A hitch-hiker's guide to invariant distributions on SL(2) over a p-adic field

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Abstract

In this expository article, we write down the admissible, unitary, and tempered duals on $SL(2)$, using the explicit parameterization of Gelfand-Graev (and results of Sally-Shalika). This allows us to determine the Tadić decompositions and natural Borel structures on the dual spaces. Then we classify the image of the Schwartz space (resp., the Hecke algebra) under the Fourier transform and define the Fourier transform of any tempered (resp., admissible) distribution on $SL(2)$. The Fourier transform of a tempered distribution can be made considerable more explicit than that of an admissible distribution. In any case, these results are then applied to an understanding of supertempered distributions on $SL(2)$.

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1 Introduction

This paper is a essentially an expository article. It is not by any means a complete survey nor a guide to the literature. It is written to work out the ideas in [J] in a special case. Proofs and pointers to the literature are often omitted, although everything should be in one of the references listed at the end of the paper. In order to make this paper appeal to a broader audience (and to make it a more useful reference for myself and hopefully others), lots of definitions are included and I've tried my best to (a) be precise, (b) be as non-technical as possible, and (c) preserve the intrinsic beauty of the subject (which is largely due to Harish-Chandra). To be honest, some of the definitions and examples in §§7-9 seem to be unpublished, so this article is not entirely expository. I hope what little new there is in those sections does not remind anyone of the old joke about the PhD thesis (it was both interesting and new, but the interesting parts where not new and the original parts were not interesting). Questions, comments or constructive flames may be sent to wdj@usna.navy.mil

1.1 Basic notation

Let $G = SL(2, F)$, where F is a p-adic field with uniformizer $\pi = \pi_F$, ring of integers O_F, residual characteristic $p = char(O_F/\pi O_F)$, $q = \#(O_F/\pi O_F)$, and normalized valuation $|...| = |...|_F$. We always assume $p > 2$. Let ε denote a fixed non-square unit so that $1, \varepsilon, \pi, \varepsilon\pi$ form a complete set of representatives of $F^\times/(F^\times)^2$. The subgroups

$$U_0 = O_F^\times,\ U_n = 1 + \pi^n O_F, \quad n > 0,$$

form a basis of open neighborhoods of the identity in $F^\times$, generating the topology on $F^\times$. The subgroups

$$K_0 = SL(2, O_F),\ K_n = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ \middle|\ a, d \in 1 + \pi^n O_F,\ b, c \in \pi^n O_F \right\}, \quad n > 0,$$

form a basis of open neighborhoods of the identity in G, generating the topology on G. A **character** or **representation** of $F^\times$ or of G will always

mean a continuous multiplicative (not necessarily unitary) homomorphism into the automorphisms of a complex vector space, where the domain gets the "compact-open" topology above and the range gets the discrete topology.

Occasionally we use **Vinogradov's notation**: given two functions $f(x)$, complex-valued and perhaps depending on some parameter λ, and $g(x) > 0$, we write

$$f(x) <<_\lambda g(x),$$

if there is a constant $c > 0$, which may implicitly depend on λ, such that $|f(x)| \leq cg(x)$ for all x in some range which depends on the context.

2 Background on the unitary dual and the admissible dual

2.1 Definitions

Let V denote a complex vector space. A representation $\pi : G \to Aut(V)$ is called **admissible** if

(1) every $v \in V$ belongs to $V^K = stab_V(K)$, for some compact open $K \subset\subset G$, (i.e., π is **smooth**),

(2) for every compact open $K \subset\subset G$, $\dim(V^K) < \infty$.

Let V' denote the dual space of all complex-valued linear functionals on V and let $< v, v' >= v'(v)$ denote the evaluation pairing on $V \times V'$. Let V'_∞ denote the subspace of V' of vectors fixed by some compact open subgroup. Define the **contragredient** representation of a smooth $\pi : G \to End(V)$ to be the representation $\pi' : G \to Aut(V'_\infty)$ satisfying $< v, v'. =< \pi(g)v, \pi'(g)v' >$, for all $v \in V, v' \in V'_\infty$, and $g \in G$. It is known that if a representation $\pi : G \to Aut(V)$ is admissible then its contragredient representation $\pi' : G \to Aut(V'_\infty)$ is admissible. A **matrix coefficient** of a representation $\pi : G \to Aut(V)$ is a function on G of the form $g \longmapsto < \pi(g)v, v' >$, for some $v \in V, v' \in V'_\infty$. The **space of matrix coefficients** of π is denoted by $\mathcal{A}(\pi)$.

An admissible representation $\pi : G \to Aut(V)$ is called **square-integrable** or **in the discrete series** if $\mathcal{A}(\pi) \subset L^2(G)$. It is called **cuspidal** if every function $f \in \mathcal{A}(\pi)$ has compact support modulo the center and satisfies, for each nilpotent radical N of a proper parabolic subgroup of G, $\int_N f(xn)dn =$

0, for all $x \in G$. Clearly, a cuspidal representation is square-integrable. A representation $\pi : G \to End(V)$ is called **unitarizable** if V has a positive-definite G-invariant inner product $(*,*)$ (that is, $(\pi(g)v, \pi(g)v') = (v, v')$ for all $v, v' \in V$). The completion of V with respect to this inner product is of course a G-invariant Hilbert space H. Any representation $\pi : G \to Aut(H)$, for some Hilbert space H whose inner product is G-invariant, is called **unitary.** It is known that every square-integrable representation is unitarizable.

The following theorem was proven by Harish-Chandra subject one of two statements, one of which was later proven by Bernstein, and the other was later proven by Clozel.

Theorem (Harish-Chandra, Bernstein, Clozel) *Any unitary representation $\pi : G \to Aut(H)$, has the property that there is a dense G-invariant subspace $V = H_\infty$ such that (π, V) is admissible.*

We say that a representation $\pi_1 : G \to Aut(V_1)$ is **equivalent** to a representation $\pi_2 : G \to Aut(V_2)$ if there is a non-zero linear transformation $A : V_1 \to V_2$ such that $A\pi_1(g)v = \pi_2(g)Av$ for all $g \in G$, $v \in V_1$. The map A is called an **intertwining map**. The **unitary dual** of G is the set $\hat{G}_u$ of all equivalence classes of irreducible unitary representations of G and the **admissible dual** of G is the set $\hat{G}_{ad}$ of all equivalence classes of irreducible admissible representations of G. The only finite dimensional irreducible admissible representation of G is the trivial representation. As a corollary to the above theorem, we have a natural inclusion $\hat{G}_u \subset \hat{G}_{ad}$.

Each admissible representation $\pi : G \to Aut(V)$ gives rise to a representation representation $\pi : C_c^\infty(G) \to End(V)$ of the Hecke algebra $C_c^\infty(G)$ of all locally constant functions with compact support on G:

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg, \quad v \in V, \quad f \in C_c^\infty(G),$$

where dg denotes a Haar measure on G. The linear transformation $\pi(f)$ is sometimes called the **operator-valued Fourier transform** of f at π. For admissible π it is known that $\pi(f)$ is finite rank as an element of $End(V)$. Let $||\pi(f)||$ denote its operator norm and let

$$||f||_* = \sup_{\pi \in \hat{G}_u} ||\pi(f)||.$$

The completion of $C_c^\infty(G)$ with respect to this norm $||...||_*$ is a C^*- algebra denoted by $C^*(G)$.

Let A denote a C^*-algebra. A primitive ideal of A is the kernel of an irreducible representation of A. Put the Jacobson topology on the set $\mathrm{Pr}(A)$ of all primitive ideals of A. Let $A^\wedge$ denote the set of equivalence classes of irreducible representations of A on the bounded operators on a Hilbert space. There is a canonical surjection $A^\wedge \hookrightarrow \mathrm{Pr}(A)$ induced by $\pi \longmapsto \ker(\pi)$. Give $A^\wedge$ the smallest topology such that this surjection is continuous. Call this the **Jacobson** topology. This topology is T_0.

We define a topology on $\hat{G}_u$ using the following

Theorem (Dixmier): *There is a canonical bijection between the set $\hat{G}_u$ and the set $C^*(G)^\wedge$ of equivalence classes of irreducible representations of $C^*(G)$.*

We give $\hat{G}_u$ the smallest topology such that this bijection is continuous. Call this the **Jacobson topology** on $\hat{G}_u$.

A **CCR algebra** is a C^*-algebra A such that every irreducible representation $\pi : A \to End(H)$ is completely continuous (i.e., compact). It is known, by a result of Fell, that if A is a CCR algebra and if we topologize $A^\wedge$ using the Jacobson topology as above then the dual space $A^\wedge$ is a T_1 topological space.

Theorem (Bernstein) *$C^*(G)$ is a CCR algebra.*

Following Mackey, we say that the dual space $A^\wedge$ is **smooth** if there is a Borel structure on it which is Borel isomorphic to the Borel structure on the real line.

Theorem (Fell, Mackey) *If A is a separable CCR algebra then $A^\wedge$ has a smooth dual. Moreover, the Borel structure may be taken to be that generated by the Jacobson topology on $A^\wedge$.*

QUESTION: Is there a natural topology on $\hat{G}_{ad}$ which generates a smooth Borel structure and which, in turn, induces the Borel structure on $\hat{G}_u$ via the inclusion $\hat{G}_u \subset \hat{G}_{ad}$?

We shall see that the answer is yes for $SL(2)$.

2.2 The Tadić decomposition

Let A denote the diagonal subgroup of G, and let $\mathcal{L}(G)$ denote the set of all standard Levi's (with respect to A), up to associates. Therefore, $\mathcal{L}(G) = \{A, G\}$. For $M \in \mathcal{L}(G)$, we call a representation $\sigma : M \to Aut(V)$ for which $A(\sigma) \subset C_c^\infty(M)$ a **cuspidal** representation of M. For $M \in \mathcal{L}(G)$, let $M_c^\wedge$ denote the set of equivalence classes of cuspidal representations $\sigma : M \to$

$Aut(V)$, and let $M^\wedge_{u,c}$ denote the set of equivalence classes of cuspidal representations $\sigma : M \to Aut(W)$ which has unitary central character ω_σ (i.e., $|\omega_\sigma(z)| = 1$ for all z belonging to the center of M). For example, if $M = A$ then

$$A^\wedge_{u,c} = \{\chi : A \to C^\times \mid \begin{array}{c} \chi(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) = \chi'(a),\ where\ \chi'(a) = \chi_0(\frac{a}{|a|}) \cdot |a|^{i\theta}, \\ \chi_0 \in (O_F^\times)^\wedge,\ -\frac{\pi}{\log(q)} \le \theta < \frac{\pi}{\log(q)} \} \\ \cong (F^\times)^\wedge_u, \end{array}$$

$$A^\wedge_c = \{\chi : A \to C^\times \mid \begin{array}{c} \chi(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) = \chi'(a),\ where\ \chi'(a) = \chi_0(\frac{a}{|a|}) \cdot |a|^{t+i\theta}, \\ \chi_0 \in (O_F^\times)^\wedge,\ t \in R,\ -\frac{\pi}{\log(q)} \le \theta < \frac{\pi}{\log(q)} \} \\ \cong (F^\times)^\wedge, \end{array}$$

where, for any commutative topological group H, $H^\wedge$ denotes the set of continuous complex-valued (not necessarily unitary) characters of H.

