b$. The associate ring $A$ is* **superassociate** *provided every subring of $A$ is associate.*

**Remark 1** *(A) We will give an example of an associate ring which is not superassociate.*

*(B) Every integral domain is superassociate.*

*(C) Exercises requesting the construction of nonassociate rings have appeared earlier in the literature as have explicit constructions of nonassociate rings. (See [A], [B], [F] and [K].)*

The following fact is easy to prove

**Theorem 6** *Let $A = \bigoplus_{i \in I} A_i$ (unrestricted direct sum). Then $A$ is associate if and only if $A_i$ is associate for all $i \in I$.*

**Proof:** If $Aa = Ab$ for some $a$ and $b \in A$, then for each $i \in I$, $A_i a_i = (Aa)1_i = (Ab)1_i = Ab_i$ implying that there is a unit $u_i \in A_i$ such that $a_i = u_i b_i$. Taking $u(i) = u_i$ for all $i \in I$ gives us a unit $u \in A$ with $a = ub$. Hence A is associate.

The other direction is similar and omitted. □

Thus, the class of associate rings is closed under the formation of unrestricted direct sums. A straightforward argument (which we omit) establishes

**Theorem 7** *The class of associate rings is closed under the formation of direct limits.*

(Here by a direct limit we can take the direct union of a family of rings - see p. 130 of [G]. However, the theorem is true more generally even in cases where the direct limit is not the direct union.)

**Lemma 8** *The class of associate rings is not closed under the formation of subrings, subdirect sums, or homomorphic images.*

**Remark 2** *The erroneous assertion that all commutative rings are associate appears in the literature (see [CF], page 118).*

**Proof**: Consider the rings $R_0 = \mathbb{Z}[X,Y]$ and $R = \mathbb{Z}[X,Y]/(X - XY^2)$. Clearly, $R_0$ is associate. We will show its homomorphic image $R$ is not associate. We will also embed $R$ as a subdirect sum in the associate ring $\mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X]$.

Letting $x$ and $y$ be the images of the indeterminates $X$ and $Y$ respectively under the canonic epimorphism $R_0 \to R$ , it is not difficult to show that every element of $R$ is uniquely of the form $xp(x) + q(y) + xyr(x)$ where $p, q$ and $r$ are integral polynomials in a single indeterminate.

If $R$ were associate then, since $xR = xyR$, there would exist a unit $u \in R$ satisfying $xy = ux$. This implies that there exist polynomials $F, G, U, V \in R_0$ such that

$$U(X,Y)V(X,Y) = 1+(X-XY^2)F(X,Y), \quad U(X,Y)X = XY+(X-XY^2)G(X,Y),$$

where $U$ maps to $u$ under the map $R_0 \to R$ and $V$ maps to $u^{-1}$. These imply $U(X,Y) - Y = (1 - Y^2)G(X,Y)$. Let $\overline{U}(Y)$ denote the image of $U(X,Y)$ under the canonical map $R_0 \to R_0/XR_0 \cong \mathbb{Z}[Y]$. Similarly for $\overline{V}(Y), \overline{G}(Y)$. The above equations imply

$$\overline{U}(Y)\overline{V}(Y) = 1, \quad \overline{U}(Y) - Y = (1 - Y^2)\overline{G}(Y).$$

The first equation implies $\overline{U}(Y) = \pm 1$. If $\overline{U}(Y) = 1$ then the second equation implies $1 = (1 + Y)\overline{G}(Y)$, a contradiction. If $\overline{U}(Y) = -1$ then the second equation implies $-1 = (1 - Y)\overline{G}(Y)$, a contradiction. These contraditions imply $R$ is not associate.

There are canonical epimorphisms

$$\alpha : R_0 \to R_0/XR_0 \cong \mathbb{Z}[Y]$$

$$\beta : R_0 \to R_0/(1 - Y)R_0 \cong \mathbb{Z}[X]$$

$$\gamma : R_0 \to R_0/(1 + Y)R_0 \cong \mathbb{Z}[X].$$

Letting $f = \alpha \oplus \beta \oplus \gamma$ we get a homomorphism

$$f : R_0 \to \mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X].$$

If $a \in R_0$ lies in the kernel of $f$ then $a$ is a multiple of $X$, of $1 - Y$, and of $1 + Y$. Since $R_0$ is a unique factorization domain ([ZS], vol I, page 32), it follows that $a$ is a multiple of $X - XY^2$. This implies that $f$ induces an injection

$$\phi : R \to \mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X].$$

Thus $R$ embeds into the direct sum $\mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X]$ of integral domains. In fact $\phi$ embeds $R$ as a subdirect sum. To see that consider $\phi(x \cdot 0 + q(y) + xy \cdot 0) = (q(Y), q(1), q(-1))$. Now let $h(X) = h(0) + Xg(X)$ be arbitrary.

