On the unitary dual of $\overline{SL(2,F)}$, F p-adic

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Abstract: This paper describes the genuine irreducible unitary representations of the n-fold metaplectic cover $\overline{SL(2,F)}$, F p-adic, with the exception of the reducible principal series. The image of the metaplectic Harish-Chandra transform is determined and then a transfer of smooth (resp., spherical) genuine functions on $\overline{SL(2,F)}$ with their analogs on $SL(2,F)$ is given such that their associated orbital integrals are equal. Finally, it is noted that the corresponding principal series representations on $SL(2,F)$ and on $\overline{SL(2,F)}$ satisfy a simple character relation.

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1 Introduction

Some of the results found in this paper:

- Using the theory of germs of orbital integrals on a metaplectic group [V], one can classify the image of the metaplectic Harish-Chandra transform (see Proposition 6 and Corollary 7). Moreover, it is not hard to describe a transfer (or "matching") of smooth (resp., spherical) genuine functions on $\overline{SL(2,F)}$ with their analogs on $SL(2,F)$ such that the associated orbital integrals are equal.

- Based on the work of Moen [Mo1], [Mo2], Ariturk [Ar], Kazhdan-Patterson [KP], and Flicker-Kazhdan [FK], one can describe the genuine irreducible unitary representations of the n-fold metaplectic cover $\overline{SL(2,F)}$, with the exception of the reducible principal series, where F is a p-adic field (see for example Propositions 19 and 24). In particular, some of the main results of Moen and Ariturk can be extended to the case where $n \geq 1$ is arbitrary. We describe the special representation of $\overline{SL(2,F)}$ in §4.6. The supercuspidal representations have already been described - see, for example, [J3] (also [M1], [M2], [J2]).

- Finally, assuming $gcd(p,n) = 1$, we note that

the corresponding principal series representations on $SL(2,F)$ and its n-fold cover (see Theorems 30 and 31) satisfy simple character relations of the usual sort. Under this correspondence, unitary (resp., spherical) representations match with unitary (resp., spherical) representations.

Y. Flicker [F] described a comparison of smooth functions on $G = GL(2,F)$ and an n-fold metaplectic cover $\overline{G}$ by matching orbital integrals. Flicker-Kazhdan [FK] (resp., the author [J1]) determined a comparison of smooth functions on $G = GL(n,F)$ and on $\overline{G}$ (resp., on Cartans T of $\overline{SL(n,F)}$ and on $\overline{SL(n,F)}$ itself) using the matching of orbital integrals. In each of these papers, the authors then obtained an analogous correspondence of certain representations using character theory. The idea here is similar but to instead compare orbital integrals, and principal series representations, on $G = SL(2,F)$ with those on $\overline{G}$.

These results, in the case $n = 2$, form a necessary step in what will hopefully be a proof of Labesse's multiplicity one conjecture for $SL(2,\mathbb{A})$ (in preparation).

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2 Background notation

Let F be a p-adic field of characteristic 0, with uniformizer π, ring of integers $\mathcal{O} = \mathcal{O}_F$, and let q denote the cardinality of the residue field. Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{Z}$ the ring of integers. Let $\mu_n(F)$ denote the group of nth roots of unity in F and assume $|\mu_n(F)| = n$. We will identify $\mu_n(F)$ with a subgroup of $\mathbb{C}^\times$ (via some fixed isomorphism $\theta : \mu_n(F) \to \mu_n(\mathbb{C})$). Write $\mu_n = \mu_n(F)$. Let $G = SL(2,F)$. Let $\overline{G} = \overline{SL(2,F)}$ denote the n-fold metaplectic cover defined by the cocycle

$$\beta_0(g_1,g_2) = (\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)}), \qquad (1)$$

for $g_1, g_2 \in G$, where

$$x(\begin{array}{cc} a & b \\ c & d \end{array}) = \begin{cases} c, & \text{if } c \neq 0, \\ d, & \text{otherwise,} \end{cases}$$

and where $(\ ,\ ) = (\ ,\ )_n$ is the nth power Hilbert symbol. By §1 of Kubota [K], β_0 is a 2-cocycle (called a "factor set" in [K]) which is trivial near the identity in $G \times G$, so $\overline{G}$ is a topological covering group of G. We denote elements of $\overline{G}$ by $\overline{x} = (x,\zeta)$, $x \in G$, $\zeta \in \mu_n$, and multiplication by

$$\overline{x}_1\overline{x}_2 = (x_1,\zeta_1)(x_2,\zeta_2) = (x_1x_2, \beta_0(x_1,x_2)\zeta_1\zeta_2).$$