Let $X_{un}(M)$ denote the group of all unramified characters of M. For example, if $M = A$ then we may either write $\chi \in X_{un}(A)$ as

$$\chi(x) = \chi_0(\frac{x}{|x|})|x|^{t+i\theta},$$

with $\chi_0 \in (O_F^\times)^\wedge$, $t \in R$, $\chi|_{1+\pi\mathcal{O}_F} = 1$, and $-\frac{\pi}{\log(q)} \le \theta < \frac{\pi}{\log(q)}$, *or* as

$$\chi(x) = z^{\nu_F(x)},$$

where $z \in C^\times$ and ν_F denotes the usual valuation map (i.e., for all $x \in F^\times$, $|x| = q^{-\nu_F(x)}$). Therefore, we may either identify $X_{un}(A)$ as a countable union of cylinders $C/\frac{2\pi}{\log(q)Z}$ *or* as $C^\times$.

Let $(\sigma, W) \in M^\wedge_c$ and let $I_M(\sigma) : G \to Aut(V)$ denote the **unitarily induced representation**: the representation of G by right translation on

$$V = \{f : G \to W \mid \begin{array}{c} (1)\ \ f(mg) = \delta_M(m)^{1/2}\sigma(m)f(g), \\ \forall g \in G,\ m \in M \\ (2)\ \ for\ some\ open\ subgp\ K \subset\subset G,\ f(gk) = f(g), \\ \forall k \in K,\ g \in G \end{array} \}.$$

Here $\delta_M(m) = |\det(Ad(m)_\mathfrak{n})|$ where $\mathfrak{n}$ denotes the Lie algebra of

$$N = \begin{cases} \{1\}, & M = G, \\ \{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in F\}, & M = A. \end{cases}$$

so $\delta_A(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) = |a|^2$. Following Tadić, we let

$$G^\wedge_{u,\sigma} = \{\textit{all inequivalent irreducible unitarizable constituents of } I_M(\chi\sigma) \mid \chi \in X_{un}(M)\},$$

$$G^\wedge_{\sigma} = \{\textit{all inequivalent irreducible constituents of } I_M(\chi\sigma) \mid \chi \in X_{un}(M)\}.$$

Theorem (Tadić) *(a)* $G^\wedge_u = \bigcup_{M\in\mathcal{L}(G)} \bigcup_{\sigma\in M^\wedge_{u,c}} G^\wedge_{u,\sigma}$,
(b) Each $G^\wedge_{u,\sigma}$ *is both open and closed in* $G^\wedge_u$.

Each "component" $G^\wedge_{u,\sigma}$ is not necessarily connected. We can choose a subset $\tilde{M}$ of $M^\wedge_{u,c}$ such that we have a disjoint union:

$$G^\wedge_u = \coprod_{M\in\mathcal{L}(G)} \coprod_{\sigma\in\tilde{M}} G^\wedge_{u,\sigma},$$

We shall see that for the admissible dual there is an analogous "decomposition" (this is due to several people, Jacquet in particular [C], and pre-dates the decomposition above)

$$G^\wedge_{ad} = \bigcup_{M\in\mathcal{L}(G)} \bigcup_{\sigma\in M^\wedge_c} G^\wedge_\sigma.$$

We will see that we may impose a Borel structure on the admissible dual as well.

3 Borel structure on the Tadić components

Let $V = C(F)_\infty$ denote the vector space of locally constant complex-valued functions on Fwhich are bi-K-invariant for some compact open subgroup $K \subset\subset F^\times$.

3.1 Principal series, reducible principal series

For $\chi \in (F^\times)^\wedge$,define $T_\chi : G \to Aut(V)$ by

$$(T_\chi(g)f)(x) = f(\frac{dx+b}{cx+a})\chi(cx+a)|cx+a|^{-1}, \qquad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G.$$

Note that this representation is smooth.

Lemma (Gelfand-Graev): *Let* $\chi, \chi_1, \chi_2 \in (F^\times)^\wedge$

(1) If $\chi_1 \neq \chi_2^{-1}, \chi_2$ *then* T_{χ_1} *is not equivalent to* T_{χ_2},

(2) $T_\chi \cong T_{\chi^{-1}}$,

(3) T_χ *is unitary if* χ *is,*

(4) T_χ *is irreducible if and only if* $\chi \neq sign_\tau$, *for each* $\tau = \varepsilon, \pi, \varepsilon\pi$, *where* $sign_\tau$ *denotes the non-trivial quadratic character on* $F^\times$ *for which* $sign_\tau(\tau) = 1$.

(5) if $\chi = sign_\tau$, *for some* $\tau = \varepsilon, \pi, \varepsilon\pi$, *then* T_χ *decomposes into a direct sum of two irreducible inequivalent unitary subrepresentations* T_χ^+, T_χ^-.

The irreducible representations in (4) are called **principal series representations**. The irreducible representations in (5) are called **reducible principal series representations**. They, along with the special representation, are called **limits of discrete series**.

For $\chi \in (F^\times)_u^\wedge$ ramified, let

$$G_{u,\chi}^\wedge = \{ \begin{array}{c} T_{\chi'}\ , \ \ if\ \chi' \neq sign_\tau,\ for\ each\ \tau = \pi, \varepsilon\pi \\ \{T_{\chi'}^+\ ,\ T_{\chi'}^-\}, \quad if\ \chi' = sign_\tau,\ for\ some\ \tau = \pi, \varepsilon\pi \\ |\ \ \chi'(x) = \chi(x) z^{\nu_F(x)},\ \ |z| = 1,\ \ z \in C^\times/\tilde{}\},\end{array}$$

where $z_1 \sim z_2$ if and only if $\chi(x) z_1^{\nu_F(x)} = \chi(x)^{-1} z_2^{-\nu_F(x)}$, $\forall x \in F^\times$. This is either isomorphic to a half-circle union two doubleton sets (if the ramified characters of order two occur in the orbit of χ under $X_{un}(A)$) or to a circle (if the ramified characters of order two don't occur in the orbit of χ under $X_{un}(A)$), where $\{\pm 1\}$ acts via the inversion map. All the $T_{\chi'}$ occurring in the above set are irreducible and inequivalent For example, if $\chi = sign_\pi$ (resp., $\chi = sign_{\varepsilon\pi}$) then $\chi' = sign_{\pi\varepsilon}$ (resp., $\chi' = sign_\pi$) is also in it's orbit under the action of the unramified characters of A.

Note that the set $G_{u,\chi}^\wedge$ and the set $G_{u,\chi^{-1}}^\wedge$ contain equivalent representations. We write $(F^\times)_u^\wedge/\tilde{}$ for the set of unitary characters modulo the equivalence relation: $\chi \sim \chi'$ if and only if $\chi' = \chi^{-1}$.

For $\chi \in (F^\times)^\wedge$ ramified, let

$$G_{ad,\chi}^\wedge = \{ \begin{array}{c} T_{\chi'}\ , \ \ if\ \chi' \neq sign_\tau,\ for\ each\ \tau = \pi, \varepsilon\pi \\ \{T_{\chi'}^+\ ,\ T_{\chi'}^-\}, \quad if\ \chi' = sign_\tau,\ for\ some\ \tau = \pi, \varepsilon\pi \\ |\ \ \chi'(x) = \chi(x) z^{\nu_F(x)}\ \ z \in C^\times/\tilde{}\},\end{array}$$

where the equivalence action is as above. We can also write this as

$$G^{\wedge}_{ad,\chi} = \{ \begin{array}{c} T_{\chi'}\ ,\ \ if\ \chi' \neq sign_\tau,\ for\ each\ \tau = \pi, \varepsilon\pi \\ \{T^+_{\chi'}\ ,\ T^-_{\chi'}\},\quad if\ \chi' = sign_\tau,\ for\ some\ \tau = \pi, \varepsilon\pi \\ |\ \ \chi'(x) = \chi(x)\chi_0(\frac{x}{|x|})|x|^{t+i\theta},\ \ some \\ (\chi_0, t, \theta) \in \{(O_F^\times)^\wedge_{un} \times R \times [-\frac{\pi}{\log(q)}, \frac{\pi}{\log(q)})\}/\tilde{\}}, \end{array}$$

where $(\chi_0, t, \theta) \sim (\chi_0', t', \theta')$ if and only if $\chi_0' = \chi_0^{-1}$, $t' = -t$, and $\theta' = -\theta$. All the $T_{\chi'}$ occurring in the above set are irreducible and inequivalent. This is either isomorphic to $C^\times/\{\pm 1\}$ with two double points (if the ramified characters of order two occur in the orbit of χ under $X_{un}(A)$) or to $\{C^\times\}/\{\pm 1\}$ (if the ramified characters of order two don't occur in the orbit of χ under $X_{un}(A)$), where $\{\pm 1\}$ acts via the inversion map.

Note that the set $G^\wedge_{ad,\chi}$ and the set $G^\wedge_{ad,\chi^{-1}}$ contain equivalent representations. We write $(F^\times)^\wedge/\sim$ for the set of characters modulo the equivalence relation: $\chi \sim \chi'$ if and only if $\chi' = \chi^{-1}$.

The unramified case contains the complementary series and special representation as well, so will be handled later.

3.1.1 Borel structure on $G^\wedge_{u,\chi}$, $\chi \in A^\wedge_u$ *ramified*

The Borel structure on $G^\wedge_{u,\chi}$, when χ does not belong to the orbit of some $sign_\tau$ under $X_{un}(A)$ is Borel isomorphic to the Borel structure on the circle. The Borel structure on $G^\wedge_{u,\chi}$, when χ *does* belong to the orbit of some $sign_\tau$ under $X_{un}(A)$ is Borel isomorphic to the Borel structure on the half-circle union that on the set consisting of two (closed) points.

3.1.2 Borel structure on $G^\wedge_{ad,\chi}$, $\chi \in A^\wedge$ *ramified*

The Borel structure on $G^\wedge_{ad,\chi}$, when χ does not belong to the orbit of some $sign_\tau$ under $X_{un}(A)$ is Borel isomorphic to the Borel structure on $\{C^\times\}/\{\pm 1\}$. The Borel structure on $G^\wedge_{ad,\chi}$, when χ *does* belong to the orbit of some $sign_\tau$ under $X_{un}(A)$ is Borel isomorphic to the Borel structure on $C^\times/\{\pm 1\}$ with two double points.

