# Representations of finite groups on Riemann-Roch spaces, II

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**Abstract**

If $G$ is a finite subgroup of the automorphism group of a projective curve $X$ and $D$ is a divisor on $X$ stabilized by $G$, then under the assumption that $D$ is nonspecial, we compute a simplified formula for the trace of the natural representation of $G$ on Riemann-Roch space $L(D)$.

## 1 Introduction

Let $X$ be a smooth projective (irreducible) curve over an algebraically closed field $k$ of characteristic zero and let $G$ be a finite subgroup of automorphisms of $X$ over $k$. If $D$ is a divisor of $X$ which $G$ leaves stable then $G$ acts on the Riemann-Roch space $L(D)$. We ask the question: is there a simple formula for the character of a (modular) representation[1] which arises in this way?

This character has been computed, for example, in the work of Borne [B] in some cases. (Others include: Chevalley-Weil, Ellingsrud-Lonsted, Nakajima, Köck, and Kani, for example. We refer to [B] for references.)

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[1]For some background on the representations which arise in this way, see for example [JT].

**Lemma 1** (**Borne's formula**) *If $D$ is a $G$-equivariant nonspecial divisor, then the (virtual) character of $L(D)$ is given by*

$$[L(D)] = (1 - g_Y)[k[G]] + [deg_{eq}(D)] - [\tilde{\Gamma}_G], \tag{1}$$

*where $Y$ is the quotient of $X$ by $G$, $g_Y$ is its genus, $\deg_{eq}(D)$ is the equivariant degree of $D$, and $\tilde{\Gamma}_G$ is the ramification module.*

The definitions of equivariant degree and ramification module will be recalled below. Our contribution to simplifying the formula is to find a simple expression for the character of the ramification module, under some rationality conditions.

Let $V$ be a $k[G]$-module and let $F$ be a subfield of $k$. We say that $V$ has an $F[G]$**-module structure** if there is an $F$-vector space $V_F$, such that (a) $V_F$ is an $F[G]$-module, and (b) $V = V_F \otimes_{F[G]} k[G]$.

**Theorem 2** *If $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure, then it decomposes into irreducible $\mathbb{Q}[G]$-modules as*

$$\tilde{\Gamma}_G \simeq \bigoplus_j \frac{1}{m_j^2 d_j} (\sum_\ell (\dim(V_j) - \dim(V_j^{H_\ell})) \frac{R_\ell}{2}) V_j.$$

*Here the first sum is over the set $G^*_{\mathbb{Q}}$ of all irreducible $\mathbb{Q}[G]$-modules, and for each irreducible $\mathbb{Q}[G]$-module $V_j$, $m_j$ is its Schur index and $d_j$ is the number of distinct irreducible $k[G]$-modules in a decomposition of $V_j \otimes k$. The second sum is over all conjugacy classes of cyclic subgroups of $G$, $H_\ell$ is a representative cyclic subgroup, $V_j^{H_\ell}$ indicates the dimension of the fixed part of $V_j$ under the action of $H_\ell$, and $R_\ell$ denotes the number of branch points in $Y$ where the inertia group is conjugate to $H_\ell$.*

In fact, we prove this as a corollary to Proposition 4 below.

In this paper, we will give the proof of this theorem and of some partial results in the case that $\tilde{\Gamma}_G$ does not have a $\mathbb{Q}[G]$-module structure.

## 2 Definitions and Proof

Let $X$ be a smooth projective curve over an algebraically closed field $k$ of characteristic zero. Let $G$ be a finite group of automorphisms of $X$ over $k$. For any point $P \in X(k)$, let $G_P$ be the inertia group at $P$ (i.e. the subgroup of $G$ fixing $P$). Since char $k$ is zero, the quotient $\pi : X \to Y = X/G$ is tamely ramified, and this group $G_P$ is cyclic. $G_P$ acts on the cotangent space of $X(k)$ at $P$ by a $k$-character. This character is the **ramification character** of $X$ at $P$.

The **ramification module** is defined by

$$\Gamma_G = \sum_{P \in X(k)_{ram}} \mathrm{Ind}_{G_P}^G (\sum_{\ell=1}^{e_P - 1} \ell \psi_P^\ell),$$

where $e_P = |G_P|$, and $\psi_P$ is the ramification character at $P$. By Theorem 2 in [N], there is a unique $G$-module $\tilde{\Gamma}_G$ such that

$$\Gamma_G = |G| \tilde{\Gamma}_G.$$

We abuse terminology and also call $\tilde{\Gamma}_G$ the **ramification module**. The goal of this paper is to compute the decomposition of $\tilde{\Gamma}_G$ into irreducible $k[G]$-modules. We do this by computing the character of $L(D)$ for a particular choice of $D$, then using Borne's formula.

Now consider a $G$-invariant divisor $D$ on $X(k)$. If $D = \frac{1}{e_P} \sum_{g \in G} g(P)$ then we call $D$ a **reduced orbit**. The reduced orbits generate the group of $G$-invariant divisors $Div(X)^G$.

**Definition 3** *The* ***equivariant degree*** *is a map from $Div(X)^G$ to the Grothendieck group $R_k(G) = \mathbb{Z}[G_k^*]$ of virtual $k$-characters of $G$,*

$$deg_{eq} : Div(X)^G \to R(G),$$

*defined by the following conditions:*

1. *$deg_{eq}$ is additive on $G$-invariant divisors of disjoint support,*
2. *If $D = r \frac{1}{e_P} \sum_{g \in G} g(P)$ is an orbit then*

$$deg_{eq}(D) = \begin{cases} Ind_{G_P}^G(\sum_{\ell=1}^{r} \psi_P^{-\ell}), & \text{if } r > 0, \\ -Ind_{G_P}^G(\sum_{\ell=0}^{-(r+1)} \psi_P^{\ell}), & \text{if } r < 0, \\ 0, & r = 0, \end{cases}$$

*where $\psi_P$ is the ramification character of $X$ at $P$.*

If $D = \pi^*(D_0)$ is the pull-back of a divisor $D_0 \in Div(Y)$ then $deg_{eq}(D)$ has a very simple form. On each orbit, $r$ is a multiple of $e_P$, so every character of the cyclic group $G_P$ appears. The equivariant degree on this orbit is induced from a multiple of the regular representation of $G_P$. Thus we have

$$deg_{eq}(D) = deg(D_0)[k[G]], \tag{2}$$

(This is also a special case of Corollary 3.10 in [B].)
Therefore, as a corollary to Theorem 2, when $D$ is a pullback of "large degree" then $L(D)$ may be described in representation-theoretic terms (that is to say, the ramification characters are not needed).

