A hitch-hiker's guide to genuine invariant distributions on metaplectic covers of SL(2) over a p-adic field

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1-4-94

Abstract

This is an expository account of certain aspects of non-abelian harmonic analysis on n-fold metaplectic covers of $SL(2)$ over a p-adic field. Here we allow n to be any number greater than 1 and p to be any prime. We will conjecturally describe the admissible, tempered, and unitary duals of the n-fold metaplectic cover of $SL(2)$. Then we will give the Paley-Weiner theorems on these dual spaces based on these conjectural descriptions.

1 Introduction

This paper would be essentially an expository article if it weren't for all the conjectures. As it stands, it is a survey of what isn't known about the representations theory of central covering groups of p-adic reductive groups in the case of $SL(2)$. It is not by any means complete (I don't know all that isn't known, ...) nor is it a guide to the literature. Proofs and pointers to the literature are often omitted, although everything which is a theorem should be in one of the references listed at the end of the paper. We will follow the pattern of the expository paper [J1] which deals with the same topics (in more detail) in the case $n = 1$ and $p > 2$.

Although this paper will concern itself only with certain issues in harmonic analysis on groups over a p-adic field F, some applications will be to harmonic analysis on groups over a global field, so our set-up will partially reflect the specific applications which need not concern us here. Our motivation is to verify the analogs of the local assumptions in [A] needed for the validity of Arthur's invariant trace formula for the n-fold metaplectic cover of $SL(2, \mathbb{A}_k)$. We have merely succeeded in reducing the local assumptions of [A] down to a list of specific conjectures regarding irreducible representations of the n-fold covering group of $SL(2, F)$. The readers will have to decide for themselves whether or not I've made already muddy waters even worst. In any case, for this reason sometimes we will be working with $SL(2, F)$ and sometimes with a group G defined over $\mathbb{Q}$ as in [A].

1.1 Basic notation

Let F be a p-adic field with uniformizer $\pi = \pi_F$, ring of integers O_F, residual characteristic $p = char(O_F/\pi O_F)$, $q = \#(O_F/\pi O_F)$, and normalized valuation $|...| = |...|_F$. Suppose that $k/\mathbb{Q}$ is a number field containing the $n-th$ roots of unity μ_n. (If $n > 2$ then this implies F has no real places, i.e, $k_v = \mathbb{C}$ for all $v \in \Sigma_k^\infty$.) For $F = k_v$ p-adic, let $O = O_F = O_v$ denote the ring of integers of F, $\pi = \pi_F = \pi_v$ a local uniformizer, $q = q_v$ the cardinality of the residue field, and $\nu = \nu_F = \nu_v : F^\times \to \mathbb{Z}$ the normalized valuation, so that $\nu(\pi) = q^{-1}$.

Let G denote the reductive group over $\mathbb{Q}$ defined from $G_0 = SL(2)/k$ via restriction of scalars $k/\mathbb{Q}$, so for example $G(\mathbb{Q})$ $=SL(2, k)$. Let $A_0 \subset G_0$ denote the diagonal subgroup and let $A = Res_{k/\mathbb{Q}}(A_0/k)$.

The subgroups

$$U_0 = O_F^\times,\ U_n = 1 + \pi^n O_F, \quad n > 0,$$

form a basis of open neighborhoods of the identity in $F^\times$, generating the topology on $F^\times$. The subgroups

$$K_0 = SL(2, O_F),\ K_n = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, d \in 1 + \pi^n O_F,\ b, c \in \pi^n O_F \right\}, \quad n > 0,$$

form a basis of open neighborhoods of the identity in $SL(2, F)$, generating its compact-open topology. A **character** or **representation** of $F^\times$ or of $SL(2, F)$ will always mean a continuous multiplicative (not necessarily unitary) homomorphism into the automorphisms of a complex vector space, where the domain gets the "compact-open" topology above and the range gets the discrete topology.

Occasionally we use **Vinogradov's notation**: given two functions $f(x)$, complex-valued and perhaps depending on some parameter λ, and $g(x) > 0$, we write

$$f(x) <<_\lambda g(x),$$

if there is a constant $c > 0$, which may implicitly depend on λ, such that $|f(x)| \leq cg(x)$ for all x in some range which depends on the context.

2 Background

2.1 Local covers of $SL(2)$

2.1.1 Over F

Let $F = k_v$, where $v \in \Sigma_k$. By our assumptions, $\mu_n \subset F^\times$. If $p < \infty$ then

$$F^\times = \pi^{\mathbb{Z}} \cdot \mu_{q-1} \cdot U_1,$$

a direct product. If $(p, 2n) = 1$ then $\mu_n \subset F^\times$ implies $q \equiv 1$ $(mod\ n)$. Let

$$N = \begin{cases} n, & n \text{ odd}, \\ n/2, & n \text{ even}. \end{cases} \tag{1}$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F)$, let

$$x(g) = \begin{cases} c, & c \neq 0, \\ d, & c = 0, \end{cases}$$

and let $\beta = \beta_v : SL_2(F) \times SL_2(F) \to \mu_n$ be defined by

$$\begin{aligned} \beta(g_1, g_2) &= (x(g_1), x(g_2))_n \cdot (-x(g_1)^{-1}x(g_2), x(g_1g_2))_n \\ &= (\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)})_n, \end{aligned} \tag{2}$$

where $(..., ...)_n = (..., ...)_{n,v} : F^\times \times F^\times \to \mu_n$ denotes the Hilbert symbol [W]. Note that this is trivial if $F = \mathbb{C}$.

Lemma 2.1 *(Kubota [K], Moore [M]) The map β defines a continuous Borel 2-cocycle (giving μ_n the discrete topology of course). Its cohomology class in the continuous cohomology group $H^2(SL_2(F), \mu_n)$ corresponds to an n-fold topological central extension $\overline{SL_2(F)}$ which fits into the short exact sequence*

$$1 \to \mu_n \to \overline{SL_2(F)} \xrightarrow{p} SL_2(F) \to 1.$$

If $H < SL_2(F)$ is a subgroup then we define $\overline{H} = p^{-1}(H) \subset \overline{SL_2(F)}$.

Elements of $\overline{SL_2(F)}$ will be denoted by (g, ς), where $g \in SL_2(F)$, $\varsigma \in \mu_n$. Multiplication on $\overline{SL_2(F)}$ is given by

$$(g_1, \varsigma_1) \cdot (g_2, \varsigma_2) = (g_1 g_2, \beta(g_1, g_2)\varsigma_1\varsigma_2). \tag{3}$$

so inverses are given by

$$(g, \varsigma)^{-1} = (g^{-1}, \beta(g, g^{-1})^{-1}\varsigma^{-1}) \tag{4}$$

and conjugation is given by

$$(g, \varsigma) \cdot (h, \varsigma') \cdot (g, \varsigma)^{-1} = (ghg^{-1}, \beta(gh, g^{-1})\beta(g, h)\beta(g, g^{-1})^{-1}\varsigma'). \tag{5}$$

We let $B_0 = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subset SL_2$ denote the standard Borel, N_0 its unipotent radical, and A_0 its Levi component.

Lemma 2.2 *For all $g \in A(F)$,*

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

$\varsigma \in \mu_n$, we have $(g, \varsigma)^{-1} = (g^{-1}, (a, a)_n \varsigma^{-1})$.

proof: This follows from the general properties: for $g_1, g_2 \in A_0(F)$,

$$g_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix},$$

we have $\beta(g_1, g_2) = (a_2^{-1}, a_1^{-1})_n$,

(a) $(a_1, a_2)_n = 1$, provided $a_1 \in A_0(F)^n$ or $a_2 \in A_0(F)^n$ (the set of n-th powers),

(b) $(a_1, a_2)_n = (a_2, a_1)_n^{-1}$,

(c) $(a_1^{-1}, a_2)_n = (a_1, a_2)_n^{-1}$. $\square$

Lemma 2.3 *(a)* $\overline{N_0(F)} = N_0(F) \times \mu_n$ *(direct product),*

(b) if p *is relatively prime to* n *then* $\overline{SL_2(O)} = SL_2(O) \times \mu_n$ *(direct product),*

(c) if $K_0 = SL_2(O)$ *and*

$$K_m = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O) \mid a-1, d-1, b, c \in \pi^m O\},\ m > 0,$$

then $\overline{K_m} = K_m \times \mu_n$ *(direct product) for all* $m \geq n$*,*

(d) if $n = 2$, $F = k_v = \mathbb{R}$, *and* $K_v = SO(2, \mathbb{R})$ *then* $\overline{K_v} \neq K_v \times \mu_n$*,*

(e) The center of $\overline{SL_2(F)}$ *is* $Z(\overline{SL_2(F)}) = \{\pm 1_2\} \times \mu_n$*,*

(f) For $a_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{-1} \end{pmatrix} \in A_0(F)$, $i = 1, 2$, *and* $\varsigma_i \in \mu_n$, *we have* $(a_1, \varsigma_1) \cdot (a_2, \varsigma_2) \cdot (a_1, \varsigma_1)^{-1} = (a_2, (t_1, t_2)_n^{-2} \varsigma_2)$*,*

(g) the largest abelian subgroup of $\overline{A_0(F)}$ *contains*

$$C_F = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in (F^\times)^N\},$$

where N *is as in (2.1). (In particular,* $\overline{A_0(F)}$ *is abelian if* $n = 1, 2$*.)*

Remark 2.4 *In contrast to* $GL(r)$*, for* $G_0 = SL(2)$ *the centralizer in* $\overline{G_0(F)}$ *of a regular element of the largest abelian subgroup of* $\overline{A_0(F)}$ *is not equal to* $\overline{A_0(F)}$*.*

proof: These are conseqences of the well-known properties of the Hilbert symbol [W].

For (d) see, for example, [Ge] or [Sa].