Let $\rho : \overline{G} \to G$ denote the natural surjection. If H is any subset of G then we define $\overline{H} = \rho^{-1}(H)$. Let $K_0 = SL(2,\mathcal{O})$. If p is relatively prime to $2n$ then it is known (see [K]) that $\overline{K_0}$ splits, i.e., that there is an isomorphism $\overline{K_0} \cong K_0 \times \mu_n$, which we fix once and for all. In this case, we sometimes identify K_0 with a subgroup of $\overline{K_0}$ via this isomorphism.

We choose a Haar measure on G, to be normalized below, and a Haar measure on μ_n normalized so that $vol(\mu_n) = 1$. Let the Haar measure on $\overline{G}$ be the product measure.

For any subset $H \subset G$, let H_{reg} denote the collection of elements of H which have distinct eigenvalues (in some algebraic extension of F).

Let A_0 denote the diagonal subgroup of G, let $W = N_G(A_0)/A_0$ denote the Weyl group, represented by the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here N_G denotes the normalizer in G as usual.

Let N_0 denote the subgroup of G consisting of upper triangular unipotent matrices and let $B = A_0N_0 = N_0A_0$ denote the upper triangular (standard) Borel subgroup. It is easy to verify that $\overline{N_0}$ splits. We often identify N_0 with the subgroup $(N_0, 1)$ of $\overline{N_0}$.

Definition 1 *Recall that a complex representation of $\overline{G}$ on a vector space V, $\overline{\pi} : \overline{G} \to \mathrm{Aut}(V)$, is called* **genuine** *if $\overline{\pi}(g,\zeta) = \theta(\zeta)\overline{\pi}(g,1)$, for $g \in G$ and $\zeta \in \mu_n$.*

Recall that a complex-valued function on $\overline{G}$ $f : \overline{G} \to \mathbb{C}$, is called **genuine** *if $f(g,\zeta) = \theta(\zeta)^{-1} f(g,1)$, for $g \in G$ and $\zeta \in \mu_n$.*

We shall omit the use of θ and simply identify μ_n with $\mu_n(\mathbb{C})$.

Here we call an irreducible admissible represention of G (or $\overline{G}$ **supercuspidal** if all its matrix coefficients are compactly supported.

3 Orbital integrals

Let $C_c^\infty(\overline{G})_{gen}$ denote the algebra of compactly supported locally constant genuine (i.e., "smooth") functions on $\overline{G}$ and let $\mathcal{H}(\overline{G}//K_0)_{gen}$ denote the algebra of compactly supported locally constant genuine bi-K_0-invariant (i.e., "spherical") functions on $\overline{G}$. Let M denote a Levi subgroup of a parabolic subgroup $P = MN$, let

$$\Delta(\overline{m}) = \Delta_M(\overline{m}) = |\det(Ad_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})|,$$

and let

$$\delta(\overline{m}) = \delta_M(\overline{m}) = |\det\ Ad_{\mathfrak{n}}(m)|,$$

where $\overline{m} \in \overline{M}$, $\overline{m} = (m,\zeta)$, and where $\mathfrak{n}$ denotes the Lie algebra of N. If $m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and M is the diagonal subgroup of G then $\Delta(\overline{m}) = |1 - a^{-2}|$ and $\delta(\overline{m}) = |a^2|$.

Let $A = A_0$ from this point on, unless stated otherwise. Let $S^{\overline{A}}$ denote the orbital integral (restricted to a certain subset of $\overline{A}$),

$$\begin{aligned} S^{\overline{A}} f(\overline{m}) &= \Delta(m)\delta(m)^{-1/2} \textstyle\int_{\overline{G}/\overline{A}} f(g\overline{m}g^{-1})\ dg \\ &= \delta(m)^{1/2} \textstyle\int_N f(n\overline{m})\ dn, \end{aligned}$$

provided $Cent(\overline{m},\overline{G}) = \overline{A}$. We will show that S defines algebra homomorphisms

$$C_c^\infty(\overline{G})_{gen} \to C_c^\infty(\overline{A}^{n_0})_{gen}^W$$

and

$$\mathcal{H}(\overline{G}//K_0)_{gen} \to \mathcal{H}(\overline{A}^{n_0}//K_0 \cap A^{n_0})_{gen}^W.$$