Let $\langle G \rangle$ denote the set of conjugacy classes of cyclic subgroups of $G$. Let $G^*_{\mathbb{Q}}$ denote the set of equivalence classes of irreducible $\mathbb{Q}[G]$-modules. By results in ([Se], §13.1, §12.4), this set has the same number of elements as $\langle G \rangle$, and the character table of $G$ over $\mathbb{Q}$ is a square matrix with rows labelled by $G^*_{\mathbb{Q}}$ and columns labelled by $\langle G \rangle$. The rows are linearly independent (as $\mathbb{Q}$-class functions), so in fact the character table is an invertible matrix.

Let $F$ be a finite extension of $\mathbb{Q}$ such that all $k[G]$-modules have $F[G]$-module structure ([Se], p. 94). For each irreducible $\mathbb{Q}[G]$-module $V_j$, $V_j \otimes_{\mathbb{Q}[G]} F[G]$ decomposes into irreducible $F[G]$-modules. The Galois group of $F$ over $\mathbb{Q}$ permutes the components transitively, so each must have the same multiplicity (the Schur index of the representation $V_j$) and the same dimension. We write $V_j \otimes_{\mathbb{Q}[G]} F[G] \simeq m_j \bigoplus_{r=1}^{d_j} W_{jr}$, where $m_j$ is the Schur index, the $W_{jr}$'s are irreducible $F[G]$-modules, and $\dim_{\mathbb{Q}} V_j = m_j d_j \dim_F W_{jr}$ for each $r$. We also have $V_j \otimes_{\mathbb{Q}[G]} k[G] \simeq m_j \bigoplus_{r=1}^{d_j} W_{jr} \otimes_{F[G]} k[G]$.

For each $F$-representation $(\rho, V)$ of $G$, $\rho : G \to Aut_F(V)$, let $\chi_V = \chi_\rho$ denote the character of $V$: $\chi_V(g) = \operatorname{tr} \rho(g)$, for $g \in G$. For brevity, let $\chi_j$ denote the character of $V_j$ over $\mathbb{Q}$ (which is equal to the character of $V_j \otimes_{\mathbb{Q}[G]} k[G]$ over $k$), and $\chi_{jr}$ denote the character of $W_{jr}$ over $F$ (which equals the character of $W_{jr} \otimes_{F[G]} k[G]$ over $k$). Then we have

$$\chi_j = m_j \sum_{r=1}^{d_j} \chi_{jr} \tag{3}$$

**Proposition 4** *Let $D = \pi^*(D_0)$ be a nonspecial divisor on $X$ and assume $L(D)$ has a $\mathbb{Q}[G]$-module structure $L(D)_{\mathbb{Q}}$. For each irreducible $\mathbb{Q}[G]$-module $V_j$, its multiplicity in $L(D)_{\mathbb{Q}}$ is given by*

$$n_j = \frac{1}{m_j^2 d_j}\left(\dim(V_j)(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(V_j)-\dim(V_j^{H_\ell}))\frac{R_\ell}{2}\right), \tag{4}$$

*where $V_j^{H_\ell}$ denotes the subspace of $V_j$ fixed by $H_\ell$.*

**Proof**: First, we recall some notation from [Ks]: For each class choose a representative cyclic subgroup $H_\ell$, $\ell = 1 \dots M$, and partially order them according to the order of the group so that $H_1$ is the trivial group. For each branch point of the cover $\pi : X \to Y$, the inertia groups at the ramification points $P$ above that branch point will be cyclic and conjugate to each other. For each $\ell$, let $R_\ell$ denote the number of branch points in $Y$ where the inertia groups are conjugate to $H_\ell$.

The proof is similar to the proof of Theorem 2.3 in [Ks]. We consider the quotients $X/H_\ell$ of $X$ by cyclic subgroups $H_\ell$. The morphism $\pi : X \to Y$ factors through this quotient, so on each $X/H_\ell$ there is a pullback divisor $D_\ell$ of $D_0$.

First, note that our assumption that $D$ is nonspecial means that for any quotient $X/H_\ell$, the pullback $D_\ell$ of $D_0$ to $X/H_\ell$ is also nonspecial. This is because

$$K_X - D = \pi_\ell^*(K_{X/H_\ell}) + R_l - \pi_\ell^*(D_\ell) = \pi_\ell^*(K_{X/H_\ell} - D_\ell) + R_l$$

where $R_\ell$ is the ramification divisor of the covering $\pi_\ell : X \to X/H_\ell$. Any element of $L(K_{X/H_\ell} - D_\ell)$ would pull back to $X$ to give an element of $L(K_X - D - R_\ell)$. Since $R_\ell$ is effective, this would also give an element of $L(K_X - D)$, contradicting our assumption that $D$ is nonspecial.

Now we decompose $L(D)_{\mathbb{Q}}$ as

$$L(D) \simeq \bigoplus_{j=1}^{M} n_j V_j \tag{5}$$

where $V_j$, $j = 1 \dots M$ are the elements of $G^*_{\mathbb{Q}}$. For each $H_\ell$ in $G^*_{\mathbb{Q}}$, consider the dimension of the piece of this module fixed by $H_\ell$. Since $L(D)^{H_\ell} = L(D_\ell)$, we get a system of equations

$$\dim L(D_\ell) = \sum_{j=1}^{M} n_j \dim(V_j^{H_\ell}), \qquad 1 \le \ell \le M. \tag{6}$$

This is a system of $M$ equations in the $M$ unknowns $n_j$. We need to show that the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is invertible, so this system has a unique solution, and that the above equation is the claimed solution.