For (f), we have

$$\begin{aligned}(a_1, \varsigma_1)^{-1} \cdot (a_2, \varsigma_2) \cdot (a_1, \varsigma_1) &= (a_1^{-1}, (t_1, t_1)_n \varsigma_1^{-1}) \cdot (a_2, \varsigma_2) \cdot (a_1, \varsigma_1) \\ &= (a_2, (t_1, t_1)_n \varsigma_2 \beta(a_1^{-1}, a_2 a_1) \beta(a_2, a_1)).\end{aligned}$$

Now we compute

$$\begin{aligned}\beta(a_2, a_1) &= (t_2^{-1}, t_1^{-1})_n \cdot (-t_2 t_1^{-1}, t_2^{-1} t_1^{-1})_n \\ &= (t_2^{-1}, t_1^{-1})_n \cdot (-t_2 t_1^{-1}, t_2^{-1})_n \cdot (-t_2 t_1^{-1}, t_1^{-1})_n \\ &= (t_2^{-1}, t_1^{-1})_n \cdot (-t_2, t_2^{-1})_n \cdot (t_1^{-1}, t_2^{-1})_n \cdot (-t_2, t_1^{-1})_n \cdot (t_1^{-1}, t_1^{-1})_n \\ &= (-t_2, t_2^{-1})_n \cdot (-t_2, t_1^{-1})_n \cdot (t_1, t_1)_n\end{aligned}$$

and

$$\begin{aligned}\beta(a_1^{-1}, a_2a_1) &= (t_1, t_2^{-1}t_1^{-1})_n \cdot (-t_2^{-1}t_1^{-2}, t_2^{-1})_n \\ &= (t_1, t_2^{-1})_n \cdot (t_1, t_1^{-1})_n \cdot (-t_2^{-1}, t_2^{-1})_n \cdot (t_1^{-2}, t_2^{-1})_n \\ &= (t_1, t_1^{-1})_n \cdot (-t_2^{-1}, t_2^{-1})_n \cdot (t_1^{-1}, t_2^{-1})_n\end{aligned}$$

The result follows since $(a_1, a_1)_n = (-a_1, a_1)_n \cdot (-1, a_1)_n = (-1, a_1)_n$.□

Corollary 2.5 *If $n = 2$ then the 2-fold metaplectic cover $\overline{A_0(F)}$ of $A_0(F)$ is abelian.*

For the purposes of the next section, we want to determine the maximal abelian subgroup of the n-fold cover of $\overline{A_0(F)}$, where $F = k_{v_i}$ for some $1 \le i \le m$. We may assume that F is non-archimedian since

(a) if $n = 2$ then $\overline{A_0(F)}$ is abelian,

(b) if $n > 2$ and if F is archimedian then $\overline{A_0(F)}$ is abelian (in fact, $\overline{A_0(F)} \cong \mathbb{C}^\times \times \mu_n$ splits).

Lemma 2.6 *A subgroup $\overline{C} \subset \overline{A_0(F)}$ is abelian if and only if $(a, a')_n^2 = 1$ for all $a, a' \in A_0(F)$.*

proof: This is an immediate consequence of Lemma 2.3(f) above. □

Note that if $C \subset A_0(F)$ is the maximal subgroup of $A_0(F)$ for which $\overline{C} \subset \overline{A_0(F)}$ is abelian then $F^{\times N} \subset C$, so $F^\times/C$ is finite. We may assume that $C = \pi^{a\mathbb{Z}}U$, for some open $U \subset O_F$. If $x = \pi^k u$ and $y = \pi^\ell v$, for $k, \ell \in \mathbb{Z}$, $u, v \in U$ then

$$(x, y)_n = (\pi, \pi)_n^{\ell k}(u^\ell/v^k, \pi)_n(u, v)_n. \tag{6}$$

If $(p, N) = 1$ then $1 + \pi O_F$ is N-divisible in the sense that $1 + \pi O_F = (1+\pi O_F)^N$. Identify μ_{q-1} with $O_F^\times/(1+\pi O_F)$. From this, the above equation, and Lemma 2.4, we have proven

Lemma 2.7 *(a) If $(p, N) = 1$ then $C = \pi^{\mathbb{Z}}O_F^{\times N} = \pi^{\mathbb{Z}}\mu_{q-1}^N(1 + \pi O_F)$ is the maximal subgroup of $F^\times$ for which $\overline{C} \subset \overline{A_0(F)}$ is abelian.*

(b) If $N = p^r d$, where $(d, p) = 1$, and if $e \ge 0$ is defined by $pO_F = \pi^e O_F$, then $C = \pi^{\mathbb{Z}}O_F^{\times N} = \pi^{\mathbb{Z}}\mu_{q-1}^d(1 + \pi^{re+1}O_F)$ is a subgroup of $A_0(F)$ for which $\overline{C} \subset \overline{A_0(F)}$ is abelian.

Remark 2.8 *The determination of C should in principle follow from known explicit reciprocity laws. Unfortunately, they become rather cumbersome when $p=2$ (which is the case we need them). See for example, [Sen].*

The order of $F^\times/\pi^{\mathbb{Z}}(1+\pi O_F)$ is $q-1$, since

$$F^\times/\pi^{\mathbb{Z}}(1+\pi O_F) \cong O_F^\times/(1+\pi O_F) \cong (O_F/\pi O_F)^\times. \tag{7}$$

The order of μ_{q-1}^N is $(q-1)/(q-1,N)$. Therefore, the order of $F^\times/\mu_{q-1}^N\pi^{\mathbb{Z}}(1+\pi O_F)$ is $(q-1,N)$. The fact that F contains the n-th roots of unity implies that $N|(q-1)$. This proves part (a) of the following

Lemma 2.9 *(a) If $(p,n)=1$ then $C=\pi^{\mathbb{Z}}(1+\pi O_F)\mu_{q-1}^N$ has index N in $F^\times$.*

(b) In general, the index of $C=\pi^{\mathbb{Z}}O_F^{\times N}=\pi^{\mathbb{Z}}\mu_{q-1}^d(1+\pi^{re+1}O_F)$ in $F^\times$ is $(q-1,d)\cdot q^{er}$.

proof: (b) The map

$$x \longmapsto 1+\pi x+\pi^{re+1}O_F$$

induces an isomorphism

$$O_F/\pi^{re}O_F \to (1+\pi O_F)/(1+\pi^{re+1}O_F).$$

Therefore, we have

$$\begin{aligned}&\#[(1+\pi O_F)/(1+\pi^{re+1}O_F)]q^{re},\\ &\#[\mu_{q-1}/\mu_{q-1}^d]=(q-1,d).\end{aligned}$$

The lemma follows. □

2.1.2 Over $\mathbb{Q}_p$

There is an $\alpha\in E$ be such that $k=\mathbb{Q}(\alpha)$ and let $f\in\mathbb{Q}[x]$ denote the minimal polynomial of α. If f factors as $f(x)=f_1(x)...f_m(x)$, where each $f_i\in\mathbb{Q}_p[x]$ is irreducible then

$$k\otimes_{\mathbb{Q}}\mathbb{Q}_p=\bigoplus_{1\le i\le m}\mathbb{Q}_p(\alpha_i),$$

where α_i is a root of f_i, $1 \leq i \leq m$. The prime ideal (p) of $\mathbb{Z}$ when extended to the ring of integers of k, $\mathcal{O}_k$, factors into a product

$$pO_k = P_1^{e_1}...P_m^{e_m}$$

Moreover, after a possible reindexing, for each place $v_i \in \Sigma_k$ (corresponding to the prime ideal P_i) over p there is a field isomorphism $\theta_i : \mathbb{Q}_p(\alpha_i) \to k_{v_i}$, $1 \leq i \leq m$. It follows that

$$G \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{1 \leq i \leq m} G_i, \tag{8}$$

where $G_i = Res_{k_{v_i}/\mathbb{Q}_p}(SL(2)/k_{v_i})$. In particular,

$$G(\mathbb{Q}_p) = \prod_{1 \leq i \leq m} k_{v_i}^{\times}. \tag{9}$$

Note that $G(\mathbb{Q}_p) = \prod_{v|p} SL_2(k_v)$. Define $\beta_p : G(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \to \mu_n$ by

$$\beta_p(g_1, g_2) = \prod_{v|p} \beta_v(g_{1,v}, g_{2,v}).$$

This is a 2-cocycle by Lemma 2.1 above. Define $\overline{G(\mathbb{Q}_p)}$ to be the n-fold covering of $G(\mathbb{Q}_p)$ associated to this 2-cocycle. We may identify $\overline{G(\mathbb{Q}_p)}$ with a subgroup of $\prod_{v|p} \overline{SL_2(k_v)}$.

Let $B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subset G$ denote the standard Borel (defined over $\mathbb{Q}$), N its unipotent radical, and A its Levi component.

Lemma 2.10 *(a)* $\overline{N(\mathbb{Q}_p)} = N(\mathbb{Q}_p) \times \mu_n$ *(direct product),*

(b) if p is relatively prime to n then $\overline{G(\mathbb{Z}_p)} = G(\mathbb{Z}_p) \times \mu_n$ *(direct product),*

(c) if $K_0 = G(\mathbb{Z}_p)$ *and*

$$K_m = \prod_{v|p} \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k_v) \mid a-1, d-1, b, c \in \pi_v^m O_v\},\ m > 0,$$

then $\overline{K_m} = K_m \times \mu_n$ *(direct product) for all* $m \geq n$,

(d) The center of $\overline{G(\mathbb{Q}_p)}$ *is* $Z(\overline{G(\mathbb{Q}_p)}) = \{\pm 1_2\} \times \mu_n$.

(e) For $a_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{-1} \end{pmatrix} \in A(\mathbb{Q}_p)$*,* $i = 1,2$*, and* $\varsigma_i \in \mu_n$*, we have* $(a_1, \varsigma_1)^{-1} \cdot (a_2, \varsigma_2) \cdot (a_1, \varsigma_1) = (a_2, \prod_{v|p} (t_1, t_2)^2_{n,v} \varsigma_2)$*,*

(f) the largest abelian subgroup of $\overline{A(\mathbb{Q}_p)}$ *contains*

$$C_{\mathbb{Q}_p} = \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \prod_{1 \leq i \leq m} (k_{v_i}^{\times})^N \},$$

where N *is as in (2.1). (In particular,* $\overline{A(\mathbb{Q}_p)}$ *is abelian if* $n = 1, 2$*.)*

proof: These are all immediate conseqences of the previous lemmas. □

2.1.3 NAK decompositions

By the above lemma, we may identify $N(\mathbb{Q}_p)$ with a subgroup of $\overline{G(\mathbb{Q}_p)}$ via $n \longmapsto (n, 1)$. We may identify $K_0 = K_{0,p}$ with a subset of $\overline{G(\mathbb{Q}_p)}$ via $k \longmapsto (k, 1)$. If $p|n$ then this image is a subgroup.

Lemma 2.11 *We have*

$$\overline{G(\mathbb{Q}_p)} = N(\mathbb{Q}_p) \cdot \overline{A(\mathbb{Q}_p)} \cdot K_p.$$

Furthermore, if $(g, \varsigma) = (n, 1) \cdot (a, \varsigma') \cdot (k, 1)$ *then the map* $(g, \varsigma) \longmapsto a \ (mod\ Z(G(\mathbb{Q}_p))$ *is well-defined.*

We call a function f of $\overline{A(F)}$ **genuine** if it satisfies

$$f(g, \varsigma) = \varsigma^{-1} \cdot f(g, 1). \tag{10}$$

Let $C_c^\infty(\overline{G_0(F)})$ denote the space of locally constant compactly supported *genuine* functions on $\overline{G_0(F)}$. We call a representation π of $\overline{G_0(F)}$ (resp., of any subgroup H of $\overline{G_0(F)}$) **genuine** if it satisfies

$$\pi(g, \varsigma) = \varsigma \cdot \pi(g, 1). \tag{11}$$

Let $\Pi(\overline{G_0(F)})$ (resp., $\Pi(H)$) denote the set of equivalence classes of genuine (continuous, not necessarily unitary, complex) irreducible representations of $\overline{G_0(F)}$ (resp., H). Let $\Pi_u(\overline{G_0(F)})$ denote the set of equivalence classes of genuine (continuous, complex) irreducible unitary representations of $\overline{G_0(F)}$ (resp., $\Pi_u(H)$).