Here, and throughout the rest of this paper, let

$$n_0 = n/gcd(n,2).$$

Moreover, the subalgebra of W-invariants,

$$C_c^\infty(\overline{A})_{gen}^W = \{f \in C_c^\infty(\overline{A}^{n_0})_{gen} \mid f((w,1)^{-1}a(w,1)) = f(a),\ \forall a \in \overline{A}^{n_0}$$

is well-defined and non-zero by a cocycle calculation (and similarly for $\mathcal{H}(\overline{A}^{n_0}//K_0 \cap A^{n_0})_{gen}^W$).

Define $D = D_{G/A}$ on A by

$$\begin{aligned} D_{G/A}(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}) &= \det(1 - Ad(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}))_{\mathfrak{g}/\mathfrak{a}} \\ &= (1-a^2)(1-a^{-2}) = -(a-a^{-1})^2, \end{aligned}$$

for $a \in F^\times$. Here $\mathfrak{g}, \mathfrak{a}$ denote the Lie algebras of G, A. This may be pulled back to $\overline{A}$ and we have $D(\overline{a}) = \Delta(\overline{a})\delta(\overline{a})^{-1/2}$, for regular $\overline{a} \in \overline{A}$. Define the **orbital integral of** $f \in C_c^\infty(\overline{G})_{gen}$ **on** $\overline{A}_{reg}$ by

$$F_f^{\overline{A^{n_0}}}(a) = |D(a)|^{1/2} \int_{\overline{A^{n_0}}\backslash\overline{G}} f(x^{-1}ax)\frac{dx}{da}, \qquad (2)$$

if $a \in \overline{A}$ is regular. The Haar measure da on $\overline{A^{n_0}}$ will be choosen later. Define the **orbital integral of** $f \in C_c^\infty(\overline{G})_{gen}$ **on** $\overline{A}_{reg}^{n_0}$ by

$$F_f^{\overline{A}}(a) = n_0|D(a)|^{1/2} \int_{\overline{A}\backslash\overline{G}} f(x^{-1}ax)\frac{dx}{da}, \qquad (3)$$

for $a \in \overline{A^{n_0}} \cap \overline{G}_{reg}$. Since $Cent(a,\overline{G}) = \overline{A}$, this is well-defined. Here da is the usual product measure on $\overline{A} = A \times \mu_n$. (The n_0 factor is to make Definitions 8 and (6) below consistent.)

We will show that $f \longmapsto F_f^{\overline{A^{n_0}}}$ defines a surjection

$$C_c^\infty(\overline{G})_{gen} \to C_c^\infty(\overline{A}^{n_0})_{gen}^W.$$

3.1 Dependence on the cocycle

Let β be a cocycle of G having values in μ_n and let $\overline{G} = \overline{G}_\beta$ denote the associated covering group. Note that the metaplectic algebras $C_c^\infty(\overline{G})_{gen}$ and $\mathcal{H}(\overline{G}//K_0)_{gen}$ depend on the cocycle β so let us denote this dependence temporarily by $C_c^\infty(\overline{G})_{gen,\beta}$ and

$\mathcal{H}(\overline{G}//K_0)_{gen,\beta}$, resp.. If the cocycle is changed to an equivalent one, say to β', where

$$\beta(g,h) = \beta'(g,h)s(g)s(h)s(gh)^{-1}$$

is the cocycle modified by s and $s : G \to \mu_n$ is any function satisfying $s(1) = 1$ (for example, the Kubota splitting [KP]), then the two algebras are isomorphic.