First, let us consider the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$. Each matrix entry is equal to the multiplicity of the trivial representation of $H_\ell$ in the restricted representation of $H_\ell$ on $V_j$. This is the inner product of characters $\langle \mathrm{Res}_{H_\ell}^{G} \chi_j, \mathbf{1}\rangle$, where $\chi_j$ is the character of $V_j$, which is defined as

$$\dim V_j^{H_\ell} = \frac{1}{|H_\ell|} \sum_{a \in H_\ell} \chi_j(a) \tag{7}$$

Thus each column of the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is a sum of columns of the character table. Each element $a$ in $H_\ell$ generates either all of $H_\ell$ or a cyclic subgroup of lower order, hence earlier in the list $\langle G\rangle$. Thus if we write our matrix in terms of the basis of columns of the character table over $\mathbb{Q}$, we get a lower triangular matrix with nonzero entries on the diagonal. This implies that our matrix is also invertible.

It remains to verify that our equation is the correct solution.

Note that

$$\dim L(D_\ell) = \frac{|G|}{|H|}\deg(D_0) + 1 - g(X/H_l), \tag{8}$$

for $1 \le \ell \le M$, by the Riemann-Roch theorem and the hypothesis that $D_\ell$ is nonspecial.

We will now substitute (4) into (6) and verify that the result agrees with (8), for each $1 \le \ell \le M$. The argument is similar to that in [Ks].

Plugging (4) into (6) gives

$$\begin{aligned}\sum_{j=1}^{M} n_j \dim(V_j^{H_\ell}) &= (\deg(D_0) + 1 - g_Y) \sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_j^{H_\ell}) \dim(V_j) \\ &\quad - \sum_{i=1}^{M} \left( \sum_{j=1}^{M} \frac{1}{m_j^2 d_j} [\dim(V_j^{H_\ell})\dim(V_j) - \dim(V_j^{H_\ell})\dim(V_j^{H_i})] \frac{R_i}{2} \right)\end{aligned}$$

Note that

$$\dim(V_j^{H_\ell}) = \langle \operatorname{Res}^G_{H_\ell} \chi_j, \mathbf{1}\rangle = m_j \sum_{r=1}^{d_j} \langle \operatorname{Res}^G_{H_\ell} \chi_{W_{jr}}, \mathbf{1}\rangle = m_j \sum_{r=1}^{d_j} \langle \chi_{W_{jr}}, \operatorname{Ind}^G_{H_\ell} \mathbf{1}\rangle, \tag{9}$$

using (3) and Frobenius reciprocity. This gives us

$$\begin{aligned} \sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim V_j^{H_\ell} \dim V_j &= \sum_{j=1}^{M} \frac{\dim V_j}{m_j d_j} \sum_{r=1}^{d_j} \langle \chi_{jr}, \operatorname{Ind}^G_{H_\ell} \mathbf{1}\rangle \\ &= \sum_{j=1}^{M} \sum_{r=1}^{d_j} \dim W_{jr} \langle \operatorname{Res}^G_{H_\ell} \chi_{jr}, \mathbf{1}\rangle \qquad (10) \\ &= \frac{1}{|H_\ell|} \sum_{a \in H_\ell} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(e) \chi_{jr}(a) \end{aligned}$$

The last part of this is summing over all irreducible $k$-characters of $G$, so the last expression is in fact the inner product of two columns of the character table for $G$ over $k$. This inner product will be zero unless $a = e$, so the sum becomes

$$\frac{1}{|H_\ell|} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(e)^2 = \frac{|G|}{|H_\ell|}. \tag{11}$$

We would like to do a similar simplification of

$$\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_j^{H_\ell}) \dim(V_j^{H_i}) \tag{12}$$

using (9) twice. The induced representation $\operatorname{Ind}^G_{H_i} \mathbf{1}$ is the action of $G$ by permutations on the cosets of $H_i$, and thus has a $\mathbb{Q}[G]$-module structure as well as an $F[G]$-module structure. It can be decomposed into irreducible $k[G]$-modules, such that for each $j$ the multiplicities of the $W_{jr} \otimes k$'s, $\langle \chi_{W_{jr}}, \operatorname{Ind}^G_{H_i} \mathbf{1}\rangle$, are all equal. Using the above, Frobenious reciprocity, and the definition of the Schur inner product, we have

$$
\begin{aligned}
\textstyle\sum_{j=1}^M \frac{1}{m_j^2 d_j} \quad & \dim(V_j^{H_\ell})\dim(V_j^{H_i}) \\
&= \textstyle\sum_{j=1}^M \frac{1}{d_j}\sum_{r=1}^{d_j}\langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle \sum_{s=1}^{d_j}\langle \chi_{js}, \mathrm{Ind}_{H_i}^G \chi_{js}\rangle \\
&= \textstyle\sum_{j=1}^M \sum_{r=1}^{d_j}\langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle\langle \chi_{jr}, \mathrm{Ind}_{H_i}^G \chi_{jr}\rangle \\
&= \textstyle\sum_{j=1}^M \sum_{r=1}^{d_j}\langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle\langle \mathrm{Res}_{H_i}^G \chi_{jr}, \mathbf{1}\rangle \\
&= \textstyle\frac{1}{|H_\ell|}\frac{1}{|H_i|}\sum_{a\in H_\ell}\sum_{b\in H_i}\sum_{j=1}^M\sum_{r=1}^{d_j}\chi_{jr}(a)\chi_{jr}(b).
\end{aligned}
\tag{13}
$$

Again, this last is an inner product of columns of the character table of $G$ over $k$, so will be zero unless $a$ and $b$ are in the same conjugacy class. Let $\mathcal{C}_G(a)$ denote the conjugacy class of $a$ in $G$. We end up with

$$
\begin{aligned}
\sum_{j=1}^M \frac{1}{m_j^2 d_j}\dim(V_j^{H_\ell})\dim(V_j^{H_i}) &= \frac{1}{|H_\ell||H_i|}\sum_{a\in H_\ell}\#(H_i\cap\mathcal{C}_G(a))\sum_{j=1}^M\sum_{r=1}^{d_j}\chi_{jr}(a)^2 \\
&= |H_\ell\backslash G/H_i|
\end{aligned}
\tag{14}
$$

the number of double cosets.