2.2 Irreducible representations of $\overline{A_0(F)}$

We use Clifford-Mackey theory to determine the genuine irreducible representations of the local covers $\overline{A_0(F)}$. We assume that all representations are smooth (the definition of smooth is recalled in the next section).

Let χ denote a genuine (smooth, one-dimensional) character of $\overline{C}$, where $\overline{C}$ is a maximal abelian subgroup of $\overline{A_0(F)}$ as in the previous subsection. Define $\chi^{a_2}(a_1) = \chi(a_2 a_1 a_2^{-1})$, for $a_1 \in \overline{C},\ a_2 \in \overline{A_0(F)}$. The conjugates χ^{a_2} are of the form

$$\begin{aligned} \chi^{a_2}(a_1) &= (t_1,t_2)_n^2 \cdot \chi(a_1) \qquad (= \chi(a_1),\ \text{if}\ a_i \in \overline{C},\ i=1,2) \\ a_1 &= (\begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, \varsigma_1) \in \overline{C},\ a_2 = (\begin{pmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{pmatrix}, \varsigma_2) \in \overline{A_0(F)}. \end{aligned} \quad (12)$$

In particular, there are no genuine 1-dimensioanl representations of $\overline{A_0(F)}$ unless $n = 1, 2$.

By Pontryagin duality, the characters of $A_0(F)/C$ are in 1-1 correspondence with $A_0(F)/C$ itself. Explicitly, the correspondence is given by $x \longmapsto \chi_x$, where $\chi_x(y) = (x,y)_n^2$. Therefore, there are $\#[A_0(F)/C\ ]$ distinct characters of $A_0(F)$ which are trivial on C. The number of distinct characters of C of the form χ_x ($x \in A_0(F)$) is therefore $\#[A_0(F)/C\ ]$. This proves the following

Lemma 2.12 *Let χ denote a genuine character of $\overline{C}$, where C is as in the previous subsection. The number of distinct conjugates χ^h of χ is $\#[A_0(F)/C]$.*

Let $C_p \subset A(\mathbb{Q}_p)$ denote a maximal subgroup such that $\overline{C_p} \subset \overline{A(\mathbb{Q}_p)}$ is abelian.

Lemma 2.13 *(Clifford-Mackey) If $\chi \in \Pi(\overline{C_p})$ then $\chi^h \neq \chi$ for all $h \in \overline{A(\mathbb{Q}_p)}/\overline{C_p}$ and*

$$Ind_{\overline{C_p}}^{\overline{A(\mathbb{Q}_p)}}(\chi) \cong Ind_{\overline{C_p}}^{\overline{A(\mathbb{Q}_p)}}(\chi^h), \quad h \in \overline{A(\mathbb{Q}_p)},$$

is both irreducible and genuine.

proof: This follows from (2.13) and an irreducibility criterion of Mackey (see [Se] for the statement of the criterion for finite groups, the proof and statement of which extends to the case considered here). □

To state more complete results, we state the extensions of some results of Clifford to the case of totally disconnected groups (see Gelbart-Knapp [GK], [BZ1, ch. I, section 2]):

Lemma 2.14 *Let N denote an open normal subgroup of a totally disconnected group H for which H/N is finite abelian and let π denote an irreducible admissible representation of H. Then*

(a) there are finitely many irreducible inequivalent admissible representations π_i, $1 \leq i \leq M$, such that

$$\pi|_N = \bigoplus_{i=1}^{M} m_i \pi_i,$$

(b) in the decomposition above, all the multipicities m_i are equal,

(c) the subgroup

$$H_i = \{h \in H \mid \pi_i^h \cong \pi_i\}$$

has the property that H/H_i acts transitively on the set of equivalence classes

$$\{\pi_i \mid 1 \leq i \leq M\} \subset \Pi(H).$$

Lemma 2.15 *Let N denote an open normal subgroup of a totally disconnected group H for which H/N is finite abelian and let π denote an irreducible admissible representation of N. Then*

(a) there is an irreducible admissible representation Π of H such that π is a constituent of $\Pi|_N$,

(b) if Π, Π' are two irreducible admissible representations of H such that (i) π is a constituent of $\Pi|_N$ and $\Pi'|_N$ and (ii) $\Pi|_N$ and $\Pi'|_N$ are multiplicity-free then

$$\Pi|_N \cong \Pi'|_N$$

and

$$\Pi' \cong \Pi \otimes \nu, \quad \text{some } \nu \in Hom(H, \mathbb{C}^\times),\ \nu|_N = 1.$$

(c) Each irreducible admissible representation Π of H (is equivalent to a representation of H which) occurs as a subrepresentation of $Ind_N^H \pi$, for some irreducible admissible representation of N.

Remark 2.16 *Part (c) is a corollary of the proof of [GK, Lemma 2.3], the other parts are as in [GK]. To be precise, part (c) is, when H/N is cyclic of prime order, an immediate corollary of the proof of [GK, Lemma 2.3]. In general, one must verfify that the properties of induction of representations (such as inducing "in stages", induction commutes with direct sums) is compatible with the process of mathematical induction used in the above-mentioned proof. This is left to the reader.*

We apply this lemma with

$$H = \overline{A_0(F)}, \quad N = \overline{C},$$

to obtain

Lemma 2.17 *Let $\chi \in \Pi(\overline{A_0(F)})$. Then*

(a)

$$\chi|_{\overline{C}} = \bigoplus_{i=1}^{m} m_i \mu_i$$

for some distinct characters μ_i of $\overline{C}$ and some integers $m_i > 0$.

(b) If $\chi, \chi' \in \Pi(\overline{A_0(F)})$ satisfy

$$\chi|_{\overline{C}} = \bigoplus_{i=1}^{m} \mu_i, \quad \chi'|_{\overline{C}} = \bigoplus_{i=1}^{m'} \mu'_i,$$

and $\mu_i = \mu'_j$ for some i, j then $m = m'$, $\{\mu_i \mid 1 \leq i \leq m\} = \{\mu'_i \mid 1 \leq i \leq m'\}$, and $\chi' \cong \mu \otimes \chi$ for some character μ of $\overline{A_0(F)}$ which is trivial on $\overline{C}$.

(c) χ is a subrepresentation of $Ind_{\overline{C}}^{\overline{A_0(F)}} \mu$ for some character μ of $\overline{C}$.

Corollary 2.18 *If $\chi = Ind_{\overline{C}}^{\overline{A_0(F)}} \mu \in \Pi(\overline{A_0(F)})$ then*

$$tr\ \chi(a) = \begin{cases} \sum_{h \in \overline{A_0(F)}/\overline{C}} \mu^h(a), & a \in \overline{C}, \\ 0, & a \in \overline{A_0(F)} - \overline{C}. \end{cases}$$

2.3 The dual spaces

In this subsection let G denote an ℓ-group in the sense of [BZ1].

2.3.1 General definitions

Let V denote a complex vector space and let $\pi : G \to Aut(V)$ denote an admissible representation in the sense of [BZ1]. Let V' denote the dual space of all complex-valued linear functionals on V, let V'_∞ denote the subspace of vectors fixed by some compact open subgroup, and let $< v, v' >= v'(v)$ denote the evaluation pairing on $V \times V'$. Recall the contragredient representation of an algebraic (i.e., smooth) $\pi : G \to Aut(V)$ is the representation $\pi' : G \to Aut(V'_\infty)$ satisfying $< v, v' >=< \pi(g)v, \pi'(g)v' >$, for all $v \in V, v' \in V'_\infty$, and $g \in G$. It is known that if a representation $\pi : G \to End(V)$ is admissible then its contragredient representation $\pi' : G \to Aut(V'_\infty)$ is admissible. A matrix coefficient of a representation $\pi : G \to Aut(V)$ is a function on G of the form $g \longmapsto < \pi(g)v, v' >$, for some $v \in V, v' \in V'_\infty$. The **space of matrix coefficients** of π is denoted by $\mathcal{A}(\pi)$.

An admissible representation $\pi : G \to End(V)$ is called **square-integrable** or **in the discrete series** if $\mathcal{A}(\pi) \subset L^2(G)$. It is called **cuspidal** if every function $f \in \mathcal{A}(\pi)$ has compact support modulo the center and satisfies, for each nilpotent radical N of a proper parabolic subgroup of G, $\int_N f(xn)dn = 0$, for all $x \in G$. Clearly, a cuspidal representation is square-integrable. A representation $\pi : G \to Aut(V)$ is called **unitarizable** if V has a positive-definite G-invariant inner product $(*, *)$ (that is, $(\pi(g)v, \pi(g)v') = (v, v')$ for all $v, v' \in V$). The completion of V with respect to this inner product is of course a G-invariant Hilbert space H. Any representation $\pi : G \to Aut(H)$, for some Hilbert space H whose inner product is G-invariant, is called **unitary.** It is known and easy to check even at this level of generality that every square-integrable representation is unitarizable.

We say that a representation $\pi : G \to Aut(V)$ is **equivalent** to a representation $\pi' : G \to Aut(V')$ if there is a non-zero linear transformation $A : V \to V'$ such that $A\pi(g)v = \pi'(g)Av$ for all $g \in G$, $v \in V$. The map A is called an **intertwining map**. The **unitary dual** of G is the set $\Pi_u(G)$ of all equivalence classes of irreducible unitary representations of G and the **admissible dual** of G is the set $\Pi_{ad}(G)$ of all equivalence classes of irreducible admissible representations of G. For the group G of all F-rational points of a reductive group over a p-adic field, it is known that $\Pi_u(G) \subset \Pi_{ad}(G)$ [S].

If $G = \overline{G_0(F)}$ then we will modify this definition slightly: the **unitary dual** of $\overline{G_0(F)}$ is the set $\Pi_u(\overline{G_0(F)})$ of all equivalence classes of genuine irreducible unitary admissible representations of $\overline{G_0(F)}$ and the **admissible**

dual of $\overline{G_0(F)}$ is the set $\Pi_{ad}(\overline{G_0(F)})$ of all equivalence classes of genuine irreducible admissible representations of G. Let $\Pi_c(\overline{G_0(F)}) \subset \Pi_u(\overline{G_0(F)})$ denote the subset of cuspidals.

Each admissible representation $\pi : G \to Aut(V)$ gives rise to a representation $\pi : C_c^\infty(G) \to End(V)$ of the Hecke algebra $C_c^\infty(G)$ of all locally constant functions with compact support on G:

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg, \quad v \in V, \;\; f \in C_c^\infty(G),$$

where dg denotes a Haar measure on G. The linear transformation $\pi(f)$ is sometimes called the **operator-valued Fourier transform** of f at π. For admissible π it is known that $\pi(f)$ is finite rank as an element of $End(V)$, so the trace of $\pi(f)$ exists. Let $||\pi(f)||$ denote its operator norm and let

$$||f||_* = \sup_{\pi \in \hat{G}_u} ||\pi(f)||.$$

The completion of $C_c^\infty(G)$ with respect to this norm $||...||_*$ is a C^*- algebra denoted by $C^*(G)$.