Lemma 2 *Let β, β' be as above, let R denote one of the algebras $C_c^\infty(\overline{G})_{gen}$ or $\mathcal{H}(\overline{G}//K_0)_{gen}$, and let $s : G \to \mu_n$ satisfy $s(1) = 1$. There is a canonical isomorphism*

$$\phi : R_\beta \cong R_{\beta'},$$

as algebras, defined by sending $f(g,\zeta) = \zeta f(g,1)$ to $\phi(f)(g,\zeta) = \zeta s(g) f(g,1)$. In other words, under this mapping, we have

$$\phi(f_1 *_{\beta'} f_2)(g,\zeta) = (\phi(f_1) *_\beta \phi(f_2))(g,\zeta),$$

where $*_\beta$ denotes the convolution with respect to the β cocycle

$$(\phi_1 *_\beta \phi_2)(g,\zeta) = \zeta \int_G \phi_1(x,1)\phi_2((x^{-1}g,1))\beta(x,x^{-1}g)^{-1}dx$$

and, similarly, $*_{\beta'}$ denotes the convolution with respect to the β' cocycle.

proof: The verification of this is straightforward. #

3.2 The Harish-Chandra transform of a smooth function

The verification of the following lemma, another cocycle calculation, is left to the reader.

Lemma 3 *The centralizer $C_{\overline{G}}(\overline{A})$ of $\overline{A}$ in $\overline{G}$ is $\overline{A}^{n_0}$, where $n_0 = n/gcd(2,n)$.*

Lemma 4 *(Weyl integration formula [S], p. 198) For each $f \in C_c^\infty(G)$*

$$\int_G f(g)dg = \sum_{T \in S} \frac{1}{|W_T|} \int_T |D(t)|^{1/2} F_f^T(t)dt,$$

where

$$F_f^T(t) = |D(t)|^{1/2} \int_{T\backslash G} f(x^{-1}tx)\frac{dx}{dt},$$

and where $D(t) = D_{G/T}(t) = \det(1 - Ad(t))_{\mathfrak{g}/\mathfrak{t}}$. Here $\mathfrak{g}$, $\mathfrak{t}$ denote the Lie algebras of G, T, resp..

The following result might be compared with Flicker's Lemma 1.1 in [F].

Lemma 5 *("Vanishing lemma") For $f \in C_c^\infty(\overline{G})_{gen}$, $a \in \overline{A}_{reg} - \overline{A^{n_0}}$, we have $F_f^{\overline{A^{n_0}}}(a) = 0$.*

proof: Since dx is a Haar measure, we may replace x by $a_1 x$, where $a_1 = (\begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, \varsigma_1) \in \overline{A}$ in the integral (2) defining $F_f^{\overline{A^{n_0}}}(a)$ without changing its value. Let $a = (\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \varsigma)$. Then

$$F_f^{\overline{A^{n_0}}}(a) = (t,t_1)_n^2 \cdot F_f^{\overline{A^{n_0}}}(a).$$

It follows that $F_f^{\overline{A^{n_0}}}(a) = 0$ or $(t,t_1)_n^2 = 1$. By the properties of the Hilbert symbol [N], if $(t,t_1)_n^2 = 1$ for all t_1 then $t \in (F^\times)^{n_0}$. #

Proposition 6 *The map $f \longmapsto F_f^{\overline{A^{n_0}}}$ defines a surjection $C_c^\infty(\overline{G})_{gen} \to C_c^\infty(\overline{A}^{n_0})_{gen}^W$.*

Remark 1

- *There's an analogous result for the Schwartz space $\mathcal{C}$ of "rapidly decreasing" functions: The map $f \longmapsto F_f^{\overline{A^{n_0}}}$ defines a continuous surjection $\mathcal{C}(\overline{G}) \to \mathcal{C}(\overline{A^{n_0}})^W$. Since it shall not be needed, we omit the proof.*
- *The map $f \longmapsto F_f^{\overline{A^{n_0}}}$ is a priori only defined as a map to functions on $\overline{A}_{reg}$. The proposition asserts (as we shall prove below) that this function extends uniquely to a smooth function on $\overline{A}$.*

proof: We shall merely sketch the idea.
Define the map Φ by $\Phi(f) = F_f^{\overline{A^{n_0}}}$.
We first verify surjectivity.
Let $H = L^2(\overline{A}^{n_0})^W$ and let V denote the closure in the L^2-topology of H of the image of $\Phi : f \longmapsto F_f^{\overline{A^{n_0}}}$ in H.

Claim: If Φ is not surjective then $V \neq H$.

The proof of claim is straightforward and left to the reader.

If $V \neq H$, we let

$$V^{\perp} = \{\phi \in H \mid \int_{\overline{A}^{n_0}} \phi(a)\overline{g(x)}dx = 0, \forall g \in V\}.$$

Claim: $H = V \oplus V^{\perp}$.

The proof of claim is straightforward and left to the reader.