From this we get

$$
\begin{aligned}
\sum_{j=1}^M n_j \dim V_j^{H_\ell} &= (\deg(D_0)+1-g_Y)\frac{|G|}{|H_\ell|} - \sum_{i=1}^M\left(\frac{|G|}{|H_\ell|} - |H_i\backslash G/H_\ell|\right)\frac{R_i}{2} \\
&= (\deg(D_0)+1-g_Y)\frac{|G|}{|H_\ell|} + 1 + \frac{|G|}{|H_\ell|}(g_Y-1) - g_{X/H_\ell} \\
&= \deg(D_0)\frac{|G|}{|H_\ell|} + 1 - g_{X/H_\ell}.
\end{aligned}
$$

where the last equalities come from applying the Hurwitz formula to the cover $X/H_\ell \to Y$ (see [Ks] for details). This is (8), as desired. $\square$

**Proof of Theorem 2.** We choose a divisor $D_0$ on $Y$ sufficiently large so that $D := \pi^*(D_0)$ is nonspecial.

Recall from (2) that Borne's formula gives us

$$[L(D)] = (1-g_Y+deg(D_0))[k[G]] - [\tilde{\Gamma}_G].$$

Since the regular representation $k[G]$ has a natural $\mathbb{Q}[G]$-module structure $\mathbb{Q}[G]$, the character $[L[D]]+[\tilde{\Gamma}_G]$ must be in the Grothendieck ring $R_{\mathbb{Q}}(G)$ of characters generated by the traces of irreducible $\mathbb{Q}[G]$-modules. This means that for any pullback divisor $D = \pi^*(D_0)$, $L(D)$ will have a $\mathbb{Q}[G]$-module structure if and only if $\tilde{\Gamma}_G$ does.

Because of this and our hypothesis, the hypotheses of Proposition 4 hold, so the conclusion is satisfied. This gives us the multiplicities of the irreducible $\mathbb{Q}[G]$-modules in $\tilde{\Gamma}_G$. $\square$.

**Corollary 5** *Suppose that $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure. Let $W$ be an irreducible $k[G]$-module which is a component of $V_j \otimes_{\mathbb{Q}[G]} k$ for an irreducible $\mathbb{Q}[G]$-module $V_j$. Then the multiplicity of $W$ in $\tilde{\Gamma}_G$, is*

$$\frac{1}{m_j d_j}\left(\sum_{\ell}(\dim(V_j) - \dim(V_j^{H_\ell}))\frac{R_\ell}{2}\right). \tag{15}$$

**Proof**: From equation (3), the multiplicity of $W_{jr}$ in $V_j \otimes_{\mathbb{Q}[G]} k$ is the Schur index $m_j$. $\square$

# 3 Action of the Galois group

Again, $k$ is algebraically closed with characteristic 0.

Now we move to the case where $\tilde{\Gamma}_G$ does not necessarily have a $\mathbb{Q}[G]$-module structure. In this case, we do not get the complete decomposition into irreducible $k[G]$-modules, but we get some useful information.

As in section 2, let $F$ denote a finite extension of $\mathbb{Q}$ over which all irreducible representations of $G$ are defined. Then there is an $F[G]$-module $L(D)_F$ such that $L(D) = L(D)_F \otimes_{F[G]} k[G]$. The Galois group $Gal(F/\mathbb{Q})$ acts on $L(D)_F$. The full orbit of $L(D)_F$ under this action will have a $\mathbb{Q}$-vector space structure, and hence a $\mathbb{Q}[G]$-module structure. In general the full Galois orbit is not needed to get a $\mathbb{Q}[G]$-module structure; instead let $\mathcal{G}$ be a smallest subgroup of $Gal(F/\mathbb{Q})$ such that

$$\bigoplus_{\gamma\in\mathcal{G}} \gamma L(D)_F$$

has a $\mathbb{Q}[G]$-module structure. (In the case of Theorem 2, $\mathcal{G}$ will be the trivial group). We denote this module by

$$\overline{L(D)_F} := \bigoplus_{\gamma\in\mathcal{G}} \gamma L(D)_F$$

and its $\mathbb{Q}[G]$-module structure as $\overline{L(D)_{\mathbb{Q}}}$.

**Proposition 6** *Let $D = \pi^*(D_0)$ be a nonspecial divisor on $X$. For each irreducible $\mathbb{Q}[G]$-module $V_j$, its multiplicity in $\overline{L(D)_{\mathbb{Q}}}$ is given by*

$$n_j = \frac{|\mathcal{G}|}{m_j^2 d_j}\left(\dim(V_j)(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(V_j)-\dim(V_j^{H_\ell}))\frac{R_\ell}{2}\right). \tag{16}$$

**Proof**: The proof is similar to that of Proposition 4. We take the fixed part of the decomposition

$$\overline{L(D)_{\mathbb{Q}}} \simeq \bigoplus_j n_j V_j$$

under the action of a cyclic subgroup $H_\ell \subset G$. Since

$$\mathrm{Res}_{H_\ell}^G \overline{L(D)_F} = \bigoplus_{\gamma \in \mathcal{G}} \mathrm{Res}_{H_\ell}^G \gamma(L(D)_F)$$

and $\mathcal{G}$ fixes the trivial representation, this will simply be

$$\dim_{\mathbb{Q}} \overline{L(D)_{\mathbb{Q}}}^{H_\ell} = \dim_F \overline{L(D)_F}^{H_\ell} = |\mathcal{G}| \dim(L(D)_F)^{H_\ell} = |\mathcal{G}| \dim L(D_\ell)_F$$

and on the other hand,

$$\dim_{\mathbb{Q}} \overline{L(D)_{\mathbb{Q}}}^{H_\ell} = \sum_j n_j \dim_{\mathbb{Q}} V_j^{H_\ell}.$$