3.2 Complementary series

This is sometimes called the supplementary series. For V as above, $-1 < \rho < 1,\ \rho \neq 0$, define $T_\rho : G \to End(V)$ by $T_\rho = T_\chi$, where $\chi(x) = |x|^{-\rho}$, for $x \in F^\times$.

Lemma (Gelfand-Graev): *Let* $-1 < \rho < 1,\ \rho \neq 0$.

(1) If $\rho_1 \neq \pm\rho_2$ *then* T_{ρ_1} *is not equivalent to* T_{ρ_2},

(2) $T_\rho \cong T_{-\rho}$,

(3) T_ρ *is unitary,*

(4) for $0 < \rho < 1,\ T_\rho$ *is irreducible.*

The irreducible representations in (4) are called **complementary series representations**. We shall see later that these representations are "non-tempered".

3.3 The special representation and the trivial representation

For V and T_χ as above, we have

Lemma (Gelfand-Graev): *Let* $\chi(x) = |x|^{-1}$, for $x \in F^\times$.

(1) T_χ is reducible,

(2) there is an irreducible, infinite-dimensional, unitary, square-integrable subrepresentation T_{sp} of T_χ,

(3) The quotient of T_χ by T_{sp} is 1-dimensional and unitary.

The representation in (2) is called the **special representation**. The representation in (3) is called the **trivial representation**. We will see later that the special representation is "tempered" but the trivial representation is not.

3.4 The $G_\sigma^\wedge$ in the unramified case

We define

$$G_{u,\chi}^\wedge = \{T_\rho,\ T_{\chi'},\ T^+_{sign_\epsilon},\ T^-_{sign_\epsilon}\ , 1, T_{sp} \mid \quad \begin{array}{c} \chi'(x) = z^{v_F(x)}, \\ z = e^{i\theta},\ \ 0 \leq \theta < \pi, \\ 0 < \rho < 1, \end{array} \quad \},$$

if χ is an unramified unitary character of A, and, if χ is unramified on A

$$G^{\wedge}_{ad,\chi} = \{T_{\chi'},\ T^{+}_{sign_\epsilon},\ T^{-}_{sign_\epsilon}, 1, T_{sp} \ \mid \ \begin{array}{c} \chi'(x) = z^{v_F(x)} \neq |...|^{-1},\ sign_\varepsilon, \\ z \in C^{\times}/\{\pm 1\}, \end{array} \},$$

where $\{\pm 1\}$ acts on $C^{\times}$ by means of inversion. Note that the set $G^{\wedge}_{u,\chi}$ and the set $G^{\wedge}_{u,\chi^{-1}}$ (resp., the sets $G^{\wedge}_{ad,\chi}$ and $G^{\wedge}_{ad,\chi^{-1}}$) contain equivalent representations.

3.4.1 Borel structure on $G^{\wedge}_{u,\chi}$, $\chi \in A^{\wedge}_{u}$ unramified

This part of the dual consists of a circle, with opposing points identified (except the point $z = 1$ is not identified with $z = -1$, since $z = -1$ corresponds to the double point) , union a set of four points, union the line segment $(0, 1)$. On each connected component, the Borel structure can be taken to be that inherited from the half-circle, the segment $(0, 1)$, or the singleton set.

3.4.2 Borel structure on $G^{\wedge}_{ad,\chi}$, $\chi \in A^{\wedge}$ unramified

This part of the dual consists of the $C^{\times}/\sim$ with one double point, union a set of two (closed) points. On each connected component, the Borel structure can be taken to be that inherited from $C^{\times}/\{\pm 1\}$ or the singleton set.

3.5 Cuspidal representations

For each quadratic extension E/F there is an infinite series of irreducible cuspidal representations.

Let $\chi \in (E^{\times 1})^{\wedge}$, where $E^{\times 1}$ denotes the kernel of the norm map, and define $T_\chi : G \to L^2(G)$ by

$$(T_\chi(g)f)(x) = \int_F K_\chi(g \mid x, y) f(y) dy,$$

where $K_\chi(g \mid x, y)$ is a certain distribution-valued kernel function [GG]. Let χ_0 denote the unique character of $E^{\times 1}$ of order 2.

Lemma (Gelfand-Graev,Sally-Shalika) *Let* $\chi, \chi_1, \chi_2 \in (E^{\times 1})^{\wedge}$.

(1) For $\chi \neq \chi_0$, T_χ *decomposes into a direct sum of 2 irreducible, inequivalent subrepresentations* T^{+}_{χ}, T^{-}_{χ},

(2) For $\chi = \chi_0$, T_χ decomposes into a direct sum of 4 irreducible, inequivalent subrepresentations T^1_χ, T^2_χ, T^3_χ, T^4_χ,

(3) For $\chi \neq \chi_0$, we have $T^+_\chi \cong T^+_{\chi^{-1}}$, $T^-_\chi \cong T^-_{\chi^{-1}}$,

(4) If $\chi \neq \chi_0$ and $\chi_1 \neq \chi_2, \chi_2^{-1}$ then $T^+_{\chi_1}$ is not equivalent to $T^+_{\chi_2}$ and $T^-_{\chi_1}$ is not equivalent to $T^-_{\chi_2}$,

(5) If $\chi_1, \chi_2 \neq \chi_0$ then $T^+_{\chi_1}$ is not equivalent to $T^-_{\chi_2}$,

(6) For each cuspidal representation σ of G,there is a quadratic extension E/F and a character $\chi \in (E^{\times 1})^\wedge$ such that $\sigma = T^\alpha_\chi$, for some $\alpha \in \{+, -, 1, 2, 3, 4\}$.

The treatment of the subrepresentations T^1_χ, T^2_χ, T^3_χ, T^4_χ given in [GG] was not correct but was rectified in [SS1].

If $\sigma \in G^\wedge_c = G^\wedge_{u,c}$ then $G^\wedge_{ad,\sigma} = G^\wedge_{u,\sigma} = \{\sigma\}$. The set of course gets the discrete topology.

3.6 Summary

The admissible dual $G^\wedge_{ad}$ is a countable union of copies of $C^\times/\{\pm 1\}$, union a $C^\times/\{\pm 1\}$ with two double points, union a $C^\times/\{\pm 1\}$ with one double point and two (closed) points, union a countable number of singleton sets.

The unitary dual $G^\wedge_u$ is a countable union of circles, union a "half-circle" with two double points, union a "half-circle" with one double point and two (closed) points, union the line segment $(0, 1) \bmod 2\pi/\ln(q)$, union a countable number of singleton sets.

4 The Fourier transform

For $f \in C^\infty_c(G)$ and $\pi \in \Pi \in G^\wedge_{ad}$, we define the **operator-valued Fourier transform** of f at π by

$$\pi(f) = \int_G f(x)\pi(x)\, dx, \quad f \in C^\infty_c(G).$$

Lemma: *If $f, g \in C^\infty_c(G)$ then $\pi(f * g) = \pi(f)\pi(g)$.*

Let $G' = \{g \in G \mid g \text{ has distinct eigenvalues}\}$, called the **regular set** of G. In particular, every element in the regular set is semi-simple (i.e., diagonalizable).

Harish-Chandra showed that there exists a locally constant function, denoted Θ_π and called the **character** of π, on the regular set G' which represents the trace of π :

$$trace(\pi(f)) = \int_G \Theta_\pi(x) f(x) dx,$$

for all $f \in C_c^\infty(G)$. This trace, which we sometimes denote simply by $\Theta_\pi(f)$, is called the **Fourier transform** of f at π.

4.1 The Schwartz space and the tempered dual

Let $||g|| = \max(|a|, |b|, |c|, |d|)$, where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, and let $\sigma(g) = \log ||g||$. For each compact open subgroup $K \subset\subset G$, let

$$\mathcal{C}_K(G) = \{f \in C_c(G//K) \mid \begin{array}{c} |f(x)| <<_{r,f} \frac{\Xi(x)}{(1+\sigma(x))^r}, \ \forall x \in G, \\ for\ each\ r > 0 \end{array} \},$$

where $C_c(G//K)$ denotes the space of compacty supported functions which are bi-K-invariant,

$$\Xi(x) = \int_{K_0} \delta_B(xk)^{-1/2} dk.$$

Here, for $x \in G'$, we have $\delta_B(x) = |\det(Ad(x_d))_{\mathfrak{n}}|$, where x_d denotes a diagonalization of x in $SL(2, \overline{F})$, where $\overline{F}$ denotes a separable algebraic closure of F and the valuation $|...|$ has been extended to $\overline{F}$. It's known that there are constants $c_1 > 0,\ c_2 > 0,\ N > 0$ such that

$$c_1 \leq \Xi(a) \leq c_2(1 + \sigma(a))^N,$$

for all $a \in A^+ = \{a \in A \mid a = \begin{bmatrix} u\pi^n & 0 \\ 0 & u^{-1}\pi^{-n} \end{bmatrix},\ u \in O_F^\times,\ n \geq 0\}$. We topologize $\mathcal{C}_K(G)$ via the semi-norms

$$\upsilon_n(f) = \sup_{x \in G} |f(x)| \frac{(1 + \sigma(x))^n}{\Xi(x)}.$$

Let

$$\mathcal{C}(G) = \bigcup_K \mathcal{C}_K(G),$$

where K runs over all compact open subgroups of G. This is the **Schwartz space** of G. Let S denote the collection of all seminorms on $\mathcal{C}(G)$ whose restriction to each $\mathcal{C}_K(G)$ is continuous. In the semi-norm topology induced by S, the Schwartz space is a complete locally convex topological vector space.

Lemma *(1)* $\mathcal{C}(G) \subset L^2(G)$,

(2) $\mathcal{C}(G)$ *is an algebra under convolution.*

We call a distribution D on $C_c^\infty(G)$ **tempered** if it extends continuously to $\mathcal{C}(G)$. We call an admissible representation π **tempered** if the distribution $f \longmapsto \Theta_\pi(f)$ extends continuously to $\mathcal{C}(G)$. Let $G_t^\wedge$ denote the set of equivalence classes of irreducible tempered representations of G, the **tempered dual**.

Theorem (Harish-Chandra): $G_t^\wedge \subset G_u^\wedge$.

