Representations of finite groups on a Riemann-Roch space of a toric variety

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Abstract

We study the action of a finite group on the Riemann-Roch space of a toric variety. Our main result is the following: if G is a finite subgroup of the automorphism group of a complete, smooth toric variety X and if D is a divisor on X fixed by G then we write the natural representation of G on $L(D)$ is a direct sum of explicitly described (permutation) representations.

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Let X be a smooth projective variety over an algebraically closed field F of characteristic 0 and let G be a finite subgroup of automorphisms of X over F. We often identify X with its set of F-rational points $X(F)$. If D is a divisor [1] of X which G leaves fixed then G acts on the Riemann-Roch space $L(D)$. We ask: which (modular) representations arise in this way?

This question was discussed in [JT] in the case when X is a curve. Here we discuss the simpler case of a (not necessarily smooth) toric variety.

[1] Since X is smooth, we need not distinguish between Weil divisors and Cartier divisors (Hartshorne [H], Prop. II.6.11, and Remark II.6.11.1A).

1 The action of G on $L(D)$

Let X be a smooth projective variety over an algebraically closed field F. Let $F(X)$ denote the function field of X (the field of rational functions on X) and, if D is any divisor on X, let

$$L(D) = \{f \in F(X)^\times \mid (f) + D \geq 0\} \cup \{0\},$$

and let $\ell(D)$ denote its dimension. This definition makes sense even if X is merely normal and D is a Weil divisor.

The action of $\mathrm{Aut}(X)$ on $F(X)$ is defined by

$$\begin{array}{rccc} \rho: & \mathrm{Aut}(X) & \to & \mathrm{Aut}(F(X)), \\ & g & \longmapsto & (f \longmapsto f^g) \end{array}$$

where $f^g(x) = (\rho(g)(f))(x) = f(g^{-1}(x))$.

Of course, $\mathrm{Aut}(X)$ also acts on the group $Div(X)$ of divisors of X, denoted $g : D \longmapsto g(D) = \sum_P d_P g(P)$, for $g \in \mathrm{Aut}(X)$ and $D = \sum_P d_P P \in Div(X)$. It is easy to show that $(f^g) = g(f)$, where (f) denotes the (principal) divisor of the function $f \in F(X)$. Because of this, if $(f) + D \geq 0$ then $(f^g) + g(D) \geq 0$, for all $g \in \mathrm{Aut}(X)$.

If $G \subset \mathrm{Aut}(X)$ leaves $D \in Div(X)$ fixed then we denote the induced representation of G in $L(D)$ again by ρ:

$$\rho : G \to \mathrm{Aut}(L(D)).$$

2 Examples

Let Δ be a fan in a lattice L. Denote by τ_1, ..., τ_n the edges or rays of the fan and let v_i denote the first (smallest) lattice point along the ray τ_i. Let D_i denote the Weil divisor

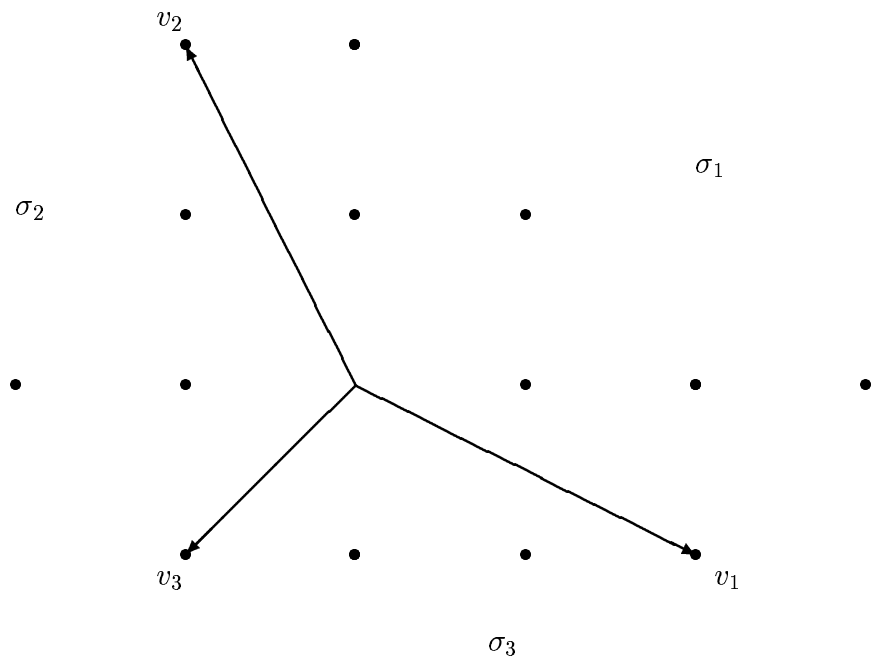
$$D_i = \mathrm{Hom}(\tau_i^\perp \cap L^\perp, \mathbb{C}^\times),$$