$$\phi(xg(x) + h(0) + xy \cdot 0) = (h(0), h(X), h(X)).$$

Since $q(Y) \in \mathbb{Z}[Y]$ and $h(X) \in \mathbb{Z}[X]$ are arbitrary, the image of $R$ is, as claimed, a subdirect sum in $\mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X]$. (Alternatively, a simple argument shows that $R$ contains no nonzero nilpotent elements; hence, $R$ is a subdirect sum of integral domains.)

Thus, although direct sums of integral domains are associate, they need not be superassociate. $\Box$

**Definition 9** *A commutative ring $A$ will be called* **domainlike** *provided every zero divisor is nilpotent.*

Note that every integral domain is domainlike.

**Lemma 10** *Every domainlike ring is superassociate.*

**Proof**: Assume $A$ is domainlike. Let $B$ be a subring of $A$ and assume $a, b \in B$ generate the same principal ideal in $B$. If $Ba = 0 = Bb$ , then $a = b = 0$ and $1 \cdot a = b$. Assume $Ba = Bb$ is a nonzero ideal in $B$. Then $a \neq 0$ and $b \neq 0$. Now there are elements $u, v \in B$ such that $ua = b$ (since $b \in Ba$) and $vb = a$ (since $a \in Bb$). Then $uvb = b$ and $(uv - 1)b = 0$. But $b \neq 0$ so $uv - 1$ is a zero divisor. Since $A$ is domainlike $r = uv - 1$ is nilpotent so there is an integer $n > 0$ such that $r^n = 0$. But then it is well-known that $1 + r$ is a unit ($(1 + r)^{-1} = 1 - r + ... + (-r)^{n-1}$) and so, since $uv = 1 + r$, each of $u$ and $v$ individually is a unit as well. $\Box$

**Remark 3** *There are superassociate rings which are not domainlike. For example, let $R$ be a unital Boolean ring with more than 2 elements. A partial order on $R$ is well-defined by $a \leq b$ provided $ab = a$. Thus, $a \in (b)$ implies $a \leq b$ and $b \in (a)$ implies $b \leq a$, whence $a = b$. Thus $R$ is superassociate. If $a \neq 0, 1$ then $a(1-a) = 0$, so $a$ is a zero divisor which is not nilpotent. Thus $R$ is not domainlike.*

**Theorem 11** *Every Noetherian commutative ring with identity is residually superassociate.*

**Proof:** From the Noether-Lasker Primary Decomposition Theorem (see [L], Theorem 4, page 154), it follows that in any Noetherian commutative ring $A$, $(0)$ has a primary decomposition, i.e., the trivial ideal is a *finite* intersection of primary ideals,

$$(0) = \bigcap_{i=1}^{r} Q_i.$$

$Q_i$ being primary (see [L] p.152) means that $A/Q_i$ is domainlike and hence is superassociate by Lemma 10. This is sufficient, according to Lemma 2, for $A$ to be residually superassociate. $\square$

We next observe that the class of associate rings is the model class $\mathbb{M}(T)$ of a set $T$ of sentences of $L_{Ring}$. $T$ may be taken to be the commutative ring axioms together with the single additional axiom

$$\forall x_1 \forall x_2 \forall y_1 \forall y_2 \exists z_1 \exists z_2 (((y_1 x_1 = x_2) \wedge (y_2 x_2 = x_1)) \rightarrow ((z_1 x_1 = x_2) \wedge (z_2 x_2 = x_1) \wedge (z_1 z_2 = 1))).$$

In particular, we prove

**Lemma 12** *Any universal sentence ($\Pi_1$-sentence) of $L_{Ring}$ true in every associate ring is true in any commutative ring. In other words, every commutative ring is a model of $T_\forall$.*

**Proof:** Let $\phi$ be any universal sentence of $L_{Ring}$ true in every associate ring. A commutative ring $A$ is the direct union of its finitely generated subrings. Since universal sentences are preserved by direct unions (see Lemma 3 (3) and subsequent discussion), it will suffice to show that $\phi$ is true in every finitely generated subring of $A$ in order to deduce that $\phi$ holds in $A$.