In fact, for our case of $SL(2)$, we have

$$G_t^\wedge = G_u^\wedge - \{1, T_\rho \mid 0 < \rho < 1\}.$$

Let

$$G_{t,\sigma}^\wedge = \{\textit{all irreducible inequivalent tempered constituents of } I_M(\chi\sigma) \ \mid \ \chi \in X_{un}(M)\},$$

We shall see that the following holds

Theorem: $G_t^\wedge = \bigcup_{M \in \mathcal{L}(G)} \bigcup_{\sigma \in M_{u,c}^\wedge} G_{t,\sigma}^\wedge$,

We will see that $G_{t,\sigma}^\wedge = G_{u,\sigma}^\wedge$ if $\sigma \in \hat{A}_u$ is ramified. If σ is unramified then $G_{t,\sigma}^\wedge$ is the same as $G_u^\wedge$, except that it doesn't contain the T_ρ or the trivial representation. In other words, the only non-tempered unitary representations are the trivial representation and the complementary series. This will follow from the character identities discussed next.

4.2 Character calculations

Let S denote a complete set of representatives of conjugacy classes of Cartan subgroups of G. We may and do assume that $A \in S$. For each $T \in S$, let $W_T = N_G(T)/T$ denote the Weyl group of T.

We begin with the

Lemma (Weyl integration formula [S, p. 198]) *For each* $f \in C_c^\infty(G)$,

$$\int_G f(g) = \sum_{T \in S} \frac{1}{\#[W_T]} \int_T |D(t)|^{1/2} F_f^T(t) dt,$$

where

$$F_f^T(t) = |D(t)|^{1/2} \int_{T\backslash G} f(x^{-1}tx)\frac{dx}{dt},$$

and where $D(t) = D_{G/T}(t) = \det(1-Ad(t))_{\mathfrak{g}/\mathfrak{t}}$. *Here* $\mathfrak{g}$, $\mathfrak{t}$ *denote the Lie algebras of* G, T.

Note that $F_f^T(t) = F_f^T(t^w)$, where $w \in W_T$, so that $F_f^A(a) = F_f^A(a^{-1})$, since $a^w = a^{-1}$, where $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

4.2.1 Principal series

Note that

$$\begin{aligned} D_{G/A}(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}) &= \det(1 - Ad(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}))_{\mathfrak{g}/\mathfrak{a}} \\ &= (1-a^2)(1-a^{-2}) = -(a-a^{-1})^2. \end{aligned}$$

Lemma *(1) The map* $f \longmapsto F_f^A$ *defines a surjection* $C_c^\infty(G) \to C_c^\infty(A)$.

(2) The map $f \longmapsto F_f^A$ *defines a continuous surjection* $\mathcal{C}(G) \to \mathcal{C}(A)$.

proof: (1) For $f \in C_c^\infty(G)$ and $a \in A' = A \cap G'$, the function $x \longmapsto f(x^{-1}ax)$ is compactly supported on $A\backslash G$ and

$$\begin{aligned} F_f^A(a) &= |D(a)|^{1/2} \textstyle\int_{A\backslash G} f(x^{-1}ax)\frac{dx}{da} \\ &= \delta_B(a)^{1/2} \textstyle\int_N \int_{K_0} f(kank^{-1})dkdn \\ &= \delta_B(a)^{1/2} \textstyle\int_N \overline{f}(an)dn, \end{aligned}$$

where

$$\overline{f}(x) = \int_{K_0} f(kxk^{-1})dk.$$

Since the function $x \longmapsto f(x^{-1}ax)$ is compactly supported on $A\backslash G$ the function $x \longmapsto \overline{f}(x^{-1}ax)$ is compactly supported on $A\backslash G/K_0$. We claim that given any compact set $C \subset\subset G$, there is an $a \in A$ with sufficiently large $\sigma(a)$ such that $an \notin C$, for all $n \in N$. Indeed, we may map C onto $\overline{C} \subset G/K_0$ via $G \to G/K_0$ and aN onto $\overline{aN}$. Since G/K_0 is non-compact but $\overline{C}$ is compact in G/K_0, the Iwasawa decomposition implies that we may find an $a \in A$ such that $\overline{aN} \cap \overline{C} = \emptyset$. From the claim it follows that $F_f^A(a)$ is compactly supported on $A' = A \cap G'$.

Moreover, since f is locally constant so is $\overline{f}$. Since $\overline{f}$ is compactly supported, $\int_N \overline{f}(an)dn$ is a locally constant function on A'.

It remains to show that $F_f^A(a)$ extends to a locally constant function on all of A. This follows from either the Shalika germ expansion. It is also an immediate corollary of part (2), which we turn to next.

(2) The existence of the orbital integral $F_f^A(a)$ follows from Clozel. By Harish-Chandra, the map $f \longmapsto f^B$ defines a continuous map $\mathcal{C}(G) \to \mathcal{C}(A)$, where

$$f^B(a) = \delta_B(a)^{1/2} \int_N f(an)dn$$

Note that both Ξ and σ are bi-K_0-invariant, so the map $f \longmapsto \overline{f}$ defines a continuous map $\mathcal{C}(G) \to \mathcal{C}(G)$. Consequently, the composition $f \longmapsto \overline{f} \longmapsto F_{\overline{f}}^A$ defines a continuous map $\mathcal{C}(G) \to \mathcal{C}(A)$. This completes the proof.

Let $\chi \in (F^\times)^\wedge$ and suppose that $\chi \neq \chi^{-1}$.

Theorem (Gelfand-Graev) *The character of the irreducible representation*

$\pi = T_\chi$ *is given on the regular set by*

$$\Theta_\pi(x) = \begin{cases} \frac{\chi(a)+\chi(a)^{-1}}{|a-a^{-1}|}, & x = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^g, \ for\ some\ g \in G,\ a \neq \pm 1, \\ 0, & if \quad x \in G' - A^G, \end{cases}$$

where $A^G = \{a^g \mid a \in A,\ g \in G\}$.

This and the Weyl integration formula imply that

$$\begin{aligned} \Theta_\pi(f) &= \textstyle\int_G \Theta_\pi(x) f(x) dx \\ &= \tfrac{1}{2} \textstyle\int_{F^\times} (\chi(a) + \chi(a)^{-1}) F_f^A(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) d^\times a \\ &= (F_f^A)^\wedge(\chi), \end{aligned}$$

where

$$\begin{aligned} \varphi^\wedge(\omega) &= \textstyle\int_{F^\times} \omega(x)\varphi(x) d^\times x \\ &= \textstyle\sum_{n=-\infty}^{\infty} \omega(\pi)^n \int_{O_F{}^\times} \omega(u)\varphi(\pi^n u) du, \qquad \omega \in (F^\times)_u^\wedge. \end{aligned}$$

denotes the Fourier transform on $F^\times$. Note that this formula for the character extends to the Schwartz space provided χ is unitary. In fact, we have

Lemma: *A principal series representation* T_χ *is tempered if and only* χ *is unitary.*

The above formulas allows us to classify the image of the Fourier transforms of a *unitary* principal series representation on either $C_c^\infty(G)$ or the Schwartz space.

First, suppose $f \in C_c^\infty(G)$, so $F_f^A \in C_c^\infty(A)$.

Lemma: *For $\varphi \in C_c^\infty(F^\times)$, the image $C_c^\infty(F^\times)^\wedge$ of the Fourier transform $\varphi \longmapsto \varphi^\wedge(\omega)$ is given by*

$$C_c^\infty(F^\times)^\wedge = \{h \in C_c((F^\times)_u^\wedge) \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } (F^\times)_u^\wedge \end{array} \}.$$

We omit the proof. (This lemma, and its Schwartz space analog below, is proven using the inverse Fourier transform and the inversion formula, which may be found in [Ta, pp. 43-44], [Ba, ch 12].) As a consequence of the above lemma, since the map $C_c^\infty(G) \to C_c^\infty(A)$ is surjective, it follows that the same description holds for the space of all functions of χ of the form $(F_f^A)^\wedge(\chi)$:

Lemma: *For $f \in C_c^\infty(G)$, the image $C_c^\infty(G)_{ps}^\wedge$ of the Fourier transform $f \longmapsto \Theta_\pi(f)$, $\pi = T_\chi$, $\chi \in A_u^\wedge$ is given by*

$$C_c^\infty(G)_{ps}^\wedge = \{h \in C_c(A_{u,c}^\wedge) \mid \begin{array}{c} h \text{ is a trig polynomial on} \\ \text{each circle in } A_{u,c}^\wedge \end{array} \}.$$

Next, suppose $f \in \mathcal{C}(G)$, so $F_f^A \in \mathcal{C}(A)$.

Lemma: *For $\varphi \in \mathcal{C}(F^\times)$, the image $\mathcal{C}(F^\times)^\wedge$ of the Fourier transform $\varphi \longmapsto \varphi^\wedge(\omega)$ is given by*

$$\mathcal{C}(F^\times)^\wedge = \{h \in C_c((F^\times)_u^\wedge) \mid \begin{array}{c} h \text{ is a trig series on each circle} \\ \text{in } (F^\times)_u^\wedge \text{ which converges absolutely} \\ \text{along with all of its derivatives} \end{array} \}.$$

Consequently, since the map $\mathcal{C}(G) \to \mathcal{C}(A)$ is surjective, it follows that the same description holds for the space of all functions of χ of the form $(F_f^A)^\wedge(\chi)$:

Lemma: *For $f \in \mathcal{C}(G)$, the image $\mathcal{C}(G)_{ps}^\wedge$ of the Fourier transform $f \longmapsto \Theta_\pi(f)$, $\pi = T_\chi$ is given by*

$$\mathcal{C}(G)_{ps}^\wedge = \{h \in C_c(A_{u,c}^\wedge) \mid \begin{array}{c} h \text{ is a trig series on each circle} \\ \text{in } A_{u,c}^\wedge \text{ which converges absolutely} \\ \text{along with all of its derivatives} \end{array} \}.$$

4.2.2 Reducible principal series

Let $\chi \in (F^\times)^\wedge$ and suppose that $\chi = sign_\tau$, for some $\tau = \varepsilon, \pi, \varepsilon\pi$. Let $\pi = T_\chi^+$ *or* $\pi = T_\chi^-$. Following [SS] or [Fr] (see also [A]), the character Θ_π of

a reducible principal series representation is given on the regular set by

$$\Theta_\pi(x) = \begin{cases} \frac{sign_\tau(x)}{|a-a^{-1}|}, & x = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^g, \ for\ some\ g \in G,\ a \neq \pm 1, \\ e_{\pi,\tau}(t)|D_T(t)|^{-1/2}, & x\ conjugate\ to\ t \in (T_\tau \cup T_\tau^\#) \cap G', \\ 0, & otherwise, \end{cases}$$

where $T_\tau, T_\tau^\#$ are two non-split Cartans (which are not conjugate if $-1 \in (F^\times)^2$ or if $\tau = \varepsilon$) and where $e_{\pi,\tau}(t)$ is a bounded locally constant function on $G' - A^G$. Using this, the Weyl integration formula, and the reasoning above, it can be shown that

Lemma *If $\pi = T_\chi^+$ or $\pi = T_\chi^-$ then π is tempered.*

4.2.3 Complementary series

Let $\chi = |...|^{-\rho}$ for some $0 < \rho < 1$. As in the case of the principal series, we see that T_ρ is not tempered because χ is not a unitary character.