which maybe regarded as the closure of the orbit of T acting on the edge τ_i.

Example 1 *Let Δ be the fan generated by*

$$v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + 2e_2, \quad v_3 = -e_1 - e_2.$$

In the notation above, the divisor $D = d_1D_1 + d_2D_2 + d_3D_3$ *is a Cartier divisor* [2] *if and only if* $d_1 \equiv d_2 \equiv d_3 \pmod 3$.



Let G *denote the automorphism group of* Δ *generated by the map* g *sending* (x, y) *to* (y, x)*, swapping* v_1 *and* v_2 *and leaving* v_3 *fixed. By Theorem 1.13 in Oda [O], this corresponds to a* T*-equivariant automorphism of* $X(\Delta)$.

Let

$$\begin{aligned} P_D &= \{(x, y) \mid \langle (x, y), v_i \rangle \geq -d_i, \ \forall i\} \\ &= \{(x, y) \mid 2x - y \geq -d_1, -x + 2y \geq -d_2, -x - y \geq -d_3\} \end{aligned}$$

denote the polytope associated to the Weil divisor $D = d_1D_1 + d_2D_2 + d_3D_3$*, where* D_i *is as above.*

Let $d_1 = d_2 = 6$ *and* $d_3 = 0$*. Then* P_D *is a triangle in the plane with vertices at* $(-6, -6)$*,* $(-2, 2)$*, and* $(2, -2)$*. Note that it remains invariant under the action of* G*. Moreover, the* G *action on* $P_D \cap L^\perp$ *has* 7 *singleton orbits (the lattice points* $(-i, -i)$*, where* $0 \leq i \leq 6$*) and* 12 *orbits of size* 2.

[2] This is an Exercise on page 65 of [F], the solution of which is an easy calculation using the Exercise on page 62, which is in turn, basically solved in the back of the book.

In this example, the patch U_{σ_1} is an affine variety with coordinates x_1, x_2, x_3 given by $x^3 - x_2x_3 = 0$. The automorphism g acts on U_{σ_1} sending (x_1, x_2, x_3) to (x_1, x_3, x_2). The torus embedding $T \hookrightarrow U_{\sigma_1}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1t_2, t_1t_2^2, t_1^2t_2)$.

The patch U_{σ_2} is an affine variety with coordinates y_1, y_2, y_3 given by $y^2 - y_1y_3 = 0$. The automorphism g does not act on U_{σ_2}. The torus embedding $T \hookrightarrow U_{\sigma_2}$ is given by sending (t_1, t_2) to $(y_1, y_2, y_3) = (t_1^{-2}t_2^{-1}, t_1^{-1}, t_1^{-1}t_2)$.

The patch U_{σ_3} is an affine variety with coordinates z_1, z_2, z_3 given by $z^2 - z_1z_3 = 0$. The automorphism g does not act on U_{σ_3}. The torus embedding $T \hookrightarrow U_{\sigma_3}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1^{-1}t_2^{-2}, t_2^{-1}, t_1t_2^{-1})$. The automorphism g sends U_{σ_2} to U_{σ_3} by sending (y_1, y_2, y_3) to (z_1, z_2, z_3).