This gives us the system of equations (6) from Proposition 4, but solving for $\frac{n_j}{|\mathcal{G}|}$ instead of $n_j$, so the solution is:

$$n_j = \frac{|\mathcal{G}|}{m_j^2 d_j}[\dim(V_j)(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(V_j)-\dim(V_j^{H_\ell}))\frac{R_\ell}{2}].$$

□

Now we would like to decompose $L(D)$ into irreducible $k[G]$-modules, or equivalently $L(D)_F$ into irreducible $F[G]$-modules:

$$L(D)_F \simeq \bigoplus_{j=1}^{M} \bigoplus_{r=1}^{d_j} n_{jr} W_{jr}$$

where $W_{jr}$ and $d_j$ are as in section 2. Given such a decomposition of $L(D)_F$, $\overline{L(D)_F}$ will decompose as

$$\overline{L(D)_F} \simeq \bigoplus_{j=1}^{M} \bigoplus_{r=1}^{d_j} n_{jr} (\bigoplus_{\gamma \in \mathcal{G}} \gamma(W_{jr})). \tag{17}$$

Since $\bigoplus_{\gamma \in \mathcal{G}} \gamma(W_{jr})$ is fixed by the Galois group, it will have a $\mathbb{Q}[G]$-module structure, and because $\mathcal{G}$ fixes $\mathbb{Q}[G]$-modules, the $\mathbb{Q}[G]$-module structure will be a multiple of the irreducible $V_j$. By counting dimensions we see that this multiple must be

$$\bigoplus_{\gamma \in \mathcal{G}} \gamma(W_{jr}) = \frac{|\mathcal{G}|}{m_j d_j} V_j \otimes_{\mathbb{Q}[G]} F[G]$$

where as in section 2, $m_j$ is the Schur index of $V_j$. Thus we have

$$\overline{L(D)} \simeq \bigoplus_{j=1}^{M} \sum_{r=1}^{d_j} n_{jr} \frac{|\mathcal{G}|}{m_j d_j} V_j.$$

**Corollary 7** *Let $D = \pi^*(D_0)$ be a nonspecial divisor on $X$, and let*

$$L(D) \simeq \bigoplus_{j=1}^{M} \bigoplus_{r=1}^{d_j} n_{jr} W_{jr} \otimes k$$

*be a decomposition of $L(D)$ into irreducible $k[G]$-modules. Then the multiplicities $n_{jr}$ for each $j$ satisfy*

$$\sum_{r=1}^{d_j} n_{jr} = \frac{1}{m_j} \left( \dim(V_j)(\deg(D_0) + 1 - g_Y) - \sum_{\ell=1}^{M} (\dim(V_j) - \dim(V_j^{H_\ell})) \frac{R_\ell}{2} \right). \tag{18}$$

**Theorem 8** *We have*

$$\sum_{\gamma\in\mathcal{G}}\gamma(\chi(\tilde{\Gamma}_G)) = \sum_j m_j(\sum_{\ell=1}^{M}(\dim V_j) - \dim V_j^{H_\ell}))\frac{R_\ell}{2})\chi_j,$$

*where $\mathcal{G}$ is the Galois group of a smallest extension of $\mathbb{Q}$ for which $\sum_{\gamma\in\mathcal{G}}\gamma(\chi(\tilde{\Gamma}_G)(g))$, $g \in G$, has all its values in $\mathbb{Q}$.*

**Proof**: This follows from the definition of $\mathcal{G}$, Corollary 5. □

# 4 Examples

**Example 9** *Consider the genus 2 curve*

$$y^2 = x(x-2)(x-4), \qquad z^2 = x+4,$$

*over $F = \mathbb{C}$. This has an action of $G = C_2 \times C_2$ given by*

$$\begin{aligned} \alpha : (x,y,z) &\longmapsto (x,-y,z), \\ \beta : (x,y,z) &\longmapsto (x,y,-z), \\ \alpha\beta : (x,y,z) &\longmapsto (x,-y,-z). \end{aligned}$$

*This group has character table*

| | 1 | $\alpha$ | $\beta$ | $\alpha\beta$ |
|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_4$ | 1 | $-1$ | $-1$ | 1 |

*The divisor*

$$D = (0,0,2) + (0,0,-2) + (-4, 8\sqrt{3}, 0) + (-4, -8\sqrt{3}, 0)$$

*is $G$-equivariant and supp$(D)/G = \{(0,0,2), (-4, 8\sqrt{3}, 0)\}$. One can show*

$$L(D) = \{a_0 + a_1\frac{1}{t} + a_2\frac{y}{xt}\}.$$

*Direct computation: This Riemann-Roch space splits up into a direct sum of irreducible 1-dimensional $G$-modules. As a matrix representation,*

$$\rho(\alpha^i\beta^j) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & (-1)^j & 0 \\ 0 & 0 & (-1)^{i+j} \end{array} \right).$$

*The character of $\rho$ agrees with $\chi_1 + \chi_2 + \chi_4$ on $G$, so*

$$L(D) \cong \rho_1 \oplus \rho_2 \oplus \rho_4.$$

*Geometric computation using the formula: We have*

$$\dim(\rho)(1 - g(X_G)) = 1,$$

*for each irreducible $\rho$.*

*Note that each orbit of $G$ in supp$(D)$ is represented by a point $P$ where the inertia group is $G_P = C_2$ and the multiplicity of the orbit is $r_P = 1$. Therefore,*

$$T_{P,\rho} = \left\{ \begin{array}{ll} 0, & \rho = \rho_i,\ i = 1, \\ 1, & \rho = \rho_i,\ i = 2, 3, \\ 2, & \rho = \rho_4, \end{array} \right.$$

*where*

$$T_{P,\rho} = \left\{ \begin{array}{ll} \sum_{\ell=1}^{r_P} n_{\rho,\ell}(G_P) = \sum_{\ell=1}^{r_P} \langle \psi_P^\ell, \mathrm{Res}_{H_\ell}^G \rho \rangle, & r_P > 0, \\ -\sum_{\ell=0}^{-(r_P+1)} n_{\rho,-\ell}(G_P) = \sum_{\ell=1}^{-(r_P+1)} \langle \psi_P^{-\ell}, \mathrm{Res}_{H_\ell}^G \rho \rangle, & r_P < 0, \end{array} \right.$$