Lemma *If $\pi = T_\rho$ then π is non-tempered.*

The character Θ_π of a reducible principal series representation is given by the same formula as in the case of the principal series. This and the Weyl integration formula imply that

$$\begin{aligned} \Theta_\pi(f) &= \textstyle\int_G \Theta_\pi(x) f(x) dx \\ &= \tfrac{1}{2} \textstyle\int_{F^\times} (|a|^{-\rho} + |a|^\rho) F_f^A(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) d^\times a \\ &= M(F_f^A)(\rho), \end{aligned}$$

where M denotes the **Mellin transform**

$$\begin{aligned} M\varphi(\rho) &= \textstyle\int_{F^\times} |x|^{-\rho} \varphi(x) d^\times x \\ &= \textstyle\sum_{n=-\infty}^{\infty} q^{\rho n} \int_{O_F{}^\times} \varphi(\pi^n u) du. \end{aligned}$$

Lemma: *For $\varphi \in C_c^\infty(F^\times)$, the image $MC_c^\infty(F^\times)$ of the Fourier transform $\varphi \longmapsto \Theta_\pi(f)$, $\pi = T_\rho$ is given by*

$$MC_c^\infty(G) = \{h \in C^\infty(0,1) \mid h\ is\ a\ finite\ Laurant\ series\ in\ q^\rho\ \}.$$

Consequently, since the map $C_c^\infty(G) \to C_c^\infty(A)$ is surjective, it follows that the same description holds for the space of all functions of χ of the form $M(F_f^A)(\rho)$:

Lemma: *For $f \in C_c^\infty(G)$, the image $MC_c^\infty(G)$ of the Mellin transform $f \longmapsto M(F_f^A)(\rho)$ is given by*

$$C_c^\infty(F^\times)^\wedge = \{h \in C^\infty(0,1) \mid h \text{ is a finite Laurant series in } q^\rho\}.$$

4.2.4 The special representation

Let $\pi = T_{sp}$. Then

$$\Theta_\pi(x) = \begin{cases} \frac{|a|+|a|^{-1}}{|a-a^{-1}|} - 1, & x = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^g, \text{ for some } g \in G,\ a \neq \pm 1, \\ -1, & x \in G' - A^G. \end{cases}$$

It follows from this and the Weyl integration formula that

$$\begin{aligned} \Theta_\pi(f) &= \textstyle\int_G \Theta_\pi(x) f(x) dx \\ &= \tfrac{1}{2} \textstyle\int_{F^\times} (|a| + |a|^{-1} - |a - a^{-1}|) F_f^A(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) d^\times a \\ &\quad - \textstyle\sum_{T \in S - \{A\}} \tfrac{1}{2} \int_T |D(t)|^{1/2} F_f^T(t) dt. \end{aligned}$$

This converges even for f in the Schwartz space, so we have

Lemma: *If $\pi = T_{sp}$ then π is tempered.*

4.2.5 The trivial representation

Let $\pi = 1$ so

$$\Theta_\pi(x) = 1.$$

To see that

$$\Theta_\pi(f) = \int_G f(x) dx$$

diverges in general on the Schwartz space, we use [S, Lemma 4.2.5] to compute

$$\begin{aligned} \textstyle\int_G \frac{\Xi(x)}{(1+\sigma(x))^r} dx &= \textstyle\sum_{a \in A^+/A^0} \mu(K_0 a K_0) \frac{\Xi(a)}{(1+\sigma(a))^r} \\ &\approx \textstyle\sum_{a \in A^+/A^0} \mu(K_0 a K_0) \delta_B(a)^{1/2} \frac{\delta_B(a)^{-1/2} \Xi(a)}{(1+\sigma(a))^r} \\ &\approx \textstyle\sum_{a \in A^+/A^0} \delta_B(a)^{1/2} \frac{1}{(1+\sigma(a))^{r-r_0}}, \end{aligned}$$

for some constant r_0 and all $r > 0$. (Here μ denotes the volume.) This diverges for all r, so the trivial representation is not tempered. (Here $f(a) \approx g(a)$ means $f(a) << g(a) << f(a)$.)

Lemma *If $\pi = 1$ then π is non-tempered.*

4.2.6 The cuspidal representations

The characters of the cuspidal representations will not be given here - we refer to [SS2]. They are all of the form

$$\Theta_\pi(x) = \begin{cases} \alpha(x) + \frac{\beta(x)}{|D(x)|^{1/2}}, & x \in G' - A^G, \\ 0, & x \notin G' - A^G, \end{cases}$$

where α, β are uniformly bounded locally constant functions on $G' - A^G$. From this and our previous estimates, it is not hard to see that the cuspidal representations are tempered (see also [S, Theorem 4.5.1, Theorem 4.5.10]).

Lemma: *If $\pi \in G_c^\wedge$ then π is tempered.*

5 The image of the Fourier transform

5.1 Admissible Paley-Wiener Theorem for smooth functions

The following result classifies the image of $C_c^\infty(G)$ under the Fourier transform:

Theorem (Bernstein-Deligne-Kazhdan) *The image of the Fourier transform $\pi \longmapsto \Theta_\pi(f)$, for $f \in C_c^\infty(G)$ and $\pi \in \Pi \in G_{ad}^\wedge$, consists of the space of functions h on $G_{ad}^\wedge$ such that*

(1) h is supported on finitely many connected components of $G_{ad}^\wedge$,

(2) h is regular on $C^\times/\{\pm 1\}$ in $G_{ad}^\wedge$, regarded as a complex algebraic variety.

Let $C_c^\infty(G)_{ad}^\wedge$ denote the space of functions described in the above theorem. We call it the **admissible Paley-Wiener space**.

5.2 Unitary Paley-Wiener Theorem for smooth functions

We use the information above to prove the following

Theorem *The image of the Fourier transform $\pi \longmapsto \Theta_\pi(f)$, for $f \in C_c^\infty(G)$ and $\pi \in \Pi \in G_u^\wedge$, consists of the space of functions h on $G_u^\wedge$ such that*

(1) h is supported on finitely many connected components of $G_u^\wedge$,

(2) h is a trigonometric polynomial on each circle (or half-circle) in $G_u^\wedge$ which it is supported on,

(3) if h is supported on the complementary series then h is a finite Laurant series in $x = q^\rho$.

proof: (1) follows from the admissible Paley-Wiener theorem.

(2) follows from the section on characters of principal series representations.

(3) follows from the section on characters of complementary series representations.

This completes the proof since h can be arbitrary on the discrete part of the unitary dual.

Let $C_c^\infty(G)_u^\wedge$ denote the space of functions described in the above theorem. We call it the **unitary Paley-Wiener space**.

5.3 Tempered Paley-Wiener Theorem for Schwartz functions

We also have the following

Theorem *The image of the Fourier transform $\pi \longmapsto \Theta_\pi(f)$, for $f \in \mathcal{C}(G)$ and $\pi \in \Pi \in G_t^\wedge$, consists of the space of functions h on $G_t^\wedge$ such that*

(1) h is supported on finitely many connected components of $G_t^\wedge$,

(2) h is a trigonometric series on each circle (or half-circle, ignoring the double point if it occurs) in $G_t^\wedge$ which it is supported on, converging absolutely along with all of its derivatives.

proof: (1) follows from the admissible Paley-Wiener theorem.

(2) follows from the section on characters of principal series representations.

This completes the proof since h can be arbitrary on the discrete part of the tempered dual.

Let $\mathcal{C}(G)^\wedge$ denote the space of functions described in the above theorem. We call it the **tempered Paley-Wiener space**.

6 Linear functionals on the tempered Paley-Wiener space

6.1 A result of L. Schwartz

The space of linear functionals on the tempered Paley-Wiener space can be successfully described using the theory of distributions on the real line:

Theorem (L. Schwartz [Sch, ch. III, Théorème XXI]) *Let S^1 denote the unit circle. If $T \in C^\infty(S^1)'$ then there is a continuous function u on S^1 and an integer $n \geq 0$ such that $T = \frac{d^n}{d\theta^n}u$ (as distributions).*

As a corollary of this and the above Paley-Wiener theorem, we have the following

Proposition: *The dual space of the tempered Paley-Wiener space $\mathcal{C}(G)^\wedge$ is isomorphic to the direct product*

$$\bigoplus_{d\in\pi_0(A^\wedge_{u,c})} \bigcup_{n\geq 0} C(S_d)^{(n)} \bigoplus \bigoplus_{\sigma\in\pi_0(G^\wedge_{lds})} C\delta_\sigma,$$

where $\pi_0(X)$ denotes the set of connected components of a topological space X, where $G^\wedge_{lds}$ denotes the union of the reducible principal series, the special representation, and the cuspidal representations, and where δ_σ denotes the evaluation functional on the space of (constant) functions on the (singleton) elements of $\pi_0(G^\wedge_{lds})$, and, if we write the unitary dual $A^\wedge_u = A^\wedge_{u,c}$ as a union of circles $S_d \cong R/\frac{2\pi}{\log(q)}Z$, where

$$C(S_d)^{(n)} = \{\frac{d^n}{d\theta^n}u \mid u \text{ continuous on } S_d\},$$

6.2 The Fourier transform of a tempered distribution

As a consequence of this, we can "completely describe" the Fourier transform of a tempered distribution:

Theorem: *If T is a tempered distribution on G then there is a*

$$T^\wedge \in \bigoplus_{d\in\pi_0(A^\wedge_{u,c})} \bigcup_{n\geq 0} C(S_d)^{(n)} \bigoplus \bigoplus_{\sigma\in\pi_0(G^\wedge_{lds})} C\delta_\sigma$$

such that

$$T(f) = T^\wedge(f^\wedge),$$

for all $f \in \mathcal{C}(G)$, where $f^\wedge(\pi) = \Theta_\pi(f)$ denotes the Fourier transform of f at $\pi \in \Pi \in G_t^\wedge$.

7 Linear functionals on the admissible Paley-Wiener space

7.1 Analytic functionals

A large part of the space of linear functionals on the admissible Paley-Wiener space can be described using the theory of analytic functionals (see Hörmander [H1, ch. IV], [H2, ch. IX]). Our description of the Fourier transform of an admissible distribution is unfortunately much less explicit than in the tempered case treated in the previous section. However, if the admissible distribution has "exponential growth" then, as we will see, more precise results can be given.