Let A denote a C^*-algebra. A primitive ideal of A is the kernel of an irreducible representation of A. Put the Jacobson topology on the set $\Pr(A)$ of all primitive ideals of A. Let $A^\wedge$ denote the set of equivalence classes of irreducible representations of A on the bounded operators on a Hilbert space. There is a canonical surjection $A^\wedge \hookrightarrow \Pr(A)$ induced by $\pi \longmapsto \ker(\pi)$. Give $A^\wedge$ the smallest topology such that this surjection is continuous. Call this the **Jacobson** topology. This topology is T_0.

We define a topology on $\Pi_u(G)$ using the following

Lemma 2.19 *(Special case of a theorem of Dixmier): Let G denote a finite central covering group of a reductive p-adic group. There is a canonical bijection between the set $\Pi_u(G)$ and the set $C^*(G)^\wedge$ of equivalence classes of irreducible representations of $C^*(G)$.*

We give $\Pi_u(G)$ the smallest topology such that this bijection is continuous. Call this the **Jacobson topology** on $\Pi_u(G)$.

A **CCR algebra** is a C^*-algebra A such that every irreducible representation $\pi : A \to End(H)$ is completely continuous (i.e., compact). It is known, by a result of Fell, that if A is a CCR algebra and if we topologize

$A^\wedge$ using the Jacobson topology as above then the dual space $A^\wedge$ is a T_1 topological space. (The converse is also true, by a result of Dixmier.)

Conjecture 2.20 *(Analog of a theorem of Bernstein) Let G denote a finite topological covering group of a reductive p-adic group. Then $C^*(G)$ is a CCR algebra.*

Remark 2.21 *Can one prove that $C^*(G_0(F))^\wedge$ is T_1 by redefining the topology on the dual using limits of matrix coefficients ([T])?*

Following Mackey, we say that the dual space $A^\wedge$ is **smooth** if there is a Borel structure on it which is Borel isomorphic to the Borel structure on the real line.

Lemma 2.22 *(Fell, Mackey) If A is a separable CCR algebra then $A^\wedge$ has a smooth dual. Moreover, the Borel structure may be taken to be that generated by the Jacobson topology on $A^\wedge$.*

2.3.2 Jacquet functors

Let $\mathcal{L}(G)$ denote the set of all standard Levi's (with respect to A), up to associates. Therefore, $\mathcal{L}(G) = \{A, G\}$.

Let $(\sigma, W) \in \Pi_c(\overline{M(F)})$ and let $I_M(\sigma) : \overline{G(F)} \to Aut(V)$ denote the **unitarily induced representation**: the representation of $\overline{G(F)}$ by right translation on

$$V = \left\{ f : \overline{G(F)} \to W \text{ genuine} \;\middle|\; \begin{array}{c} (1)\;\; f(mg) = \delta_M(m)^{1/2}\sigma(m)f(g), \\ \forall g \in \overline{G(F)},\; m \in \overline{M(F)} \\ (2)\;\; \text{for some open subgp } K \subset\subset \overline{G(F)},\; f(gk) = f(g), \\ \forall k \in K,\; g \in \overline{G(F)} \end{array} \right\}.$$

Here $\delta_M(m) = |\det(Ad(m)_{\mathfrak{n}})|$.

It is remarked in [BD, §2.2] that the arguments of [BZ1, chapter 2] carry over to finite central extensions of reductive groups over a p-adic field (see also [KP, §1.2]). I claim that the arguments of [BZ2, section 2], and the corresponding sections of [Ca], carry over to finite central extensions of split reductive groups over a p-adic field. Perhaps this should be regarded as a conjecture since they will not be repeated here and those parts which we do need here can probably be proven directly with less effort. In any case, we shall state such results as needed for $\overline{G_0(F)}$ without proof, merely referencing [BZ1], [BZ2] and [Ca].

If (π, V) denotes a $\overline{G_0(F)}$-module and if $P = MN$ denotes a standard parabolic of G_0, let $V_{\overline{N(F)}}$ denote the Jacquet module

$$V_{\overline{N(F)}} = V/\{\pi(n)v - v \mid v \in V,\ n \in \overline{N(F)}\}.$$

This sends admissible representations to admissible representations (this is due to Jacquet, see [BZ1, sections 3.16-3.17] for the $GL(r)$ case).

Lemma 2.23 *(Jacquet [Ca, Theorem 5.2.1]) If $V_{\overline{N(F)}} = 0$ for all proper standard parabolics P of G_0 then (π, V) is cuspidal.*

Proposition 2.24 *(Jacquet [BZ1, section 3.19]) If $\pi \in \Pi_{ad}(\overline{G_0(F)})$ then there is a Levi $M \in \mathcal{L}(G_0)$ and a cuspidal $\sigma \in \Pi_{ad}(\overline{M(F)})$ such that π is a constituent of $I_M(\sigma)$.*

In particular, every $\pi \in \Pi_{ad}(\overline{G_0(F)})$ which is not a constituent of an induced representation of a cuspidal representation of a proper Levi occurs discretely in the decomposition of the right regular representation of $\overline{G_0(F)}$ on $L^2(\overline{G_0(F)})$.

Let $\chi, \chi' \in \Pi(\overline{A_0(F)})$. If $\chi^w \neq \chi$ for all $w \in W - \{1\}$ then we call χ **regular**. We say that χ, χ' are W-**conjugate** if $\chi' = \chi^w$ for some $w \in W$. The following two results were proven for metaplectic covers of $GL(r)$ in [KP].

Lemma 2.25 *([BZ2, Corollary 2.13]) Let $\chi \in \Pi(\overline{A_0(F)})$. The Jordan-Holder series of $I_{A_0}(\chi)_{\overline{N(F)}}$ has as its composition factors*

$$Ind_{\overline{C}}^{\overline{A_0(F)}}(\chi^w \cdot \delta_{A_0}^{1/2}), \quad w \in W.$$

If χ is regular then

$$I_{A_0}(\chi)_{\overline{N(F)}} = \oplus_{w \in W} Ind_{\overline{C}}^{\overline{A_0(F)}}(\chi^w \cdot \delta_{A_0}^{1/2}).$$

Proposition 2.26 *([BZ2, Theorem 2.9(b)]) Let $\chi, \chi' \in \Pi(\overline{A_0(F)})$. If χ is regular then*

$$dim\ Hom_{\overline{G_0(F)}}(I_{A_0}(\chi), I_{A_0}(\chi')) \leq 1,$$

with equality if and only if χ', χ are W-conjugate.

In other words, distinct W-conjugacy classes of $\chi \in \Pi(\overline{A_0(F)})$ yield inequivalent representations.

3 Description of the dual spaces

3.1 Principal series, reducible principal series

Let $\chi \in \Pi(\overline{A_0(F)})$. The induced representation $I_{A_0}(\chi)$ is in general not irreducible. However, we do have the following

Theorem 3.1 *(Moen [Mo2]) If n is even and $(p,n) = 1$ then $I_{A_0}(\chi)$ is irreducible for all $\chi \in \Pi_u(\overline{A_0(F)})$.*

In fact, C. Moen [Mo1] explicitly computes the intertwining operators as matrices using the Kirillov model. In the notation of the next section, we have

Proposition 3.2 *(Moen) If n is even and $(p,n) = 1$ then*
(1) if $n = 2$ then $J_\mu J_{-\mu} = 4\Gamma(2\mu)\Gamma(-2\mu)$,
(2) if $n = 2$ then $J_\mu J_{\mu^w} = 16\Gamma(4\mu)\Gamma(-4\mu)$,
(3) if $n > 4$ and $n/2$ is even then

$$\det(J_\mu) = n^{n/2}|y|^{-\alpha n/2}\beta q^{-\alpha n/2}\Gamma(n\alpha)^{\frac{n}{4}+1}\Gamma(-n\alpha)^{\frac{n}{4}-1}\Gamma_\gamma(-\frac{n\alpha}{2}+\frac{1}{2}),$$

(4) if $n > 4$ and $n/2$ is odd then

$$\det(J_\mu) = n^{n/2}|y|^{-\alpha n/2}q^{-\alpha n/2}\Gamma(n\alpha)^{\frac{n}{4}+\frac{1}{2}}\Gamma(-n\alpha)^{\frac{n}{4}-\frac{1}{2}}.$$

Conjecture 3.3 *If n is even then $I_{A_0}(\chi)$ is irreducible for all $\chi \in \Pi_u(\overline{A_0(F)})$.*

Theorem 3.4 *(Moen [Mo2]) If n is odd and $(p, 2n) = 1$ then $I_{A_0}(\chi)$ is irreducible for all $\chi \in \Pi_u(\overline{A_0(F)})$ such that $\chi^w \neq \chi$ where $w = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1)$. If $\chi^w = \chi$ then $I_{A_0}(\chi)$ is reducible and has two irreducible constituents.*

Conjecture 3.5 *If n is odd then $I_{A_0}(\chi)$ is irreducible for all $\chi \in \Pi_u(\overline{A_0(F)})$ such that $\chi^w \neq \chi$ where $w = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1)$. If $\chi^w = \chi$ then $I_{A_0}(\chi)$ is reducible and has two irreducible constituents.*

Proposition 3.6 *(Moen) If n is odd and $(p, 2n) = 1$ then*

$$\det(J_\mu) = n^n |y|^{-\alpha n} q^{-\alpha n} \Gamma(n\alpha)^{\frac{n}{2}+\frac{1}{2}} \Gamma(-n\alpha)^{\frac{n}{2}-\frac{1}{2}}.$$

We can investigate whether or not two induced representations are equivalent by using the Jacquet functor. In general, we have the following corollary to Proposition 2.22 above.

Lemma 3.7 *If $\chi, \chi' \in \Pi(\overline{A_0(F)})$ are not W-conjugate then $I_{A_0}(\chi)$ is not equivalent to $I_{A_0}(\chi')$.*

Suppose that $\pi \in \Pi_u(\overline{G_0(F)})$. We call π a (unitary) **principal series** representation if $\pi = I_{A_0}(\chi)$ for some $\chi \in \Pi_u(\overline{A_0(F)})$. These representations are tempered. (We define tempered in the next section.) We call π a **complementary series** representation if $\pi = I_{A_0}(\chi)$ for some $\chi \in \Pi(\overline{A_0(F)}) - \Pi_u(\overline{A_0(F)})$. These representations are not tempered.

3.2 Complementary series

In this subsection, we shall review the results of Ariturk [Ar], or at least those which easily generalize to the n-fold cover.

Let $\mu \in \Pi(\overline{C})$, $\chi = Ind_{\overline{C}}^{\overline{A_0(F)}} \mu \in \Pi(\overline{A_0(F)})$. If $\mu(x) = \mu_0(x)|x|^s$, for some character μ_0 of finite order and some $s \in \mathbb{C}$ then we write $s = s(\mu) = s(\chi)$.

Let $K(\mu)$ denote the space of locally constant functions $f : F \times \overline{A_0(F)} \to \mathbb{C}$ such that

(i) $f(x, a_1 a_2) = \mu(a_1) f(x, a_2)$, $a_1 \in \overline{C}$, $a_2 \in \overline{A_0(F)}$,

(ii) $|x|\chi(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, 1) f(x, a)$ is constant for $|x|$ large.