*For $\rho = 1$, we therefore have*

$$\dim(L(D)_\rho) = 1 + 0 - 0 = 1.$$

*This means that $L(D)$ contains the trivial representation $\rho_1$ with multiplicity 1, as predicted.*

*The cover $X \to X/G$ has 5 branch points: three with inertia group $G_P = C_2 = \{1, \alpha\}$ (at $x = 0, 2, 4$), one with inertia group $G_P = C_2 = \{1, \beta\}$ (at $x = -4$), and one with inertia group $G_P = C_2 = \{1, \alpha\beta\}$ (at $x = \infty$). This means*

$$R(\{1, \alpha\}) = 3, \quad R(\{1, \beta\}) = 1, \quad R(\{1, \alpha\beta\}) = 1.$$

*Plugging these into Borne's formula Lemma 1 and our Theorem 2 yields $\chi_1 + \chi_2 + \chi_4$ for the character of $L(D)$.*

**Example 10** *If $k$ contains the cyclotomic field $\mathbb{Q}(\zeta_\ell)$ of $\ell^{th}$ roots of unity and if $X = \mathbb{P}^1/k$ is the projective line, regarded as the complex plane with the point added at $\infty$. Let $g \in G = C_\ell \subset k$ act by $g : z \longmapsto gz$. The only fixed points are the points $P_1 = 0$ and $P_2 = \infty$. At $P_1$, the action on the cotangent space is via the identity character $\psi_\ell$ of $G$. At $P_2$, the action on the cotangent space is via the character $\psi_\ell^{-1} = \psi_\ell^{\ell-1}$. Therefore, the ramification module is given by*

$$\begin{aligned}\Gamma_G &= Ind_G^G(\textstyle\sum_{i=1}^{\ell-1} i\psi_\ell^i + \sum_{i=1}^{\ell-1} i\psi_\ell^{i(\ell-1)}) \\ &= \textstyle\sum_{i=1}^{\ell-1} i\psi_\ell^i + \sum_{i=1}^{\ell-1} i\psi_\ell^{\ell-i}) = \ell \sum_{i=1}^{\ell-1} \psi_\ell^i.\end{aligned}$$

*Since $\sum_{i=0}^{\ell-1} \psi_\ell^i(g) = 0$ for $g \neq 1$, $\Gamma_G(g)$ is $= 0$ if $g \in G - \{1\}$, and $= -\ell$ at $g = 1$. In any case, it is rational-valued, though the action of $G$ on $X$ is not defined over $\mathbb{Q}$, if $\ell > 2$.*

The following example illustrates Corollary 7.

**Example 11** *If $k$ contains the cyclotomic field $\mathbb{Q}(\zeta_7)$ of $7^{th}$ roots of unity and if $C$ is the Klein quartic $X^3Y + Y^3Z + Z^3X = 0$ then the cyclic group of order 3 acts on $X$ by cyclically permuting the coordinates and the cyclic group of order 7 acts by sending $(X : Y : Z) \longmapsto (\zeta_7 X : \zeta_7^4 Y : \zeta_7^2 Z)$ [E].*

*Let $G \cong C_3 \rtimes C_7$ be the semi-direct product of these groups. This has character table[2]:*

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | $\zeta_3^2$ | 1 | $\zeta_3$ | 1 |
| 1 | $\zeta_3$ | 1 | $\zeta_3^2$ | 1 |
| 3 | 0 | $\zeta_7^3 + \zeta_7^5 + \zeta_7^6$ | 0 | $\zeta_7 + \zeta_7^2 + \zeta_7^4$ |
| 3 | 0 | $\zeta_7 + \zeta_7^2 + \zeta_7^4$ | 0 | $\zeta_7^3 + \zeta_7^5 + \zeta_7^6$ |

*The conjugacy class for the columns are as given by GAP's* `ConjugacyClasses` *command (the same ordering is used by MAGMA).*

*There are 23 points of degree 1 on $X/k$, among them being $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$. The divisor $P_1 + P_2 + P_3$, and all its multiples, is $G$-equivariant. The functions[3]:*

$$1, \quad X/Z, \quad Y/X, \quad Z/Y,$$

[2] This was obtained using [Gap]. Incidently, there is only one non-cyclic group of order 21, up to isomorphism.

[3] This was obtained "by hand" but has been verified using [MAGMA].

*form a basis of $L(D)$, where $D = 2(P_1 + P_2 + P_3)$. The functions*[4]*:*

$$1, \quad X/Z, \quad Y/X, \quad Z/Y, \quad X/Y, \quad Z/X, \quad Y/Z,$$

*form a basis of $L(D)$, where $D = 3(P_1 + P_2 + P_3)$. The first space is, as a $G$-module, not defined over $\mathbb{Q}$. In fact, it is the direct sum of the trivial representation $\rho_1$ (associated to the $1^{st}$ row of the character table above) and the 3-dimensional representation $\rho_4$ (associated to the $4^{th}$ row of the table). The second space is rational as a $G$-module. In fact, it is the direct sum of $\rho_1$, $\rho_4$ and $\rho_5$. (These facts may be verified by computing the trace directly using the above basis.)*

*If $k$ does not contain $\zeta_7$ then only $C_3$ acts on the curve $X/k$. The space $L(D)$, where $D = 2(P_1 + P_2 + P_3)$, decomposes into the $C_3$-modules*

$$\langle 1 \rangle \oplus \langle x/z, y/x, z/y \rangle = \langle 1 \rangle \oplus \langle x/z + y/x + z/y \rangle \oplus V,$$

*where $V$ is a 2-dimensional $C_3$-module. If $k$ contains the cyclotomic field $\mathbb{Q}(\zeta_3)$ then $V$ decomposes into a direct sum of two irreducible 1-dimensional $C_3$-modules. If $k$ does not contain the cyclotomic field $\mathbb{Q}(\zeta_3)$ then $V$ is irreducible as a $C_3$-module.*