In spite of what we have seen already in the previous section, the following may still be of interest

QUESTION Is there is an analogous description for the linear functionals on the unitary Paley-Wiener space using the theory of (Sato) hyperfunctions?

(This question is inspired by the relationship between analytic functionals and hyperfunctions discussed in [H2, §9.2].)

We begin with some definitions:

Let $\Omega \subset G_\sigma^\wedge \hookrightarrow X_{un}(M) \cong (C^\times)^n$, $\sigma \in M_c^\wedge$, some $n \geq 1$, denote a connected component of $G_{ad}^\wedge$, which we may regard as an complex algebraic variety or as a complex analytic manifold. Let $R(X_{un}(M))$ denote the space of regular functions on $X_{un}(M)$, regarded as a variety, and $A(X_{un}(M))$ denote the space of analytic functions on $X_{un}(M)$, regarded as a manifold. We give $A(X_{un}(M))$ the topology of uniform convergence on compacta, so that $R(X_{un}(M)) \subset A(X_{un}(M))$ is dense. Following Hörmander ([H1, ch. IV]), an **analytic functional** on $X_{un}(M)$ is an element of the topological dual of $A(X_{un}(M))$, denoted $A(X_{un}(M))'$, where $A(X_{un}(M))$ is given the topology of uniform convergence. We say that an analytic functional λ is **carried on** a compact set $K \subset\subset X_{un}(M)$ if for every neighborhood $U \supset K$, there is a constant $C_U > 0$ such that

$$|\lambda(f)| \leq C_U \sup_{z \in U} |f(z)|.$$

It follows from the definitions that for any $\lambda \in A(\Omega)'$, there exists a compact K such that λ is carried on K. The analytic functionals carried on K are in one-to-one correspondence, via the Laplace transform, with those entire functions with exponential growth (a result due to Pólya- Ehrenpreis-Martineau, see [H1, Theorem 4.5.3]).

Let $PW(G) = C_c^\infty(G)^\wedge_{ad}$ denote the Paley-Wiener space and let $PW(G)'$ denote its algebraic dual space (as a vector space). From the bijection $f \longmapsto f^\wedge$, $C_c^\infty(G) \to PW(G)$, we define the **Fourier transform** of an invariant distribution D on $C_c^\infty(G)$ to be the linear functional $D^\wedge \in PW(G)'$ such that

$$D(f) = D^\wedge(f^\wedge),$$

for all $f \in C_c^\infty(G)$. We say that the distribution D is **analytic** if the linear functional $D^\wedge$ restricted to $R(X_{un}(M)) \subset PW(G)$ extends continuously an analytic functional on $A(\Omega)$.

Lemma: *If D is analytic then there is a compact $K \subset\subset \Omega$ such that $D^\wedge$ is carried on K.*

Define a function $E_{M,a}$ on $X_{un}(M)$, by $E_{M,a}(I_M(\chi\sigma)) = \chi(a)$ for each $\chi \in X_{un}(M)$, $a \in A_M$, where A_M denotes the split component of the center of M (in M). Let

$$H_K(a) = max_{\chi \in K} \log(||a||^{Re(s_\chi)}), \quad \forall a \in A_M$$

where $s_\chi \in C^n$ is an element representing $\chi \in X_{un}(M)$.

Lemma *The vector space span of all linear combinations of the functions $E_{M,a}$, $a \in A_M$, is dense in $A(X_{un}(M))$.*

proof The only cases we need to prove this are $M = G$ and $M = A$. The statement is trivial if $M = G$ so assume $M = A$.

The trace Paley-Wiener theorem of Bernstein-Deligne-Kazhdan mentioned above implies that each regular function h on $X_{un}(A)$ may be realized as the trace: there is an $f \in C_c^\infty(G)$ such that $h(\chi) = \Theta_{I_A(\chi)}(f)$ for all $\chi \in X_{un}(A)$ such that the induced representation $I_A(\chi)$ is irreducible. In this case, the character formulas in the principal series case imply that

$$\begin{gathered} h(\chi) = \textstyle\int_G \Theta_{I_A(\chi)}(g) f(g) dg = (F_f^A)^\wedge(\chi) \\ = \textstyle\int_A F_f^A(a)\chi(a) da = \sum a \in A \\ finite c_a \chi(a), \end{gathered}$$

for all $\chi \in X_{un}(A)$, where the sum over $a \in A$ runs over a finite set (since $F_f^A(a)$ is compactly supported). From this the result easily follows.

Therefore to verify that an analytic functional carried on $K \subset\subset X_{un}(M)$ it suffices to check that the condition $|\lambda(f)| \leq C_U \sup_{z\in U} |f(z)|$ holds for all f belonging to this span. Since

$$|E_{M,a}(\chi)| \leq ||a||^{Re(s_\chi)},$$

this implies the following

Proposition: *If D is analytic and $D^\wedge$ is carried on a compact $K \subset\subset X_{un}(M)$ then for each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$|D^\wedge(E_{M,a})| \leq C_\varepsilon \exp((1+\varepsilon)H_K(a)), \qquad \forall a \in A_M.$$

7.2 The Fourier transform of an analytic distribution

The result below describes the Fourier transform of an analytic distribution on G.

Theorem: *If D is an analytic distribution then there is a linear functional $D^\wedge \in PW(G)'$ such that $D(f) = D^\wedge(f^\wedge)$,for all $f \in C_c^\infty(G)$, and, for each Levi M of G, and each $\epsilon > 0$ there are constants $C_\epsilon > 0$, $C_M > 0$, such that*

$$|D^\wedge(E_{M,a})| \leq C_\varepsilon \exp((1+\varepsilon)\sigma(a)^{C_M}), \qquad \forall a \in A_M.$$

Conversely, let D be an invariant distribution such that its Fourier transform $D^\wedge \in PW(G)'$ satisfies the following property: for each Levi M of G, and each $\epsilon > 0$ there are constants $C_\epsilon > 0$, $C_M > 0$, such that

$$|D^\wedge(E_{M,a})| \leq C_\varepsilon \exp((1+\varepsilon)\sigma(a)^{C_M}), \qquad \forall a \in A_M.$$

Then D is analytic.

7.3 "Nice" admissible distributions are analytic

We shall define admissible distributions, which appear in the result below, in the next section. Roughly speaking, the theorem below says that admissible distributions of "exponential growth" on $SL(2)$, which are very common but not well classified, are analytic.

Theorem: *If D is admissible and if there is a constant $C > 0$ such that $|D(a)|^{1/2}F_D(a) << ||a||^C$, for all $a \in A$ then D is analytic.*

proof: We need only check the condition of analyticity for the Levi component $M = A$ since the case $M = G$ is essentially trivial due to the fact that $A_M = \{\pm 1\}$ in that case. Furthermore, we need only prove the estimate

$$|D^\wedge(E_{A,a})| \leq C_\varepsilon \exp((1+\varepsilon)\sigma(a)^{C_A}), \qquad \forall a \in A,$$

for "sufficiently large" $a \in A$ (that is, $a \in A$ for which $\sigma(a)$ is sufficiently large).

Fix an $a \in A$ and pick $f = f_a \in C_c^\infty(G)$ such that

(1) f is supported in a neighborhood of the conjugacy class of a in G, and

(2) $F_f^A \in C_c^\infty(A)$ is supported in a neighborhood U of a in A such that

$$\begin{gathered}(F_f^A)^\wedge(\chi) = \chi(a) + \chi(a)^{-1} \\ |F_f^A(x)| << |a| + |a|^{-1}, \quad \forall x \in U.\end{gathered}$$

Since D is admissible there is (by a result of Harish-Chandra stated above) a function F_D such that

$$D(f) = \int_G F_D(g)f(g)dg.$$

Therefore, by definition of $f = f_a$, the definition of $E_{A,a}$, and the Weyl integration formula, we have the estimate

$$\begin{gathered}D^\wedge(E_{A,a})| = 2|D^\wedge(f^\wedge)| = 2|D(f)| \\ \leq 2\int_G |F_D(g)f(g)|dg \\ << \sum T \in S \\ T \neq A \frac{1}{\#[W_T]} \int_T |D(t)|^{1/2}|F_D(t)F_f^T(t)|dt + \int_A |D(a)|^{1/2}|F_D(a)F_f^A(a)|da \\ << 1 + \int_A |D(a)|^{1/2}|F_D(a)F_f^A(a)|da \\ << 1 + |D(a)|^{1/2}(|a| + |a|^{-1}) \max_{x\in U} |F_D(x)| << ||a||^{3+C}.\end{gathered}$$

(The fact that

$$\sum_{T \in ST \neq A} \frac{1}{\#[W_T]} \int_T |D(t)|^{1/2}|F_D(t)F_f^T(t)|dt << 1$$

follows from the fact that the $T \neq A$ are compact, bounded in number, and f is supported in a neighborhood of the conjugacy class of a in G. If we take a "sufficiently large" in the sense above then this estimate follows easily.)

This implies that D is analytic by the previous theorem.

Corollary: *The space of all analytic distributions on G includes the characters Θ_π, for $\pi \in \hat{G}_{ad}$.*

proof: There is a result of Harish-Chandra and Howe which implies that the character of every irreducible admissible representation is an admissible distribution. The bound $|D(a)|^{1/2}|F_D(a)| << ||a||^C$, for some $C = C(D) > 0$, follows from the explicit character formulas above.

8 Admissible distributions

8.1 Definitions

Let $K \subset\subset G$ be a compact open subgroup and for each irreducible representation σ of K, let χ_σ denote its character.

An invariant distribution D on G (not to be confused with the Weyl discriminant function $D(x)$ on G) is called (G, K)-**admissible at** $\gamma \in G$ if for any compact open subgroup $K' \subset K$ and any irreducible representation σ of K', we have $D * \chi_\sigma = 0$ on γK unless G intertwines the trivial representation on K with σ. In other words, for each $f \in C_c^\infty(G)$ with $supp(f) \subset \gamma K$, we have $D(f * \chi_\sigma^0) = 0$ on γK, unless G intertwines the trivial representation on K with σ. Here $\chi_\sigma^0(x)$ denotes the extended character of $\sigma : \chi_\sigma^0(x) = \chi_\sigma(x)$ if $x \in K'$ and $\chi_\sigma^0(x) = 0$ if $x \in G - K'$. If U is a G-invariant open set in G then we say that D is **admissible on** U if it is (G, K)-admissible at each $\gamma \in U$ in G for some sufficiently small K (which may depend on γ). Note that it suffices to verify this for each regular $\gamma \in U \cap G'$.