Now we complete the proof of surjectivity. Let $\phi \in V^{\perp}$. We must show that $\phi = 0$. For any $F \in V$, we have

$$\int_{\overline{A}^{n_0}} \phi(a)F(a)da = 0.$$

Let

$$\tilde{\phi}(g) = \begin{cases} \phi(a), & \text{if } g^x = a \in \overline{A}^{n_0}_{reg} \text{ some } x \in \overline{G}, \\ 0, & \text{otherwise.} \end{cases}$$

Since ϕ is W-invariant, $\tilde{\phi}$ is well-defined. Pick $\epsilon > 0$ and approximate F by $F_f^{\overline{A^{n_0}}}$, for some $f \in C_c^{\infty}(\overline{G})$ such that

$$\int_{\overline{A}^{n_0}} |\phi(a)(F(a) - F_f^{\overline{A^{n_0}}}(a))|da < \epsilon. \qquad (4)$$

By Lemma 4 and (4), we have

$$|\int_{\overline{G}} \tilde{\phi}(g)f(g)dg| < \epsilon/2.$$

Since $f \in C_c^{\infty}(\overline{G})$ and $\epsilon > 0$ were choosen arbitrarily, we have that $\tilde{\phi} = 0$ a.e. on $\overline{G}$. Thus $\phi = 0$ a.e. on A. This proves that $V^{\perp} = 0$, hence $V = H$, hence Φ is surjective.

For $f \in C_c^{\infty}(\overline{G})_{gen}$ and $a \in \overline{A^{n_0}}_{reg} = \overline{A^{n_0}} \cap \overline{G}_{reg}$, the function $x \longmapsto f(x^{-1}ax)$ is compactly supported on $\overline{A}\backslash\overline{G}$ and

$$\begin{aligned} F_f^{\overline{A^{n_0}}}(a) &= |D(a)|^{1/2} \textstyle\int_{\overline{A}\backslash\overline{G}} f(x^{-1}ax)\frac{dx}{da} \\ &= \delta(a)^{1/2} \textstyle\int_N \int_{K_0} f(kank^{-1})dkdn \\ &= \delta(a)^{1/2} \textstyle\int_N f^{K_0}(an)dn, \end{aligned}$$

where

$$f^{K_0}(x) = \int_{K_0} f(kxk^{-1})dk.$$

From the claim it follows that $F_f^{\overline{A^{n_0}}}(a)$ is compactly supported on $\overline{A}_{reg}$.

Moreover, since f is locally constant so is f^{K_0}. Since f^{K_0} is compactly supported, $\int_N f^{K_0}(an)dn$ is a locally constant function on $\overline{A}_{reg}$.

It remains to show that $F_f^{\overline{A^{n_0}}}(a)$ extends to a locally constant function on all of $\overline{A}$. This follows from the Shalika germ expansion for covering groups [V]. #

The Proposition above implies the following result.

Corollary 7 *Assume that the cocycle β is trivial on K_0. The map S defines an algebra homomorphism*

$$\mathcal{H}(\overline{G}//K_0)_{gen} \to \mathcal{H}(\overline{A}^{n_0}//K_0 \cap A^{n_0})^W_{gen}.$$

Definition 8 *Define $N : A \to \overline{A}$ by $N(a) = (a^{n_0}, 1)$. We say that $\overline{f} \in C_c^{\infty}(\overline{G})_{gen}$* **transfers**[1] *to $f \in C_c^{\infty}(G)$ if they satisfy*

$$F_{\overline{f}}^{\overline{A}}(N(a)) = F_f^A(a),$$

for all $a \in A$ such that $N(a) \in \overline{A}_{reg}$, where the measures have been normalized as above.

A similar definition, for $\overline{f} \in \mathcal{H}(\overline{G}//K_0)_{gen}$ and $f \in \mathcal{H}(G//K_0)$, is left to the reader.

4 Representations

The goal of this section is to describe the unitary dual of $\overline{G}$. The general idea in the case of the non-algebraic group $\overline{G}$ is basically the same as for the group G: using Jacquet functors, one may show that an irreducible admissible representation is either a constituent of some representation induced from (a supercuspidal representation of a Levi component of) a proper parabolic subgroup or else is supercuspidal. As indicted above, the supercuspidal representations have been described elsewhere. To capture those irreducible representations which are unitary, one using results in [FK] (for $\overline{GL(2,F)}$), [Mo1], [Mo2], and [Ar].