Let $R \subset \overline{A_0(F)}$ denote a complete set of representatives of $\overline{A_0(F)}/\overline{C}$, and let r denote the cardinality of R. The elements $f \in K(\mu)$ may be identified with the r-tuple $(f(x, a))_{a \in R}$.

Let

$$F_i = \{x \in F \mid v(x) \equiv i \ (mod\ n)\}$$

and let $\mathcal{S}(F)$ denote the Schwartz space of F.

Lemma 3.8 *([Ar, Lemma 3.1]) For $\phi \in \mathcal{S}(F)$, $\mu_s(x) = |x|^s$ and $0 < \mathrm{Re}(s) < 1$, $1 \le j \le n-1$, then*

$$\int_{F_i} \phi(x)\mu_s(x)(x,\pi^j)_n d^\times x = c_j q^{s-1/2} \int_{F_{(n-1)i-1}} \widehat{\phi}(x)\mu_{1-s}(x)(x,\pi^{-j})_n d^\times x,$$

where $|c_j| = 1$ and $c_1 \ldots c_{n-1} = 1$.

Lemma 3.9 *([Ar]) For $\phi \in \mathcal{S}(F)$, $\mu_s(x) = |x|^s$ and $0 < \mathrm{Re}(s) < 1$, then*

$$\int_{F_i} \phi(x)\mu_s(x)d^\times x = \int_F \widehat{\phi}(x)\mu_{1-s}(x)K_s(x)d^\times x,$$

where K_s is a locally constant function.

Let $V(\mu)$ denote the space of all locally constant functions $\varphi : \overline{G_0(F)} \times \overline{A_0(F)} \to \mathbb{C}$ such that
(i) $\varphi(g, a_1a_2) = \mu(a_1)\varphi(g, a_2)$, $a_1 \in \overline{C}$, $a_2 \in \overline{A_0(F)}$,
(ii) $\varphi(a_1g, a_2) = \delta(a_1)\varphi(g, a_2a_1)$, where $a_1 \in \overline{A_0(F)}$, $a_2 \in \overline{A_0(F)}$.
Here δ denotes the usual modulus function (extended to $\overline{A_0(F)}$ via the obvious pull-back). For $\varphi \in V(\mu)$, let

$$I\varphi(g,a) = \int_F \varphi(w \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot g, waw^{-1})dx, \quad Re(s(\mu)) > 0.$$

Lemma 3.10 *I intertwines $I_{A_0}(\mu)$ and $I_{A_0}(\mu^w)$.*

Let $L(\overline{G_0(F)}, \overline{B_0(F)})$ denote the space of all locally constant functions φ on $\overline{G_0(F)}$ such that

$$\varphi((\begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix}, \varsigma) \cdot g) = |a|^2\varphi(g).$$

For $\varphi_1 \in V(\mu)$, $\varphi_2 \in V(\mu^w)$, the function

$$g \longmapsto \int_{\overline{A_0(F)}/\overline{C}} \varphi_1(g,a)\varphi_2(g,a)da$$

belongs to $L(\overline{G_0(F)}, \overline{B_0(F)})$. Therefore,

$$
\begin{aligned}
< \ \varphi_1, \varphi_2 >= & \int_{\overline{B_0(F)} \backslash \overline{G_0(F)}} \int_{\overline{A_0(F)}/\overline{C}} \varphi_1(g,a)\varphi_2(g,a) da dg \\
= \ & \int_F \int_{\overline{A_0(F)}/\overline{C}} \varphi_1(w^{-1} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a)\varphi_2(w^{-1} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a) da d^\times x
\end{aligned}
$$

gives a non-degenerate bilinear form on $V(\mu) \times V(\mu^w)$.

Lemma 3.11 *$I_{A_0}(\mu^w)$ is the contragredient of $I_{A_0}(\mu)$.*

Lemma 3.12 *For φ_1, $\varphi_2 \in V(\mu)$, $\mu(x) = |x|^s$, we have*

$$
< \varphi_1, I\varphi_2 >= \int_F \int_{\overline{A_0(F)}/\overline{C}} f_1^\wedge(x,a)\overline{(Jf_2^\wedge)(x,a)} da d^\times,
$$

where $J = J_\mu$ is a linear transformation on $K(\mu)^\wedge$ and

$$
f_i(x,a) = \varphi_i(w^{-1} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a), \quad i = 1, 2.
$$

We may identify J with an $r \times r$ matrix which we still denote by J.

Conjecture 3.13 *If $0 \le Re(s(\mu)) \le 1$ and $|Im(s(\mu))| \le \pi/n \ln(q)$ then* $\det(J) = 0$ *if and only if $\mu(x) = |x|^s$ and $s = 1/n$.*

Lemma 3.14 *(Langlands, Ariturk) If $0 \le Re(s(\mu)) \le 1/n$ and $|Im(s(\mu))| \le \pi/n \ln(q)$ then the image of J_μ is an irreducible representation of $\overline{G_0(F)}$.*

Conjecture 3.15 *If $0 < s(\mu) < 1/n$ and $\mu(x) = |x|^s$ then J_μ is a positive definite matrix and $< \varphi_1, I\varphi_2 >$ is a positive definite form. In particular, $I_A(\mu)$ is unitary in this range.*

3.3 Special representation, non-tempered representation

In the notation of the previous subsection, we have the following

Conjecture 3.16 *If $s = 1/n$ and $\mu(x) = |x|^s$ then the image of J_μ is "the special representation" π_{sp}. It is elliptic, square-integrable and spherical. If $n > 1$ then the kernel of J_μ is an infinite-dimensional, elliptic, non-tempered, non-spherical representation π_{nt} which contains and Iwarori fixed vector.*

It seems reasonable to expect that

(1) the representation π_{sp} is the only square-integrable representation in $\Pi_u(\overline{G_0(F)})$ which is not cuspidal.

(2) π_{sp} is tempered.

(3) The representation π_{nt} and the complementary series comprise the only non-tempered representations in $\Pi_u(\overline{G_0(F)})$.

3.4 Cuspidal representations

Thanks to ideas originally introduced by Weil and Shale, much is known about the cuspidal representations in $\Pi_u(\overline{G_0(F)})$ in the case where $n = 2$ and $p > 2$. In the general case, a great deal of information can be obtained from [F], [Bl], by restricting from an n-fold cover $\overline{GL_2(F)}$ of $GL(2, F)$ to the n-fold cover of $SL(2, F)$.

Lemma 3.17 *If $\Pi \in \Pi_u(\overline{GL_2(F)})$ is cuspidal then there are finitely many cuspidal representations $\pi_i \in \Pi_u(\overline{G_0(F)})$, $1 \leq i \leq M$, such that*

$$\Pi|_{\overline{G_0(F)}} = \bigoplus_{i=1}^{M} \pi_i.$$

Remark 3.18 *The number M is independent of Π if $n > 1$ but not if $n = 1$!*

We call the set $\{\pi_i \mid 1 \leq i \leq M\}$ a **packet** associated to Π or a Π-**packet**. It is natural to ask if every cuspidal representation $\pi \in \Pi_u(\overline{G_0(F)})$ occurs in some Π-packet. In the case $n = 2$, $p > 2$, this question is investigated in [GPS].

3.5 Summary

We sketch the conjectural structure of the dual spaces. For further details in a special case, see [J1].

The admissible dual may be regarded as a real manifold which consists of an countably infinite number of disjoint non-compact "cylinders" each being analytically isomorphic to $\mathbb{C}/\mathbb{Z}$ (the principal series) disjoint union with an countably infinite number of distinct points (the reducible unitary principal series, the special representation, its non-tempered "complement", and the cuspidal representations).

The unitary dual may be regarded as a real manifold which consists of an countably infinite number of disjoint compact "circles" each being isomorphic to $\mathbb{R}/\mathbb{Z}$ (the unitary principal series and the complementary series) disjoint union with an countably infinite number of distinct points (the reducible unitary principal series, the special representation, its non-tempered "complement", and the cuspidal representations).

The tempered dual may be regarded as a real manifold which consists of an countably infinite number of disjoint compact "circles" (the unitary principal series) disjoint union with an countably infinite number of distinct points (the reducible unitary principal series, the special representation, and the cuspidal representations).

4 The Fourier Transform

For $f \in C_c^\infty(\overline{G_0(F)})$ and $\pi \in \Pi(\overline{G_0(F)})$, we define the **operator-valued Fourier transform** of f at π by

$$\pi(f) = \int_{\overline{G_0(F)}} f(x)\pi(x)\, dx, \quad f \in C_c^\infty(\overline{G_0(F)}).$$

Lemma 4.1 *If $f, g \in C_c^\infty(\overline{G_0(F)})$ then $\pi(f * g) = \pi(f)\pi(g)$.*

Let $\overline{G_0(F)}' = \{(g,\varsigma) \in \overline{G_0(F)} \mid g \text{ has distinct eigenvalues}\}$, called the **regular set** of $\overline{G_0(F)}$. In particular, every element in the regular set is semisimple (i.e., diagonalizable). The following result is a corollary of a result of Howe and Harish-Chandra [S]:

Lemma 4.2 *There exists a genuine locally constant function denoted Θ_π on the regular set $\overline{G_0(F)}'$ which represents the trace of π :*

$$trace(\pi(f)) = \int_{\overline{G_0(F)}} \Theta_\pi(x) f(x) dx,$$

for all $f \in C_c^\infty(\overline{G_0(F)})$.

This trace, which we sometimes denote simply by $\Theta_\pi(f)$, is called the **Fourier transform** of f at π. The function $\Theta_\pi(x)$ is called the **character** of π.

Sketch of proof: We modify the proof of [S, Corollary 4.8.2].

By Lemma 2.3, over each sufficiently small compact open subgroup K of $G_0(F)$, the cover $\overline{G_0(F)} \to G_0(F)$ splits. For such a group, we may and will identify the dual group $\mathcal{E}(K)$ (in the notation of [S]), with the subset of the dual group $\mathcal{E}(\overline{K})$ consisting of representations which are trivial on μ_n. This allows us to extend the "intertwines" definition of [S, section 4.8] to $\overline{G_0(F)}$.