According to Harish-Chandra, if D is admissible then there is a function F_D on G which is locally constant on the regular set G' such that

(1) $|D(x)|^{1/2}F_D$ is locally bounded,

(2) for each $f \in C_c^\infty(G)$,

$$D(f) = \int_G F_D(x)f(x)dx.$$

In general, if D is a distribution on G which is given by an integral as above, we say that D is **representable**. For representable distributions,

the (G,K)-admissibility condition at γ becomes: for each $f \in C_c^\infty(G)$ with $supp(f) \subset \gamma K$, we have

$$\begin{aligned}\textstyle\int_G F_D(x)(f * \chi_\sigma^0)(x)dx &= \textstyle\int_U F_D(x) \int_{K'} f(xk^{-1})\chi_\sigma^0(k)dkdx \\ &= \textstyle\int_G (F_D * \chi_\sigma^0)(x)f(x)dx = 0,\end{aligned}$$

unless G intertwines the trivial representation on K with σ. Therefore, the (G,K)-admissibility condition for represented distributions is: $(F_D * \chi_\sigma^0)(x) = 0$ on γK, unless G intertwines the trivial representation on K with σ.

8.2 A descent to the Levis

By the Weyl integration formula, we may write

$$D(f) = \sum_{T \in S} \frac{1}{|W_T|} \int_T |D(t)|^{1/2} F_D(t) F_f^{G/T}(t)dt = \sum_M D_M(F_{\overline{f}^P}^{M/T}).$$

Moreover, here

$$D_M(h) = \sum TM - elliptic \frac{1}{|W_T|} \int_T |D(t)|^{1/2} F_D(x)h(t)dt,$$

for all $h \in C_c^\infty(M)$, where the sum runs over all Cartans T in G which are G-conjugate to a Cartan of M but not to any smaller $M' \in \mathcal{L}(G)$.

In the case of $SL(2)$ this amounts to saying that

$$D_G(f) = \int_G F_D(x)f(x)ch_{G-A^G}(x)dx,$$

and

$$D_A(f) = \int_G F_D(x)f(x)ch_{A^G}(x)dx = \frac{1}{2}\int_A |D(a)|^{1/2} F_D(a) F_f^A(a)da,$$

where ch_S denotes the characteristic function of the set $S \subset G$.

Lemma: *If D is G-admissible and if $M \in \mathcal{L}(G), M \neq G$, then D_M is M-admissible.*

Remark: D_G is admissible at γ provided

$$(ch_{G-A^G} \cdot F_D * \chi_\sigma^0)(x) = 0, \text{ on } \gamma K \text{ unless } G \text{ intertwines the trivial representation on } K \text{ with } \sigma.$$

In this case, since $G - A^G$ is open in G, we see that for each $\gamma \in G - A^G$ there is a compact open $K \subset\subset G$ such that $\gamma K \subset G - A^G$. Since $ch_{G-A^G} = 1$ on γK, this condition is a consequence of the condition for D itself. It follows that D_G is admissible on $G - A^G$.

proof: The result follows for D_A provided for each $a \in A$ and for some sufficiently small compact open subgroup $U_m \subset\subset A$, and each non-trivial character χ of U_m, we have

$$\begin{aligned} &\textstyle\int_{U_m} |D(k)|^{1/2} F_D(k)\chi(k)dk \\ &= q^{-m} \textstyle\int_{O_F} |D((1+\pi^m u)a)|^{1/2} F_D((1+\pi^m u)a)\chi(1+\pi^m u)du \\ &= \textstyle\sum_{n\geq m} d_n q^{-n} \int_{O_F^\times} F_D((1+\pi^n u)a)\chi(1+\pi^n u)du = 0, \end{aligned}$$

where d_n denotes the common value of $|D((1+\pi^n u)a)|^{1/2}$. (These integrals exists since $|D(k)|^{1/2}F_D(k)$ is locally constant on the regular set and bounded on U_m.) If $a \neq \pm 1$ then this follows from the fact that F_D is locally constant on the regular set.

This completes the proof.

8.3 Questions and an example

QUESTION: Is is possible to classify the admissible representations D in terms of the functions F_D? In other words, what conditions on F_D guarantee that D is admissible?

This question is related to the characterization of the algebraic dual of the admissible Paley-Wiener space so would appear to be rather difficult.

QUESTION: Is is possible to classify which representable *tempered* distributions are admissible? In other words, what conditions on the *tempered* functions F_D guarantee that D is admissible?

This question is related to the characterization of the algebraic dual of the tempered Paley-Wiener space and so seems to be more tractable. In fact, we will explore this question further in the case of $SL(2)$ below. As a start, we prove the following

Lemma *Let $C \subset G_\sigma^\wedge \subset G_t^\wedge$ denote a compact Borel measurable set, where $\sigma \in M_c^\wedge$ and M is a Levi component of G, and let $d\lambda$ denote a bounded Borel measure on C. The distribution defined by*

$$D(f) = \int_C \Theta_\pi(f)d\lambda(\pi), \qquad f \in C_c^\infty(G),$$

exists and is admissible.

proof: By Harish-Chandra, each distribution $f \longmapsto \Theta_\pi(f)$ is admissible. This implies that

$$(\Theta_\pi * \chi_\sigma^0)(x) = 0, \text{ on } U \text{ unless } G \text{ intertwines the trivial representation on } K \text{ with } \sigma,$$

for σ, K, U as above. Since C is compact we may apply Fubini's theorem to verify this admissibility condition for

$$F_D(x) = \int_C \Theta_\pi(x) d\lambda(\pi).$$

This completes the proof.

8.4 Distributions on the tempered Paley-Wiener space, revisited

Let $d\mu$ denote the Plancherel measure on the tempered dual $G_t^\wedge$ and let $m(\pi)d\mu(\pi)$ denote a distribution on the tempered Paley-Wiener space $\mathcal{C}(G)^\wedge$ such that

(1) $m(\pi)$ is supported on finitely many components of $G_t^\wedge$,

(2) there is a continuous function h on $G_t^\wedge$ such that on each component $G_\sigma^\wedge$, with $\sigma \in M_c^\wedge$ being a cuspidal representation of some Levi M of G, there is a function h_σ on $X_{un}(M)$ such that $h(I_M(\omega\sigma))$ is the restriction of $h_\sigma(\omega)$ on the set

$$\{\omega \in X_{un}(M) \mid I_M(\omega\sigma) \in G_t^\wedge\},$$

such that, as distributions on $\{\omega \in X_{un}(M) \mid I_M(\omega\sigma) \in G_t^\wedge\}$, we have

$$m(I_M(\omega\sigma)) = \frac{\partial^I}{\partial\omega^I} h_\sigma(\omega),$$

where $I = (i_1, ..., i_r)$ denotes a multi-index, r being the real dimension of $X_{un}(M)$, and $\frac{\partial^I}{\partial\omega^I}$ denotes partial differentiation on the real manifold $X_{un}(M)$. Note that by (1) the function h in (2) is compactly supported. We call such a distribution a **distribution of finite type** on $G_t^\wedge$. A distribution satisfying (2) but not (1) will be called a **distribution of quasi-finite type** on $G_t^\wedge$. The maximum of the integers $|I| = i_1 + ... + i_r$, where I runs over all multi-indices occurring in (2), is called the **order** of the distribution.

The theorem of section 6 implies the following

Theorem: *If D is a tempered distribution on G then there is a distribution of quasi-finite type $m(\pi)d\mu(\pi)$ on the tempered Paley-Wiener space $\mathcal{C}(G)^\wedge$ such that*

$$D(f) = \int_{G_t^\wedge} \Theta_\pi(f) m(\pi) d\mu(\pi), \qquad f \in \mathcal{C}(G).$$

QUESTION: Is any such distribution is admissible?

By the previous lemma, the answer is yes for distributions of order 0.

8.5 Quasi-finite distributions and the Schwartz space

We shall end this section with a proof, using a completely different argument, of a similar statement for $H = GL(2, F)$. (We use $GL(2)$ rather than $SL(2)$ since the presence of a non-compact center introduces a key idea which will be expanded on and generalized in [J].) Let Z denote the center of H for the remainder of this section.

Theorem: *If D is a tempered distribution on the space $\mathcal{C}(H)$ then there is a distribution of quasi-finite type $m(\pi)d\mu(\pi)$ on tempered Paley-Wiener space $\mathcal{C}(H)^\wedge$ such that*

$$D(f) = \int_{H_t^\wedge} \Theta_\pi(f) m(\pi) d\mu(\pi), \qquad f \in \mathcal{C}(H).$$

proof (Really, more of a sketch since some details are left out.): For each character $\omega \in Z_u^\wedge$, consider the map $\mathcal{C}(H) \to \mathcal{C}_*(H, \omega)$ defined by $f \longmapsto f_\omega$, where

$$f_\omega(x) = \int_Z f(xz)\omega(z)^{-1} dz,$$

and where

$$\mathcal{C}_*(H, \omega) = \{f \in \mathcal{C}_*(H) \mid f(xz) = \omega(z) f(x), \quad x \in H,\ z \in Z\}.$$

We have $\mathcal{C}(H) = \int_{\omega \in Z_u^\wedge} \mathcal{C}_*(H, \omega)$, so D has a Fourier expansion

$$D(f) = \int_{\omega \in Z_u^\wedge} D_\omega(f_\omega) d\omega,$$

each D_ω being a distribution on $\mathcal{C}_*(H, \omega)$. Following Kazhdan, let $W(\omega)$ denote the space of all invariant tempered distributions on $\mathcal{C}_*(H, \omega)$ and, for

each compact open subgroup $K \subset\subset H$, let $\mathcal{C}_*(H,\omega)_K$ denote the space of all linear combinations of translates of bi-K-invariant functions in $\mathcal{C}_*(H//K,\omega)$, and let

$$\mathcal{C}_*(H,\omega)_K^{\perp} = \{f \in \mathcal{C}_*(H,\omega) \mid \int_K f(k_1kk_2)dk = 0, \quad k_1,k_2 \in K\}.$$