[1] In other words, the Harish-Chandra transform of f corresponds with that of $\overline{f}$.

The only cases not falling into this net are the non-tempered representations and, if any, the reducible principal series.

For a reductive p-adic group H, let $\Pi(H)$ denote the set of equivalence classes of irreducible admissible representations of H and let $\Pi_u(H)$ denote the unitary ones. If $H \subset G$ is a subgroup, let $\Pi_{gen}(\overline{H})$ denote the set of equivalence classes of irreducible admissible genuine representations $\overline{H}$.

Let $M = A_0$ and $N = N_0$. Each $(\sigma, W) \in \Pi(\overline{M})$ expends trivially to a representation of $\overline{B} = \overline{M}N$. Let $I_{\overline{M}}(\sigma) : \overline{G} \to Aut(V)$ denote the **unitarily induced representation**: the representation of $\overline{G}$ by right translation on

$$\begin{aligned} V \;=\; \{ & f : \overline{G} \to W \text{ smooth, genuine} \mid \\ & (1)\ f(bg) = \delta_M(b)^{1/2}\sigma(b)f(g), \quad \forall g \in \overline{G},\ b \in \overline{B} \\ & (2) \text{ for some compact open subgroup } K \subset \overline{G}, \\ & \qquad f(gk) = f(g),\ \forall k \in K,\ g \in \overline{G}\ \}. \end{aligned}$$

Of course, if σ is genuine, so is $I_{\overline{M}}(\sigma)$.

4.1 Maximal abelian subgroups of $\overline{A}$

For the purposes of explicitly computing the representations of $\overline{A}$, we want to determine, as best as we can, a maximal abelian subgroup of the n-fold cover of $\overline{A}$. This turns out to be hard if $p|n$. Since the (incomplete) results in the case $p|n$ will not needed for our final results, we merely state the result in the case p and n are relatively prime.

Let

$$h(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \qquad x \in F^\times.$$

Lemma 9 *A subgroup $\overline{C} \subset \overline{A}$ is abelian if and only if $(a, a')_n^2 = 1$ for all $h(a), h(a') \in C$.*

The (easy) proof of this is omitted.

For any p-adic field, we have

$$F^\times = \pi^{\mathbb{Z}} \cdot \mu_{q-1} \cdot U_1,$$

a direct product (as abelian groups - see for example [N], Proposition 1.1). Here

$$U_k = \{x \in \mathcal{O}_F \mid x \equiv 1 \pmod{\pi^k}\}.$$

If $gcd(p, 2n) = 1$ then $\mu_n \subset F^\times$ implies $q \equiv 1 \pmod{n}$.

Lemma 10 *(a) If $gcd(p, n_0) = 1$ then $C = h(\pi^{\mathbb{Z}}\mathcal{O}_F^{\times n_0}) = h(\pi^{\mathbb{Z}}\mu_{q-1}^{n_0}(1 + \pi\mathcal{O}_F))$ and $C = h(\pi^{n_0\mathbb{Z}}\mathcal{O}_F^\times)$ are maximal subgroups of $F^\times$ for which $\overline{C} \subset \overline{A}$ is abelian.*

(b) If $gcd(p, n) = 1$ then $C = h(\pi^{\mathbb{Z}}(1 + \pi\mathcal{O}_F)\mu_{q-1}^{n_0})$ (resp. $C = h(\pi^{n_0\mathbb{Z}}\mathcal{O}_F^\times)$) has index n_0 in $F^\times$.

Remark 2 *If $n = 1$ or $n = 2$ then $n_0 = 1$ and $C = A$.*

The proofs are omitted.

4.2 Irreducible representations of $\overline{A}$

We assume $gcd(p, 2n) = 1$ throughout this subsection unless otherwise stated.

Let $C = h(\pi^{n_0\mathbb{Z}}\mathcal{O}_F^\times)$ and $A_n = h(\pi^{n\mathbb{Z}}\mathcal{O}_F^\times)$. We know that $\overline{A_n}$ splits, so we may identify $\overline{A_n}$ with $A_n \times \mu_n$. Moreover, Clifford-Mackey theory implies that

$$\begin{array}{ccc} \Pi(\overline{C})_{gen} & \to & \Pi(\overline{A})_{gen} \\ \chi & \longmapsto & Ind_{\overline{C}}^{\overline{A}}(\chi) \end{array}$$

is a surjection [Mo1]. The conjugates χ^g, for $g \in \overline{A}/\overline{C}$, are distinct, and A/C is order n_0, so the dimension of each $Ind_{\overline{C}}^{\overline{A}}(\chi)$ is n_0.