The following result is a consequence of a theorem of Howe [S, Theorem 4.8.1]:

Lemma 4.3 *Let* T *be a Cartan subgroup of* $G_0(F)$ *and let* $\omega \subset T' = G_0(F)' \cap T$. *There exists a compact open subgroup* $K_1 \subset G_0(F)$ *with the following properties:*

1. $\overline{K_1}$ *splits,*
2. *Fix a compact open subgroup* $K_2 \subset G_0(F)$ *such that* $\overline{K_2}$ *splits, and an element* $\mathbf{d}_2 \in \mathcal{E}(K_2)$. *Let* F *denote the set of all* $\mathbf{d}_1 \in \mathcal{E}(K_1)$ *such that*

   (i) $\overline{G_0(F)}$ *intertwines* $\mathbf{d}_1$ *with* $\mathbf{d}_2$,

   (ii) ω *intertwines* $\mathbf{d}_1$ *with itself.*

   Then F *is finite.*

Replace the use of [S, Theorem 4.8.1] in the proof of [S, Corollary 4.8.2] by the above lemma. This will yield the claimed result. □

4.1 The Schwartz space and the tempered dual

Let $||g|| = \max(|a|, |b|, |c|, |d|)$, where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, and let $\sigma(g) = \log ||g||$. For each compact open subgroup $K \subset\subset \overline{G_0(F)}$, let

$$\mathcal{C}_K(\overline{G_0(F)}) = \{ f \in C_c(\overline{G_0(F)}//K) \mid \begin{array}{c} f \text{ genuine} \\ |f(x)| <<_{r,f} \frac{\Xi(x)}{(1+\sigma(x))^r}, \ \forall \overline{x} = (x, \varsigma) \in \overline{G_0(F)}, \\ \text{for each } r > 0 \end{array} \},$$

where $C_c(\overline{G_0(F)}//K)$ denotes the space of compacty supported functions which are bi-K-invariant,

$$\Xi(x) = \int_{K_0} \delta_B(xk)^{-1/2} dk.$$

Here, for $x \in \overline{G_0(F)}'$, we have $\delta_B(\overline{x}) = |\det(Ad(x_d))_{\mathfrak{n}}|$, where x_d denotes a diagonalization of x in $SL(2, \overline{F})$, where $\overline{F}$ denotes a separable algebraic closure of F and the valuation $|...|$ has been extended to $\overline{F}$. It's known that there are constants $c_1 > 0,\ c_2 > 0,\ N > 0$ such that

$$c_1 \le \Xi(\overline{a}) \le c_2(1 + \sigma(a))^N, \quad \overline{a} = (a, \varsigma),$$

for all $a \in A^+ = \{a \in A \mid a = \begin{bmatrix} u\pi^n & 0 \\ 0 & u^{-1}\pi^{-n} \end{bmatrix},\ u \in O_F^\times,\ n \ge 0\}$. We topologize $\mathcal{C}_K(\overline{G_0(F)})$ via the semi-norms

$$\upsilon_n(f) = \sup_{x \in \overline{G_0(F)}} |f(x)| \frac{(1 + \sigma(x))^n}{\Xi(x)}.$$

Let

$$\mathcal{C}(\overline{G_0(F)}) = \bigcup_K \mathcal{C}_K(\overline{G_0(F)}),$$

where K runs over all compact open subgroups of $\overline{G_0(F)}$. This is the **Schwartz space** of $\overline{G_0(F)}$. Let S denote the collection of all seminorms on $\mathcal{C}(\overline{G_0(F)})$ whose restriction to each $\mathcal{C}_K(\overline{G_0(F)})$ is continuous. In the semi-norm topology induced by S, the Schwartz space is a complete locally convex topological vector space.

Lemma 4.4 *(1)* $\mathcal{C}(\overline{G_0(F)}) \subset L^2(\overline{G_0(F)})$,
(2) $\mathcal{C}(\overline{G_0(F)})$ *is an algebra under convolution.*

Let

$$D_{G_0/A_0}(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}) = \det(1 - Ad(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}))_{\mathfrak{g}_0/\mathfrak{a}_0}$$
$$= (1 - a^2)(1 - a^{-2}) = -(a - a^{-1})^2.$$

Here $\mathfrak{g}, \mathfrak{a}_0$denote the Lie algebras of G_0, A_0. For $t \in \overline{G_0(F)}'$, let $T = Cent(t, \overline{G_0(F)}) = \overline{G_0(F)}_t$, denote the centralizer. Note T need not be equal to the metaplectic cover of a centralizer of G_0. In other words, if $t = (x, \varsigma)$ then in general $Cent((x,1), \overline{G_0(F)}) \neq \overline{Cent(x, G_0(F))}$. Define the **orbital integral** of $f \in C_c^\infty(\overline{G_0(F)})$ by

$$F_f^T(t) = |D(t)|^{1/2} \int_{T \backslash \overline{G_0(F)}} f(x^{-1}tx) \frac{dx}{dt}. \tag{13}$$

(This exists as a simple consequence of a well-known result of Harish-Chandra [HC].) We define D as above by identifying T with A_0 over the algebraic closure. If $a \in \overline{A_0(F)}$, then define

$$F_f^{A_0^N}(a) = |D(a)|^{1/2} \int_{\overline{A_0(F)^N} \backslash \overline{G_0(F)}} f(x^{-1}ax) \frac{dx}{da}. \tag{14}$$

Lemma 4.5 *For $f \in C_c^\infty(\overline{G_0(F)})$, $a \in \overline{A_0(F)} - \overline{A_0(F)^N}$, we have $F_f^{A_0^N}(a) = 0$.*

proof: Since dx is a Haar measure, we may replace x by $a_1 x$, where $a_1 = (\begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, \varsigma_1) \in \overline{A_0(F)}$, and the integral defining $F_f^{A_0^N}(a)$ must remain invariant. Let $a = (\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \varsigma)$. Then

$$F_f^{A_0^N}(a) = (t, t_1)_n^2 \cdot F_f^{A_0^N}(a).$$

It follows that $F_f^{A_0^N}(a) = 0$ or $(t, t_1)_n^2 = 1$. By the properties of the Hilbert symbol, if $(t, t_1)_n^2 = 1$ for all t_1 then $t \in (F^\times)^N$. □

The following result will not be needed but is included for completeness.

Lemma 4.6 *(1) The map $f \longmapsto F_f^{A_0^N}$ defines a surjection $C_c^\infty(\overline{G_0(F)}) \to C_c^\infty(\overline{A_0(F)^N})$.*

(2) The map $f \longmapsto F_f^{A_0^N}$ defines a continuous surjection $\mathcal{C}(\overline{G_0(F)}) \to \mathcal{C}(\overline{A_0(F)^N})$.

proof: Because of Lemma 4.5, we may write, for $a \in \overline{A_0(F)^N} \cap \overline{G_0(F)}'$,

$$F_f^{A_0^N}(a) = |D(a)|^{1/2} \int_{\overline{A_0(F)}\backslash\overline{G_0(F)}} f(x^{-1}ax)\frac{dx}{da}, \tag{15}$$

since $Cent(a, \overline{G_0(F)}) = \overline{A_0(F)}$. Here da is a suitably normalized Haar measure on $\overline{A_0(F)}$.

(1) For $f \in C_c^\infty(\overline{G_0(F)})$ and $a \in \overline{A_0(F)^N}' = \overline{A_0(F)^N} \cap \overline{G_0(F)}'$, the function $x \longmapsto f(x^{-1}ax)$ is compactly supported on $\overline{A_0(F)}\backslash\overline{G_0(F)}$ and

$$\begin{aligned} F_f^{A_0^N}(a) &= |D(a)|^{1/2} \textstyle\int_{\overline{A_0(F)}\backslash\overline{G_0(F)}} f(x^{-1}ax)\frac{dx}{da} \\ &= \delta_B(a)^{1/2} \textstyle\int_N \int_{K_0} f(kank^{-1})dkdn \\ &= \delta_B(a)^{1/2} \textstyle\int_N f^{K_0}(an)dn, \end{aligned}$$

where

$$f^{K_0}(x) = \int_{K_0} f(kxk^{-1})dk. \tag{16}$$

Since the function $x \longmapsto f(x^{-1}ax)$ is compactly supported on $\overline{A_0(F)}\backslash\overline{G_0(F)}$ the function $x \longmapsto f^{K_0}(x^{-1}ax)$ is compactly supported on $\overline{A_0(F)}\backslash\overline{G_0(F)}/K_0$. We claim that given any compact set $Y \subset\subset \overline{G_0(F)}$, there is an $a \in \overline{A_0(F)}$ with sufficiently large $\sigma(a)$ such that $an \notin Y$, for all $n \in N$. Indeed, we may map Y onto $\overline{Y} \subset \overline{G_0(F)}/K_0$ via $\overline{G_0(F)} \to \overline{G_0(F)}/K_0$ and aN onto $\overline{aN}$. Since $\overline{G_0(F)}/K_0$ is non-compact but $\overline{Y}$ is compact in $\overline{G_0(F)}/K_0$, the Iwasawa decomposition implies that we may find an $a \in \overline{A_0(F)}$ such that $\overline{aN} \cap \overline{Y} = \emptyset$. From the claim it follows that $F_f^{A_0^N}(a)$ is compactly supported on $\overline{A_0(F)}'$.

Moreover, since f is locally constant so is f^{K_0}. Since f^{K_0} is compactly supported, $\int_N f^{K_0}(an)dn$ is a locally constant function on $\overline{A_0(F)}'$.

It remains to show that $F_f^{A_0^N}(a)$ extends to a locally constant function on all of $\overline{A_0(F)}$. This follows from the Shalika germ expansion for covering groups [V]. It is also an immediate corollary of part (2), which we turn to next.

(2) The existence of the orbital integral $F_f^{A_0^N}(a)$ follows directly from the corresponding result on $G_0(F)$ (which is a consequence of a more general result of Clozel). As a consequence of a result of Harish-Chandra and Lemma 2.3(a), the map $f \longmapsto f^B$ defines a continuous map $\mathcal{C}(\overline{G_0(F)}) \to \mathcal{C}(\overline{A_0(F)})$,

where

$$f^B(a) = \delta_B(a)^{1/2} \int_N f(an)dn \tag{17}$$

Note that both Ξ and σ are bi-K_0-invariant, so the map $f \longmapsto f^{K_0}$ defines a continuous map $\mathcal{C}(G) \to \mathcal{C}(G)$. Consequently, the composition $f \longmapsto f^{K_0} \longmapsto F^{A_0^N}_{f^{K_0}}$ defines a continuous map $\mathcal{C}(\overline{G_0(F)}) \to \mathcal{C}(\overline{A_0(F)})$. This completes the proof. □

We call a distribution D on $C_c^\infty(\overline{G_0(F)})$ **tempered** if it extends continuously to $\mathcal{C}(\overline{G_0(F)})$. We call a genuine locally constant function h on $\overline{G_0(F)}$ **tempered** if the distribution

$$f \longmapsto \int_{\overline{G_0(F)}} f(x)\overline{h(x)}dx$$

is tempered (here $\overline{h(x)}$ denotes the complex conjugate, so $\overline{h(g,\varsigma)} = \varsigma \cdot \overline{h(g,1)}$, for all $(g,\varsigma) \in \overline{G_0(F)}$). We call an admissible representation π **tempered** if each $h \in \mathcal{A}(\pi)$ is tempered.

Lemma 4.7 *(Analog of [S, Lemma 4.5.2]) Assume the conjectures of section 3. $\pi \in \Pi_{ad}(\overline{G_0(F)})$ is tempered if and only if the distribution $f \longmapsto \Theta_\pi(f)$ extends continuously to $\mathcal{C}(\overline{G_0(F)})$.*

We will partially verify this below. It is clear from the definitions that if $\pi \in \Pi_{ad}(\overline{G_0(F)})$ is square-integrable then π is tempered.