Example 2 *Let $X = \mathbb{P}^n$ with homogeneous coordinates $x_0, ..., x_n$, and let H denote the hyperplane divisor $x_0 = 0$. In §III.1.5 of [Sh], we see that $L(dH)$ is isomorphic to the space of all (inhomogeneous) polynomials in $X_1 = x_1/x_0$, ..., $X_n = x_n/x_0$ of degree less than or equal to d. Of course, X is a toric variety associated to the lattice*

$$L = \{a \in \mathbb{Z}^{n+1} \mid a_1 + ... + a_{n+1} = 0\},$$

and the fan Δ constructed from the cones

$$\sigma_j = \mathbb{R}_{\geq 0}[e_1 - e_j, ..., e_{n+1} - e_j],$$

for $1 \leq j \leq n+1$. Of course, the automorphism group of X (as a projective variety) is $PGL(n+1)$. (The toric automorphism group of X is the subgroup of $PGL(n+1, \mathbb{Z})$ preserving this fan and the form $x_1 + ... + x_{n+1}$ (up to a scalar) - namely the symmetric group S_{n+1}.) If G is any subgroup of $PGL(n+1)$ which leaves x_0 fixed then G also acts on $L(dH)$. Clearly,

$$\mathbb{C} \subset L(H) \subset L(2H) \subset ... \subset L(dH)$$

is a sequence of invariant subspaces. Assume

$$G = \{g \in \mathrm{Aut}(X) = PGL(n+1) \mid g(e_1) = \lambda \cdot e_1\}$$

so does not fix any other x_i, $i \neq 0$, but G fixes H. The space $L(H)$ is $(n+1)$-dimensional, so $L(H)/\mathbb{C}$ is n-dimensional. It is clearly an irreducible representation of G. Indeed, each quotient $L((k+1)H)/L(kH)$ is irreducible, so $L(dH)$ is a direct sum of these irreducibles.

3 The main result

The following is our main result. Let Δ be a fan associated to an integral lattice L in $\mathbb{R}^n$. Let P_D denote the polytope associated to the Weil divisor D on $X = X(\Delta)$.

Theorem 3 *Let $X = X(\Delta)$ be a complete toric variety defined over an algebraically closed field F of characteristic 0. Suppose $G \subset \mathrm{Aut}(X)$ is a finite subgroup and that the T-invariant Weil divisor D on X is fixed by G. Let P_D denote the polytope associated to D and let $S = P_D \cap L^{\perp}$. Let*

$$S = S_1 \cup S_2 \cup ... \cup S_k$$

denote the decomposition of S into disjoint primitive G-orbits. For each $m \geq 1$, the natural representation π of G on $L(D)$ decomposes into a direct sum of k irreducible permutation representations

$$\pi = \oplus_{i=1}^{k} \pi_i,$$

where π_i is $|S_i|$-dimensional.

proof: All T-equivariant automorphisms of X correspond to automorphisms of L preserving Δ (see Theorem 1.13 in Oda [O]). So $g \in G$ acts on $L(D)$ via its action on S.

Note that there is a natural isomorphism $L(D) \cong \Gamma(X, \mathcal{O}(D))$ (see for example Griffiths and Harris [GH], page 136). We use this map to pull back the action of G on $\Gamma(X, \mathcal{O}(D))$ to an action on $L(D)$. By Fulton [F], page 66, we have

$$L(D) = \oplus_{u \in S} \mathbb{C} \cdot \chi^u,$$

where $\chi = (x_1, ..., x_n)$ and χ^u is the associated monomial in multi-index notation. This shows that π acting on $L(D)$ has an $|S_i|$-dimensional subrepresentation π_i acting on $\oplus_{u \in S_i} \mathbb{C} \cdot \chi^u$. Every finite dimensional representation of G is semi-simple (Mascke's Theorem, Thrm 3.14, [CR], or [Se], §15.7). ¿From these, the theorem follows. □

Question: Given G, X, and D as above. Do the (equivalence classes of) representations of G occurring in $L(D)$ determine D up to linear equivalence?

Corollary 4 *In the notation of the theorem above, the trace of $\pi(g)$ is the number of fixed points of g on S.*

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