But a finitely generated subring $A_0$ of $A$ is Noetherian. By Theorem 6 and Lemma 1, such a ring $A_0$ is embeddable into an unrestricted direct sum $B$ of an indexed family $\{B_i\}_{i \in I}$ of associate rings. By Theorem 6, $B$ is associate. Thus, $\phi$ is true in $B$. Since universal sentences hold in subrings whenever they hold in an extension, $\phi$ must be true in $A_0$. Since $A_0$ was an arbitrary finitely generated subring of $A$ and since $A$ is the direct union of such, $\phi$ must be true in $A$. Since $A$ was an arbitrary commutative ring and since $\phi$ was an arbitrary universal sentence of $L_{Ring}$ true in every associate ring we must have that every commutative ring is a model of $T_\forall$. □

By Lemmas 4 and 12, we have therefore proven

**Theorem 13** *Every commutative ring is embeddable in an associate ring.*

**Corollary 14** *Every algebraically closed commutative ring with* 1 *is associate.*

**Proof**: Let $A$ be an algebraically closed commutative ring. By Theorem 13 there is an associate ring $B$ into which $A$ embeds. Suppose $a, b \in A$ are such that $Aa = Ab$. Then there are elements $c, d \in A$ such that $ca = b$ and $db = a$. But then $Ba = Bb$ and since $B$ is associate the system of equations

$$\begin{cases} xa = b \\ yb = a \\ xy = 1 \end{cases} \tag{2}$$

has a solution in $B$. Since $A$ is algebraically closed in the class of commutative rings, (2) must have a solution in $A$. Thus, there is a unit $u$ of $A$ such that $ua = b$. Hence, $A$ is associate. □

Let $R$ be a commutative ring with identity. What conditions on $R$ insure that whenever $a, b \in R$ generate the same ideal, $aR = bR$, then there is a unit $u \in R$ such that $a = ub$? This condition is true if $R$ is either an integral domain (as is well-known) or if $R$ is (as is less well-known) a unital principal ideal ring.

**Theorem 15** *A principal ideal ring with identity is associate.*

The theorem is proven in [B], page 194. For completeness, we provide a brief sketch.

By Theorem 33 of [ZS], a principal ideal ring is a direct sum of principal ideal domains and “special principal ideal rings” (see [ZS], page 245). It can be shown without too much trouble using a result on page 245 of [ZS] that a special principal ideal ring is an associate ring. We conclude from this and Theorem 6 that a principal ideal ring with 1 is associate.

# 4 A Purely Algebraic Proof and an Open Question

This section sketches a second proof of our main result Theorem 13. It uses standard ring-theoretical arguments involving localization, rather than the model-theoretic proof above using the interplay between logic and ring theory. We thank an anonymous referee for this sketch.

We make four observations from which Theorem 13 follows. Let $R$ denote a unital commutative ring.

1. In a commutative ring, any ideal embeds in a maximal ideal.

   This is a trivial consequence of Zorn's lemma.

2. Localization of $R$ at a maximal ideal $M$ (or any prime ideal) gives a local ring $R_M$.

   See [L], (iii), page 69.

3. If $M$ is a maximal ideal containing the annihilator of $a$, then $a$ does not map to zero under the canonical map $R \to R_M$.

   This follows from Proposition 2.1 in [E].

4. If $R$ is a local ring then $R$ is associate.

   Let $R$ be a local ring, $M$ the maximal ideal of $R$, and let $a, b \in R$ satisfy $aR = bR$. If $a = bx$ and $b = ay$, for some $x, y \in R$, then $a(1 - xy) = 0$. If $1 - xy \notin M$ then it must be invertible, so $a = b = 0$. If $1 - xy \in M$ then $x$ and $y$ must be units.

For each nonzero $a \in R$, let $M_a$ denote a maximal ideal of $R$ containing the annihilator of $a$ and let $R_a$ denote the localization of $R$ at $M_a$. As was observed above, the image of $a$ under the canonical map $R \to R_a$ is nonzero. This implies that the map

$$R \hookrightarrow \bigoplus_{a \in R-\{0\}} R_a \tag{3}$$

is an injection.

Now each $R_a$ is associate. Thus by Theorem 6, $\bigoplus_{a\in R-\{0\}} R_a$ is associate. So from (3) we conclude our result, Theorem 13.

We now end our paper with the following question which to the best of our knowledge remains open:

**Question:** If A is an associate ring and X is an indeterminate over A, is the polynomial ring A[X] still an associate ring?

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