We will verify parts of the following statements below.

Conjecture 4.8 *(a) A genuine tempered representation of $\overline{G_0(F)}$ is unitary.*

(b) A genuine square-integrable representation of $\overline{G_0(F)}$ is tempered.

To determine which representations of the unitary dual are tempered, we compute their characters and verify directly that they do not grow "too rapidly" in a neighborhood of the singular set.

4.2 Character calculations

Let S denote a complete set of representatives of conjugacy classes of Cartan subgroups of $G_0(F)$. We may and do assume that $A \in S$. For each $T \in S$, let $W_T = N_{G_0(F)}(T)/T$ denote the Weyl group of T. If $t \in T$ is regular then

$$T \subset G_0(F)_t = Cent(t, G_0(F))$$

is finite index.

We begin with the

Lemma 4.9 *(Weyl integration formula [S, p. 198]) For each $f \in C_c^\infty(G_0(F))$,*

$$\int_{G_0(F)} f(g)dg = \sum_{T\in S} \frac{1}{\#[W_T]} \int_T |D(t)|^{1/2} F_f^T(t)dt,$$

where

$$F_f^T(t) = |D(t)|^{1/2} \int_{T\backslash G_0(F)} f(x^{-1}tx)\frac{dx}{dt},$$

and where $D(t) = D_{G_0/T}(t) = \det(1 - Ad(t))_{\mathfrak{g}/\mathfrak{t}}$. Here $\mathfrak{g}$, $\mathfrak{t}$ denote the Lie algebras of G_0, T, resp., and S denotes a complete set of representatives of $G_0(F)$-conjugacy classes of Cartan subgroups of G_0 defined over F.

Note that $F_f^T(t) = F_f^T(t^w)$, where $w \in W_T$, so that $F_f^A(a) = F_f^A(a^{-1})$, since $a^w = a^{-1}$, where $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, in the non-metaplectic case.

4.2.1 Principal series

Lemma 4.10 *Assume $\chi = Ind_{\overline{C}}^{\overline{A_0(F)}}\mu \in \Pi(\overline{A_0(F)})$ and suppose that $\chi \neq \chi^w$. The character of the irreducible representation $\pi = I_{A_0}(\chi)$ is given on the regular set by*

$$\Theta_\pi(x) = \begin{cases} \frac{tr\chi(a)+tr(\chi(a)^{-1})}{|a-a^{-1}|}, & x = a^g, \text{ for some } g \in \overline{G_0(F)},\ a \in \overline{C} - \{\pm 1\}, \\ 0, & \text{if} \quad x \in \overline{G_0(F)}' - \overline{C}^{\overline{G_0(F)}}, \end{cases}$$

where

$$\overline{C}^{\overline{G_0(F)}} = \{a^g \mid a \in \overline{C},\ g \in \overline{G_0(F)}\}.$$

proof: We will identify K with the subset $(K, 1) \subset \overline{G_0(F)}$. Let $V(\chi)$ denote the space of $\pi = I_{A_0}(\chi)$. We have, for $k \in K$, $\varphi \in V(\chi)$,

$$\begin{aligned} (\pi(f)\varphi)(k) &= \int_{\overline{G_0(F)}} f(x)(\pi(x)\varphi)(k)dx \\ &= \int_{\overline{G_0(F)}} f(x)\varphi(kx)dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\overline{G_0(F)}} f(k^{-1}x)\varphi(x)dx \\
&= \int_K \int_{\overline{B_0(F)}} f(k^{-1}bk_1)\varphi(bk_1)d_l b dk_1 \\
&= \int_K \int_{\overline{B_0(F)}} \delta(b)^{1/2}\chi(b)f(k^{-1}bk_1)d_l b \cdot \varphi(k_1)dk_1,
\end{aligned}
$$

so

$$
\begin{aligned}
tr\pi(f) &= \int_K \int_{\overline{B_0(F)}} \delta(b)^{1/2} tr\chi(b) f(k^{-1}bk)d_l b dk \qquad (18) \\
&= \int_K \int_{\overline{A_0(F)}} \int_{N_0(F)} \delta(a)^{1/2} tr\chi(a) f(k^{-1}ank)dndadk.
\end{aligned}
$$

If $n_\alpha = (\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}, 1),\ a = (\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \varsigma)$ then we have

$$
n_{-\alpha} \cdot a \cdot n_\alpha = (\begin{pmatrix} t & \alpha x(t-t^{-1}) \\ 0 & t^{-1} \end{pmatrix}, \varsigma) = a \cdot n_1
$$

if $\alpha = t/(t-t^{-1})$. After a simple change-of-variables, we have

$$
\begin{aligned}
tr\pi(f) &= \int_{\overline{A_0(F)}} \int_K \int_{N_0(F)} \frac{|t-t^{-1}|}{|t|} \delta(a)^{1/2} tr\chi(a) f(k^{-1}n^{-1}ank)dndkda \qquad (19) \\
&= \int_{A_0(F)} \int_K \int_{N_0(F)} \frac{|t-t^{-1}|}{|t|} \delta(a)^{1/2} tr\chi(a,1) f(k^{-1}\cdot(n^{-1}an,1)\cdot k)dndkda.
\end{aligned}
$$

On the other hand, by the Weyl integration formula,

$$
\begin{aligned}
tr\pi(f) &= \int_{\overline{G_0(F)}} tr\pi(g)f(g)dg \\
&= \int_{G_0(F)} tr\pi(g,1)f(g,1)dg \\
&= \frac{1}{2}\sum_{T\in S} \int_T \Delta(t)^2 \int_{T\backslash G_0(F)} f(x^{-1}tx,1) \\
&= \frac{1}{2}\int_{A_0(F)} \Delta(a)^2 \int_K \int_{N_0(F)} f(k^{-1}n^{-1}
\end{aligned}
$$

$$+\frac{1}{2}\sum T \in S$$

$$T \neq A_0(F) \int_T \Delta(t)^2 \int_{T\backslash G_0(F)} f(x^{-1}tx, 1) tr\pi(x^{-1}tx, 1) \frac{dx}{dt} dt$$

$$= \frac{1}{2}\int_{A_0(F)} \Delta(a)^2 \int_K \int_{N_0(F)} f(k^{-1} \cdot (n$$

$$\times tr\pi(k^{-1} \cdot (n^{-1}an, 1) \cdot k) dndkda$$

$$+\frac{1}{2}\sum T \in S$$

$$T \neq A_0(F) \int_T \Delta(t)^2 \int_{T\backslash G_0(F)} f(x^{-1}tx, 1) tr\pi(x^{-1}tx, 1) \frac{dx}{dt} dt.$$

The result claimed follows by comparing equations (4.7), (4.8), and Corollary 2.15. $\square$

Let $\chi = Ind_{\overline{C}}^{\overline{A_0(F)}} \mu \in \Pi(\overline{A_0(F)}), \pi = I_{A_0}(\chi) \in \Pi(\overline{G_0(F)})$. Then

$$\mathrm{tr}\chi(a) = \sum_{h \in \overline{A_0(F)}/\overline{C}} \mu^h(a, 1) \tag{23}$$

This lemma and the Weyl integration formula imply that, for $f \in C_c^\infty(\overline{G_0(F)})$,

$$\begin{gathered} \Theta_\pi(f) = \int_{\overline{G_0(F)}} \Theta_\pi(x) f(x) dx \\ = \tfrac{1}{2}\int_{A_0(F)} (\mathrm{tr}\chi(a,1) + \mathrm{tr}(\chi(a,1)^{-1})) F_f^{A_0(F)}(a,1) da \\ = (F_f^{A_0(F)})^\wedge(\chi) \\ = \textstyle\sum_{h \in \overline{A_0(F)}/\overline{C}} (F_f^{A_0(F)})^\wedge(\mu^h), \end{gathered} \tag{24}$$

where

$$\begin{gathered} \varphi^\wedge(\chi) = \int_{\overline{A_0(F)}} \mathrm{tr}\chi(x)\varphi(x) dx \\ \chi \in \Pi(\overline{A_0(F)}), \end{gathered} \tag{25}$$

and

$$\begin{gathered} \varphi^\wedge(\mu) = \int_{A_0(F)} \mu(x)\varphi(x) dx \\ \mu \in \Pi(A_0(F)), \end{gathered} \tag{26}$$

If $\chi = Ind_{\overline{C}}^{\overline{A_0(F)}} \mu$ is irreducible then

$$\varphi^\wedge(\chi) \quad = \quad \sum_{h \in \overline{A_0(F)}/\overline{C}} \sum_{m=-\infty}^{\infty} (t_h, \pi^m)_n \times \tag{27}$$

$$\times\mu(\pi)^n \int_{O_F{}^\times} (t_h, u)_n \cdot \mu(u)\varphi(\pi^n u)du, \tag{28}$$
$$h = (\begin{pmatrix} t_h & 0 \\ 0 & t_h^{-1} \end{pmatrix}, 1) \in \overline{A_0(F)}/\overline{C}$$

denotes the Fourier transform of $\varphi \in C_c(\overline{A_0(F)})$. Note that this formula for the character extends to the Schwartz space provided μ (and hence π) is unitary.

4.2.2 Reducible principal series

We shall not say much about this case, except to note that some information about the characters can be obtained by global means [J2], [F].

4.2.3 Complementary series

The fact that these representations are not tempered follows directly from the character formulas (Lemma 4.10) as in the non-metaplectic case [J1].

4.2.4 The non-tempered representation

This occurs as the "complement" of the special representation. Again, we merely observe that some information about the characters can be obtained by global means [J2], [F].

4.2.5 Square-integrable representations

These are all tempered and occur discretely. Since the characters are not really needed we will not say more, except to remark that some information about the characters can be obtained by global means [J2], [F].

5 The image of the Fourier transform

The above formulas allows us to classify (at least conjecturally) the image of the Fourier transforms of a "generic" *unitary* principal series representation $\pi = I_A(\chi), \chi = Ind_{\overline{C}}^{\overline{A_0(F)}}\mu \in \Pi(\overline{A_0(F)})$ on either $C_c^\infty(\overline{G_0(F)})$ or the Schwartz space. Note that $H = \overline{A_0(F)}/\overline{C}$ acts on $C_c^\infty(\overline{C})$ and on $\mathcal{C}(\overline{C})$ by conjugation.