*Points of ramification are determined as follows:*

- *$C_3$ action: The points ramified from the cyclic action on the coordinates $X, Y, Z$ of $\mathbb{P}^2$ are: $(1:1:1)$, $(1:\zeta_3:\zeta_3^2)$, and $(1:\zeta_3^2:\zeta_3)$. Of these, only $P_4 = (1:\zeta_3:\zeta_3^2)$ and $P_5 = (1:\zeta_3^2:\zeta_3)$. are in $X(k)$ (assuming $k$ contains $\zeta_3$). We have*

$$G_{P_4} \cong C_3, \qquad G_{P_5} \cong C_3.$$

- *$C_7$ action: The points ramified from the above action of $C_7$ on the coordinates $X, Y, Z$ of $\mathbb{P}^2$ are: $P_1$, $P_2$, and $P_3$, all of which belong to $C(k)$. We have $G_{P_i} \cong C_7$, for $i = 1, 2, 3$.*

*There is one orbit of points where the inertia group is isomorphic to $C_7$. There are two orbits of points where the inertia group is isomorphic to $C_3$. This allows us to compute the number of branch points in $X/G$ where the inertia group is conjugate to $H_\ell$:*

[4] This was obtained using [MAGMA].

$$\deg R_\ell = \begin{cases} 2, & \text{if } H_\ell = C_3, \\ 1, & \text{if } H_\ell = C_7, \\ 0, & \text{otherwise.} \end{cases}$$

*How does $C_7$ act on the cotangent space $T_{P_1}(X)^*$ at $P_1$? Local coordinates on the patch containing $P_1$ are $y = Y/X$, $z = Z/X$, and the equation for $C$ on the patch is $y + y^3 z + z^3 = 0$. Since $(3y^2z + 1)dy + (y^3 + 3z^2)dz = 0$, the cotangent space has basis $dz/(3y^2z + 1) = -dy/(y^3 + 3z^2)$. The map $\tau : (X : Y : Z) \longmapsto (X : \zeta_7^3 Y : \zeta_7 Z)$ maps $\frac{dy}{y^3+3z^2} \longmapsto \zeta_7 \frac{dy}{y^3+3z^2}$.*

*How does $C_7$ act on the cotangent space $T_{P_2}(X)^*$ at $P_2$? Local coordinates on the patch containing $P_2$ are $x = X/Y$, $z = Z/Y$, and the equation for $C$ on the patch is $x^3 + z + z^3 x = 0$. Since $(3y^2z+1)dy + (y^3+3z^2)dz = 0$, the cotangent space has basis $dx/(3z^2x+1)$. The map $\tau : (X : Y : Z) \longmapsto (X : \zeta_7^3 Y : \zeta_7 Z)$ maps $\frac{dx}{3z^2x+1} \longmapsto \zeta_7^4 \frac{dx}{3z^2x+1}$.*

*How does $C_7$ act on the cotangent space $T_{P_3}(X)^*$ at $P_3$? Local coordinates on the patch containing $P_2$ are $x = X/Z$, $y = Y/Z$, and the equation for $C$ on the patch is $x^3y + x + y^3 = 0$. Since $(3y^2z+1)dy + (y^3+3z^2)dx = 0$, the cotangent space has basis $dy/(3x^2y+1)$. The map $\tau : (X : Y : Z) \longmapsto (X : \zeta_7^3 Y : \zeta_7 Z)$ maps $\frac{dy}{3x^2y+1} \longmapsto \zeta_7^2 \frac{dy}{3x^2y+1}$.*

*How does $C_3$ act on the cotangent space $T_{P_4}(X)^*$ at $P_4$? The patch $Z = 1$ with local coordinates $x = X/Z$ and $y = Y/Z$ contains $P_4 = (\zeta_3 : \zeta_3^2 : 1)$. Again, the cotangent space has basis $dy/(3x^2y+1) = -dx/(y^3+3z^2)$. The action of the group $C_3$ is generated by by $(X : Y : Z) \longmapsto (Y : Z : X)$. In terms of the local coordinates, this sends*

$$x \longmapsto y/x, \quad y \longmapsto 1/x.$$

*Note $d(y/x) = -\frac{y}{x^2}dx + \frac{1}{x}dy = -dx + \zeta_3^2 dy$ and $d(1/x) = -\frac{1}{x^2}dx = -\zeta_3 dx$ at $P_4$. Therefore, this $C_3$ action is generated by the map which sends*

$$dx \longmapsto d(y/x) = dx + \zeta_3^2 dy, \quad dy \longmapsto d(1/x) = -\zeta_3 dx,$$

*and hence*

$$\frac{dy}{3x^2y+1} \longmapsto \frac{-\zeta_3 dx}{3(y/x)^2(1/x)+1} = \zeta_3 \frac{dy}{3x^2y+1}.$$

*The ramification module is determined as follows. By definition,*

$$\Gamma_G = \sum_{\mathcal{O}} \sum_{P \in \mathcal{O}} Ind_{G_P}^G \left( \sum_{\ell=1}^{e_P - 1} \ell \psi_P^\ell \right),$$

*where the outer sum runs over all $G$-orbits*[5] $\mathcal{O}$, $e_P$ *denotes the inertia degree*[6], *and* $\psi_P$ *denotes the character of* $G_P$ *acting on the cotangent space. Let us denote by* $\psi_7$ *the identity character on* $G_P = C_7$ *and similarly for* $\psi_3$ *on* $G_P = C_3$. *On the $G$-orbit of* $P = (1:0:0)$, *we have*

$$\begin{array}{c}\sum_{P\in\mathcal{O}} Ind_{G_P}^G(\sum_{\ell=1}^{e_P-1}\ell\psi_P^\ell) = Ind_{C_7}^G(\psi_7 + 2\psi_7^2 + ... + 6\psi_7^6)+\\ Ind_{C_7}^G(\psi_7^2 + 2\psi_7^4 + ... + 6\psi_7^{12}) + Ind_{C_7}^G(\psi_7^4 + 2\psi_7^4 + ... + 6\psi_7^{24})\\ = (1+2+4)Ind_{C_7}^G(\psi_7 + \psi_7^2 + \psi_7^4) + (3+5+6)Ind_{C_7}^G(\psi_7^3 + \psi_7^5 + \psi_7^6).\end{array}$$