By analogy with [K, section 4] (where it is assumed that the center is compact), we have a direct sum decomposition $\mathcal{C}_*(H,\omega) = \mathcal{C}_*(H,\omega)_K \bigoplus \mathcal{C}_*(H,\omega)_K$ and a corresponding decomposition of the space of invariant distributions: $W(\omega) = W(\omega)_K \bigoplus W(\omega)_K^{\perp}$. We *claim* that $W(\omega) = \bigcup_K W(\omega)_K$, but we shall not prove this here. (Argue by contradiction ...) By analogy with [K, section 4, Lemma 5], there is a decomposition

$$D_\omega = \int_{G^\wedge_{c,\omega}} \Theta_\pi m_G(\pi) d\mu_\omega(\pi) + D^0_\omega,$$

where D^0_ω is supported off the elliptic set, the distribution m_G is a finite linear combination of Dirac delta functions, and the dual $G^\wedge_{c,\omega}$ is the set of equivalence classes of cuspidal representations with unitary central character ω. By the Weyl integration formula, D^0_ω must be supported on A^H where A denotes the diagonal subgroup of H. In the decomposition

$$D_\omega(f) = (D_\omega)_H(f) + (D_\omega)_A(F^A_f),$$

discussed in the Lemma of section 8.2 above (where B here denoted the upper triangular subgroup of H), we may identify $D^0_\omega(f)$ with $(D_\omega)_A(F^A_f)$. The distribution $(D_\omega)_A$ is (tempered and) admissible on A by (analogy with) the Lemma in section 8.2. By the Plancherel formula in the abelian case, $(D_\omega)_A$ may be represented by a distribution of finite type on the tempered dual of A:

$$D^0_\omega(f) = (D_\omega)_A(F^A_f) = \int_{A^\wedge_t} (F^A_f)^\wedge(\chi) m_{A,\omega}(\chi) d\mu_A(\chi).$$

On the other hand, by the character formulas for the principal series, $(F^A_f)^\wedge(\chi) = \Theta_{I_A(\chi)}(f)$, so

$$D^0_\omega(f) = (D_\omega)_A(F^A_f) = \int_{A^\wedge_t} \Theta_{I_A(\chi)}(f) m_A(\chi) d\mu_A(\chi).$$

Therefore,

$$D = \int_{\omega \in Z_u^\wedge} D_\omega = \int_{\omega \in Z_u^\wedge} \int_{G_{c,\omega}^\wedge} \Theta_\pi m_{G,\omega}(\pi) d\mu_\omega(\pi) + \int_{\omega \in Z_u^\wedge} \int_{A_t^\wedge} \Theta_{I_A(\chi)} m_{A,\omega}(\chi) d\mu_A(\chi),$$

as desired.

This finishes the proof/sketch.

9 Supertempered distributions

We close by discussing the supertempered distributions on $SL(2)$. The definition of a supercuspidal distribution requires some preliminaries which we present first.

9.1 The constant term

Let $P = MN$ denote the Levi decomposition and let $\overline{P} = M\overline{N}$ denote the Levi decomposition of the opposite parabolic. We shall consider the maps $\mathcal{A}(G) \to \mathcal{A}(M)$,defined by $f \longmapsto f_P$, ${}_w\mathcal{A}(G) \to {}_w\mathcal{A}(M)$, defined by $f \longmapsto_w f_P$. We say that a locally constant function h on G satisfies the **weak inequality** if

$$|h(x)| << \Xi(x)(1 + \sigma(x))^r,$$

for some $r > 0$. Here ${}_w\mathcal{A}(G)$ denotes the space of matrix coefficients satisfying the weak inequality.

For $t > 0$, define $A^+(t) = \{a \in A \mid |\alpha(a)| \geq t\}$, where $\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^2$, and let $A^+ = A^+(1)$. Let $S \subset G$ be a set such that $S \cdot A^+(t_0) \subset S$, for some $t_0 > 0$, and let $f : S \to V$ be a vector-valued function, where V is some complex vector space. We say that f is **negligible along** P, written $f \approx_P 0$, if for every compact subset $C \subset\subset G$ there is a $t \geq t_0$ such that $f(xa) = 0$ for all $x \in C \cap S,\ a \in A^+(t)$. Given two vector-valued functions $f_i : S \to V,\ i = 1, 2$, we write $f_1 \approx_P f_2$ if $f_1 - f_2 \approx_P 0$. Let V be a normed vector space with norm $||...||_V$. For a fixed $x \in S$ and $L \in V$, we write

$$f(xa) \to_P L,\ a \to_P \infty, \qquad or \qquad \lim_{a \to_P \infty} f(xa) = L,$$

if for every $\varepsilon > 0$ there is a $t_\varepsilon > 0$ such that $||f(xa) - L||_V < \varepsilon$.

For each $f \in \mathcal{A}(G)$, define $f_{(B)}(a) = \delta_B(a)^{1/2} f(a)$, $a \in A$.

Theorem (Harish-Chandra) *For each $f \in \mathcal{A}(G)$, there is a unique $f_B \in \mathcal{A}(A)$, such that $f_B \approx_P f_{(B)}$ with $S = A$.*

The function f_B in this theorem is called the **constant term of f along B**. The construction of the constant term is in [S, section 2.6].

Theorem (Harish-Chandra) *For each $f \in_w \mathcal{A}(G)$, there is a unique ${}_w f_B \in \mathcal{A}(A)$, such that $\lim_{a \to_P \infty} |{}_w f_B(aa') - f_{(B)}(aa')| = 0$ for all $a' \in A$.*

Following [S, section 1.10], we call a function f (right) **Hecke finite** if, for each compact open subgroup $K \subset\subset G$, the vector space spanned by $f * g$, with $g \in C_c(G//K)$ (i.e., g is compactly supported and bi-K-invariant), is finite dimensional. We call an invariant distribution D on G **Hecke finite** if, for each $f \in C_c^\infty(G)$, the function $D_f(x) := D(r(x^{-1})f)$ is right Hecke finite as a function of $x \in G$. Here r denotes right translation: $(r(x)f)(g) = f(gx^{-1})$. Examples of Hecke finite distributions are characters of admissible representations of finite length (this is implicit in [S, section 1.11]). In fact, a Hecke finite distribution is a virtual character ([J, section 3]), hence automatically admissible.

Lemma: *If D is Hecke finite then $D_f \in \mathcal{A}(G)$. If moreover D is tempered then $D_f \in {}_w\mathcal{A}(G)$.*

For each $f \in C_c^\infty(G)$, define $h_f \in C_c^\infty(M)$ by

$$h_f(m) = \delta_P(m)^{1/2} \int_N \int_{\overline{N}} f(\overline{n}mn)\delta_P(\overline{n})^{-1} d\overline{n}dn.$$

Proposition: (1) *Let D be a Hecke finite distribution. For each parabolic P, there is an admissible distribution D_P on M such that $(D_f)_P(1) = D_P(h_f)$, for all $f \in C_c^\infty(G)$.*

(2) *Let D be a Hecke finite tempered distribution. For each parabolic P, there is an admissible tempered distribution D_P on M such that ${}_w(D_f)_P(1) = D_P(h_f)$, for all $f \in C_c^\infty(G)$.*

We will prove this in [J]. The distribution D_P on M will be called the **constant term of D along P**. This is not to be confused with the D_M introduced in section 7 above.

9.2 Definition of a supertempered distribution

Following Harish-Chandra in the real case, we say that a Hecke finite admissible distribution D is **supertempered** if $D_P = 0$ for all proper parabolics P of G. Examples of supertempered distributions are characters of square-integrable representations (see [S, Corollary 4.4.7]). Let G_e denote the set of **elliptic elements**: elements which are not conjugate to an element in some proper parabolic subgroup of G. Other examples of supertempered distributions arise from the lemma below:

Lemma: *If D is a Hecke finite distribution and $supp(F_D) \subset G_e$ then D is supertempered and tempered.*

One proof of this, which is omitted, is to use the construction of the constant term to verify directly that $(D_f)_P(1) = 0$, for all proper parabolics P (see [J, section 2]). The temperedness is an immediate consequence of a result of Kazhdan [K, section 5, Lemma 3].

QUESTION If D is supertempered and $F_D|_{G_e} = 0$ then does it necessarily follow that $D = 0$?

QUESTION: Does supertempered imply tempered?

Lemma: *If D is a Hecke finite distribution on G then D is a virtual character.*

This is not especially hard to prove (see [J, section 3]).

Corollary: *If D is a supertempered distribution then D is a virtual character.*

9.3 An example

Let $\chi = sign_\tau$, for some $\tau = \varepsilon, \pi, \pi\varepsilon$, and let $\pi = T_\chi^+$, $\pi' = T_\chi^-$.

Proposition: *The distribution*

$$D = \Theta_\pi - \Theta_{\pi'}$$

is supertempered.

proof The character computation of section 4.2.2 above shows that D is supported in the elliptic set, so by the above lemma it must be supertempered. It is instructive to give a second proof directly from the definition.

Consider the matrix coefficients

$$f_1(x) = \int_g \Theta_\pi(g)h(gx^{-1})dg, \quad f_2(x) = \int_g \Theta_{\pi'}(g)h(gx^{-1})dg, \quad h \in C_c^\infty(G),$$

and let

$$f_{1,(B)}(a) = \delta_B(a)^{1/2} f_1(a) = \delta_B(a)^{1/2} \int_G \Theta_\pi(g) h(ga^{-1}) dg,$$
$$f_{2,(B)}(a) = \delta_B(a)^{1/2} f_2(a) = \delta_B(a)^{1/2} \int_G \Theta_{\pi'}(g) h(ga^{-1}) dg.$$

The character formulas above imply that

$$f_{1,(B)}(a) - f_{2,(B)}(a) = \delta_B(a)^{1/2} \int_G (\Theta_\pi(g) - \Theta_{\pi'}(g)) h(ga^{-1}) dg$$
$$= \delta_B(a)^{1/2} \sum_{T \neq A} \tfrac{1}{2} \int_T |D(t)|^{1/2} (\Theta_\pi(t) - \Theta_{\pi'}(t)) h(ta^{-1}) dt.$$

In fact, the sum over the non-split Cartans T runs only over one or two Cartans. To finish the proof of this proposition, we use the following

Lemma: *If h is supported on $C \subset\subset G$, there exists an $r = r_C > 0$ such that, for all Cartans T in the above sum and all $a \in A^+(r)$, we have $C \cap aT = \emptyset$.*

proof: To see this, note that there is an $N > 0$ such that $\sigma(x) < N$ for all $x \in C \cup \{T \mid T \neq A \textit{ occurs in the sum above}\}$. This is because C is compact and each T is compact. On the other hand, for each Cartan $T \neq A$, from the explicit description of T as a matrix group ([SS2]) and the definition of $\sigma(x)$, it follows that there is an $r > 0$ (depending on N, T) such that $\sigma(ta) > N$ for all $a \in A^+(r)$, $t \in T$. From this the lemma follows. (*lemma*)

From the lemma and the definition of negligible, it follows that $f_{1,(B)}(a) - f_{2,(B)}(a) \approx_B 0$ along A. This implies that $D_B = 0$. (*proposition*)

Combining a special case of a result in [H] along with the above corollary in section 9.2, we find

Theorem: (Herb) A tempered, supertempered distribution of $SL(2, F)$ is a finite linear combination of discrete series characters and the distributions in the previous proposition.

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