Let $C_c^\infty(\overline{C})^H$ (resp., $\mathcal{C}(\overline{C})^H$) denote the subspace of invariant functions. The map

$$\begin{array}{rcl} f & \longmapsto & f^H, \\ f^H(a) & = & \sum_{h\in H} f(hah^{-1}), \end{array}$$

defines a surjection $C_c^\infty(\overline{C}) \to C_c^\infty(\overline{C})^H$ (resp., $\mathcal{C}(\overline{C}) \to \mathcal{C}(\overline{C})^H$). Recall

$$\begin{array}{c} \varphi^\wedge(\omega) = \int_{F^\times} \omega(x)\varphi(x)dx \\ \omega \in \Pi(F^\times). \end{array}$$

Lemma 5.1 *For $\varphi \in C_c^\infty(F^\times)$, the image $C_c^\infty(F^\times)^\wedge$ of the Fourier transform $\varphi \longmapsto \varphi^\wedge(\omega)$ is given by*

$$C_c^\infty(F^\times)^\wedge = \{h \in C_c(\Pi_u(F^\times)) \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } \Pi_u(F^\times) \end{array} \}.$$

We omit the proof. (This lemma, and its Schwartz space analog below, is proven using the inverse Fourier transform and the inversion formula, which may be found in [Ta, pp. 43-44], [Ba, ch 12].) A similar lemma holds for A_0 :

Lemma 5.2 *For $\varphi \in C_c^\infty(A_0(F))$, the image $C_c^\infty(A_0(F))^\wedge$ of the Fourier transform $\varphi \longmapsto \varphi^\wedge(\omega)$ is given by*

$$C_c^\infty(A_0(F))^\wedge = \{h \in C_c(\Pi_u(A_0(F))) \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } \Pi_u(A_0(F)) \end{array} \}.$$

A similar lemma holds for $\overline{C}$ as well:

Lemma 5.3 *For $\varphi \in C_c^\infty(\overline{C})$, the image $C_c^\infty(\overline{C})^\wedge$ of the Fourier transform $\varphi \longmapsto \varphi^\wedge(\omega)$ is given by*

$$C_c^\infty(\overline{C})^\wedge = \{h \in C_c(\Pi_u(\overline{C})) \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } \Pi_u(\overline{C}) \end{array} \}.$$

As a consequence of the above lemma, since the map $C_c^\infty(\overline{G_0(F)}) \to C_c^\infty(\overline{C})$, given by $f \longmapsto F_f^{A_0(F)}$, is surjective, it follows that the same description holds for the space of all functions of χ of the form $(F_f^{A_0(F)})^\wedge(\mu)$, $\mu \in \Pi(\overline{C})$:

Lemma 5.4 *For* $f \in C_c^\infty(\overline{G_0(F)})$, *the image* $C_c^\infty(\overline{G_0(F)})^\wedge_{ps}$ *of the Fourier transform* $f \longmapsto \Theta_\pi(f)$, $\pi = I_{A_0}(\chi)$, $\chi \in \Pi_u(\overline{A_0(F)})$, $\chi = Ind_{\overline{C}}^{A_0(F)}\mu$, *is given by*

$$C_c^\infty(\overline{G_0(F)})^\wedge_{ps} = \{h \in C_c(\Pi_u(\overline{C}))^H \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } \Pi_u(\overline{C}) \end{array} \}.$$

Next, suppose $f \in \mathcal{C}(\overline{G_0(F)})$, so $F_f^{A_0(F)} \in \mathcal{C}(\overline{C})$.

Lemma 5.5 *For* $\varphi \in \mathcal{C}(F^\times)$, *the image* $\mathcal{C}(F^\times)^\wedge$ *of the Fourier transform* $\varphi \longmapsto \varphi^\wedge(\omega)$, $\omega \in \Pi_u(F^\times)$, *is given by*

$$\mathcal{C}(F^\times)^\wedge = \{h \in C_c(\Pi_u(F^\times)) \mid \begin{array}{c} h \text{ is a trig series on each circle} \\ \text{in } \Pi_u(F^\times) \text{ which converges absolutely} \\ \text{along with all of its derivatives} \end{array} \}.$$

Similarly, we have

Lemma 5.6 *For* $\varphi \in \mathcal{C}(\overline{C})$, *the image* $\mathcal{C}(\overline{C})^\wedge$ *of the Fourier transform* $\varphi \longmapsto \varphi^\wedge(\mu)$, $\mu \in \Pi_u(\overline{C})$, *is given by*

$$\mathcal{C}(\overline{C})^\wedge = \{h \in C_c(\Pi_u(\overline{C})) \mid \begin{array}{c} h \text{ is a trig series on each circle} \\ \text{in } \Pi_u(\overline{C}) \text{ which converges absolutely} \\ \text{along with all of its derivatives} \end{array} \}.$$

Consequently, since the map $\mathcal{C}(\overline{G_0(F)}) \to \mathcal{C}(\overline{C})$ is surjective, it follows that the same description holds for the space of all functions of χ of the form $(F_f^{A_0(F)})^\wedge(\chi)$:

Lemma 5.7 *For* $f \in \mathcal{C}(\overline{G_0(F)})$, *the image* $\mathcal{C}(\overline{G_0(F)})^\wedge_{ps}$ *of the Fourier transform* $f \longmapsto \Theta_\pi(f)$, $\pi = I_{A_0}(\chi)$ *is given by*

$$\mathcal{C}(\overline{G_0(F)})^\wedge_{ps} = \{h \in C_c(\Pi_u(\overline{C}))^H \mid \begin{array}{c} h \text{ is a trig series on each circle} \\ \text{in } \Pi_u(\overline{C}) \text{ which converges absolutely} \\ \text{along with all of its derivatives} \end{array} \}.$$

5.1 Admissible Paley-Wiener Theorem for smooth functions

We verify an analog of a result of Bernstein-Deligne-Kazhdan [BDK] for $\overline{G_0(F)}$. The following result classifies the image of $C_c^\infty(\overline{G_0(F)})$ under the Fourier transform:

Theorem 5.8 *(Assume the conjectures of section 3 above.) The image of the Fourier transform $\pi \longmapsto \Theta_\pi(f)$, for $f \in C_c^\infty(\overline{G_0(F)})$ and $\pi \in \Pi \in \Pi_{ad}(\overline{G_0(F)})$, consists of the space of functions h on $\Pi_{ad}(\overline{G_0(F)})$ such that*

(1) h is supported on finitely many connected components of $\Pi_{ad}(\overline{G_0(F)})$,

(2) h is regular on $\Pi_{ad}(\overline{G_0(F)})$, regarded as a complex algebraic variety.

Let $C_c^\infty(\overline{G_0(F)})^\wedge_{ad}$ denote the space of functions described in the above theorem. We call it the **admissible Paley-Wiener space**.

sketch of proof: This follows from the (conjectural) classification above. $\square$

5.2 Unitary Paley-Wiener Theorem for smooth functions

We use the information above to prove the following

Theorem 5.9 *(Assume the conjectures of section 3 above.) The image of the Fourier transform $\pi \longmapsto \Theta_\pi(f)$, for $f \in C_c^\infty(\overline{G_0(F)})$ and $\pi \in \Pi \in \Pi_u(\overline{G_0(F)})$, consists of the space of functions h on $\Pi_u(\overline{G_0(F)})$ such that*

(1) h is supported on finitely many connected components of $\Pi_u(\overline{G_0(F)})$,

(2) h is a trigonometric polynomial on each circle in $\Pi_u(\overline{G_0(F)})$ which it is supported on,

(3) if h is supported on the complementary series then h is a finite Laurant series in $x = q^\rho$.

proof: (1) follows from the admissible Paley-Wiener theorem.

(2) follows from the section on characters of principal series representations.

(3) follows from the section on characters of complementary series representations.

This completes the proof since h can be arbitrary on the discrete part of the unitary dual. $\square$

Let $C_c^\infty(\overline{G_0(F)})_u^\wedge$ denote the space of functions described in the above theorem. We call it the **unitary Paley-Wiener space**.

5.3 Tempered Paley-Wiener Theorem for Schwartz functions

We also have the following

Theorem 5.10 *(Assume the conjectures of section 3 above.) The image of the Fourier transform* $\pi \longmapsto \Theta_\pi(f)$, *for* $f \in \mathcal{C}(\overline{G_0(F)})$ *and* $\pi \in \Pi \in \Pi_t(\overline{G_0(F)})$, *consists of the space of functions* h *on* $\Pi_t(\overline{G_0(F)})$ *such that*

(1) h *is supported on finitely many connected components of* $\Pi_t(\overline{G_0(F)})$,

(2) h *is a trigonometric series on each circle in* $\Pi_t(\overline{G_0(F)})$ *which it is supported on, converging absolutely along with all of its derivatives.*

proof: (1) follows from the admissible Paley-Wiener theorem.

(2) follows from the section on characters of principal series representations.

This completes the proof since h can be arbitrary on the discrete part of the tempered dual. $\square$

Let $\mathcal{C}(\overline{G_0(F)})^\wedge$ denote the space of functions described in the above theorem. We call it the **tempered Paley-Wiener space**.

6 Linear functionals on the tempered Paley-Wiener space

6.1 A result of L. Schwartz

The space of linear functionals on the tempered Paley-Wiener space can be successfully described using the theory of distributions on the real line:

Proposition 6.1 *(L. Schwartz [Sch, ch. III, Théorème XXI]) Let* S^1 *denote the unit circle. If* $T \in C^\infty(S^1)'$ *then there is a continuous function* u *on* S^1 *and an integer* $m \geq 0$ *such that* $T = \frac{d^m}{d\theta^m} u$ *(as distributions).*

As a corollary of this and the above Paley-Wiener theorem, we have the following

Proposition 6.2 *The dual space of the tempered Paley-Wiener space $\mathcal{C}(\overline{G_0(F)})^\wedge$ is isomorphic to the direct product*

$$\bigoplus_{d\in\pi_0(\Pi_u(\overline{A_0(F)}))} \bigcup_{m\geq 0} C(S_d)^{(m)} \bigoplus \bigoplus_{\sigma\in\pi_0(\Pi_{lds}(\overline{G_0(F)}))} \mathbf{C}\delta_\sigma,$$

where $\pi_0(X)$ denotes the set of connected components of a topological space X, where $\Pi_{lds}(\overline{G_0(F)})$ denotes the union of the reducible principal series, the special representation, and the cuspidal representations, and where *δ_σ denotes the evaluation functional on the space of (constant) functions on the (singleton) elements of $\pi_0(\Pi_{lds}(\overline{G_0(F)}))$, and, if we write the unitary dual $A_u^\wedge$ as a union of circles $S_d \cong \mathbf{R}/\frac{2\pi}{\log(q)}\mathbf{Z}$,* where

$$C(S_d)^{(m)} = \{\frac{d^m}{d\theta^m}u \mid u \text{ continuous on } S_d\},$$

6.2 The Fourier transform of a tempered distribution

As a consequence of this, we can "completely describe" the Fourier transform of a tempered distribution:

Theorem 6.3 *If T is a tempered distribution on $\overline{G_0(F)}$ then there is a*

$$T^\wedge \in \bigoplus_{d\in\pi_0(\Pi_u(\overline{A_0(F)}))} \bigcup_{m\geq 0} C(S_d)^{(m)} \bigoplus \bigoplus_{\sigma\in\pi_0(\Pi_{lds}(\overline{G_0(F)}))} \mathbf{C}\delta_\sigma$$

such that

$$T(f) = T^\wedge(f^\wedge),$$

for all $f \in \mathcal{C}(\overline{G_0(F)})$, where $f^\wedge(\pi) = \Theta_\pi(f)$ denotes the Fourier transform of f at $\pi \in \Pi \in \Pi_t(\overline{G_0(F)})$.

7 References

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