*On the $G$-orbit of* $P = (1:\zeta_3:\zeta_3^2)$, *we have*

$$\sum_{P\in\mathcal{O}} Ind_{G_P}^G(\sum_{\ell=1}^{e_P-1}\ell\psi_P^\ell) = Ind_{C_3}^G(7(\psi_3 + 2\psi_3^2)),$$

*and on the $G$-orbit of* $P = (1:\zeta_3:\zeta_3^2)$, *we have*

$$\sum_{P\in\mathcal{O}} Ind_{G_P}^G(\sum_{\ell=1}^{e_P-1}\ell\psi_P^\ell) = Ind_{C_3}^G(7(\psi_3^2 + 2\psi_3)).$$

*Putting these together, we have*

$$\Gamma_G = 7Ind_{C_7}^G(\psi_7 + \psi_7^2 + \psi_7^4) + 14Ind_{C_7}^G(\psi_7^3 + \psi_7^5 + \psi_7^6) + 21Ind_{C_3}^G(\psi_3 + \psi_3^2).$$

*The character table of $G$ can be used to determine the decomposition of this representation into irreducibles. We have*

$$\Gamma_G = 21\chi_2 + 21\chi_3 + 63\chi_4 + 84\chi_5,$$

*where* $\chi_i$ $(i = 1, 2, ..., 5)$ *denote the irreducible representation, as ordered by GAP (or MAGMA) in the table above. This gives*

$$\tilde{\Gamma}_G = \frac{1}{|G|}\Gamma_G = \chi_2 + \chi_3 + 3\chi_4 + 4\chi_5.$$

*Under the action of the Galois group $C$ generated by $\sigma$ and $\tau$, where* $\sigma : \psi_3 \mapsto \psi_3^{-1}$ *and* $\tau : \psi_7 \mapsto \psi_7^{-1}$, *we have* $\tau(\tilde{\Gamma}_G) = \chi_2 + \chi_3 + 4\chi_4 + 3\chi_5$ *and* $\sigma(\tilde{\Gamma}_G) = \tilde{\Gamma}_G$.

[5] Namely, the $G$-orbit of $(1:0:0)$ with inertia group $C_7$, the $G$-orbit of $(1:\zeta_3:\zeta_3^2)$ with inertia group $C_3$, and the $G$-orbit of $(1:\zeta_3^2:\zeta_3)$ with inertia group $C_3$.

[6] That is, the size of the inertia group, $e_P = |G_P|$.

*Next, we compute the equivariant degree of the $G$-equivariant divisors $D_r = r(P_1 + P_2 + P_3)$, for $r = 2, 3$. Recall that $P_1 + P_2 + P_3$ is the $G$-orbit of $P_3$. Using the ramification character $\psi_7$ of $G_{P_3} \cong C_7$, we have*

$$deg_{eq}(D_2) = Ind_{C_7}^G(\psi_7^{-1} + \psi_7^{-2}) = Ind_{C_7}^G(\psi_7^5 + \psi_7^6),$$

*and*

$$deg_{eq}(D_3) = Ind_{C_7}^G(\psi_7^{-1} + \psi_7^{-2} + \psi_7^{-3}) = Ind_{C_7}^G(\psi_7^4 + \psi_7^5 + \psi_7^6).$$

*Using the character table of $G$, we find*

$$deg_{eq}(D_r) = \begin{cases} \chi_5, & r = 1, \\ 2\chi_5, & r = 2, \\ \chi_4 + 2\chi_5, & r = 3, \\ \chi_4 + 3\chi_5, & r = 4, \\ 2\chi_4 + 3\chi_5, & r = 5, \\ 3\chi_4 + 3\chi_5, & r = 6, \\ 3\chi_1 + 3\chi_4 + 3\chi_5, & r = 7, \end{cases}$$

*and $deg_{eq}(D_{r+7}) = deg_{eq}(D_r) + deg_{eq}(D_7)$. Now we use Borne's formula to compute the class of the Riemann-Roch space of $D_r$, $r = 1, 2$. We have,*

$$\begin{gathered} [L(D_2)] = k[G] + deg_{eq}(D_2) - \tilde{\Gamma}_G \\ = (\chi_1 + \chi_2 + \chi_3 + 3\chi_4 + 3\chi_5) + 2\chi_5 - (\chi_1 + \chi_2 + \chi_3 + 3\chi_4 + 4\chi_5) \\ = \chi_1 + \chi_5, \end{gathered}$$

*and*

$$\begin{gathered} [L(D_3)] = k[G] + deg_{eq}(D_2) - \tilde{\Gamma}_G \\ = (\chi_1 + \chi_2 + \chi_3 + 3\chi_4 + 3\chi_5) + (\chi_4 + 2\chi_5) - (\chi_1 + \chi_2 + \chi_3 + 3\chi_4 + 4\chi_5) \\ = \chi_1 + \chi_4 + \chi_5. \end{gathered}$$

*This is enough data to use our formula to compute the Galois orbit of the class of the ramification module for $G$. We have*

$$\begin{gathered} dim(\chi_1 + \chi_2)^{C_3} = 0, \qquad dim(\chi_1 + \chi_2)^{C_7} = 2, \\ dim(\chi_4 + \chi_5)^{C_3} = 2, \qquad dim(\chi_4 + \chi_5)^{C_7} = 0. \end{gathered}$$

*We have already computed* $R_{C_3} = 2$, $R_{C_7} = 2$, *so our formula computes* $\tilde{\Gamma}_G + \tau\tilde{\Gamma}_G$ *to be*

$$2\chi_1 + 2\chi_2 + 7\chi_4 + 7\chi_5,$$

*which agrees with our direct computation above.*

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