Lemma 11 *([Mo2], §1.2) There is a 2-1 correspondence*

$$\Pi(A_n) \to \Pi(\overline{A})_{gen}$$

given by $\eta_j : \Pi(A_n) \to \Pi(\overline{C})_{gen}$, $\iota : \Pi(\overline{C})_{gen} \to \Pi(\overline{A})_{gen}$, where

$$\iota(\chi) = Ind_{\overline{C}}^{\overline{A}}(\chi),$$

and

$$\eta_j(\chi)(a, \zeta) = \begin{cases} \zeta\chi(a), & a \in A_n, \\ \zeta(x, \pi)_n^{n_0}(\pi, \pi)_n^{n_0}\alpha_j\chi(a), & a = h(x) \in C - A_n \end{cases},$$

where α_1, α_2 are the two roots of $\alpha^2 = (\pi, \pi)_n^{n_0^2}$.

Lemma 12 *Let $\mu \in \Pi(\overline{C})_{gen}$ and let $\chi = \chi_\mu = Ind_{\overline{C}}^{\overline{A}}\mu$. Then*

$$\begin{aligned} \operatorname{tr}\chi(a) &= \textstyle\sum_{h\in\overline{A}/\overline{C}} \mu^h(a) \\ &= \begin{cases} n_0\chi(a), & a \in \overline{A^{n_0}}, \\ 0 & a \in \overline{A} - \overline{A^{n_0}}. \end{cases} \end{aligned} \tag{5}$$

proof: The first part of this lemma is a special case of the more general Frobenius formula for the trace of an induced representation. The second part is true, by the orthogonality of characters, since the character $x \longmapsto (a,x)_n^2$ is non-trivial if and only if $a \in F^\times - F^{\times n_0}$. #

4.2.1 Corresponding functions on $\overline{A}_{reg}$

Define the Fourier transform on $\overline{A}$ by

$$\phi^\wedge(\chi) = \int_{\overline{A}} \operatorname{tr}\chi(x)\phi(x)dx \qquad \chi \in \Pi(\overline{A}),\ \phi \in C_c^\infty(\overline{A}).$$

If H is an abelian subgroup of G or of $\overline{G}$ then we define the **Fourier transform** by

$$\phi^\wedge(\mu) = \int_H \mu(x)\phi(x)dx \qquad \mu \in \Pi(H),\ \phi \in C_c^\infty(H).$$

We investigate the conditions on $\phi \in C_c^\infty(A)$, $\overline{\phi} \in C_c^\infty(\overline{A})_{gen}$ under which

$$\phi^\wedge(\chi) = \overline{\phi}^\wedge(\overline{\chi}),$$

if $\chi \in \Pi(A)$ and $\overline{\chi}$ are corresponding representations (in a sense made precise below).

Definition 13 *We say that $\phi \in C_c^\infty(A)$* **corresponds** *to $\overline{\phi} \in C_c^\infty(\overline{A})_{gen}$ if*

$$\phi^\wedge(\chi) = \overline{\phi}^\wedge(\overline{\chi}),$$

for all $\chi \in \Pi(A_n)$, regarded as an element of $\Pi(A)$ trivial on A/A_n, where $\overline{\chi} = \iota(\eta_j(\chi))$, for some $j = 1,2$.

Definition 14 *We say that $f \in C_c^\infty(G)$* **corresponds on the hyperbolic set** *to $\overline{f} \in C_c^\infty(\overline{G})_{gen}$ if $F^A \in C_c^\infty(A)$ corresponds to $F_{\overline{f}}^{\overline{A}} \in C_c^\infty(\overline{A}^{n_0})_{gen}$ in the sense above.*

4.3 Jacquet functors

It is remarked in [BD], §2.2 that the arguments of [BZ1], chapter 2 carry over to finite central extensions of reductive groups over a p-adic field (see also [KP], §1.2). The arguments of [BZ2], section 2 and the corresponding sections of [Ca] also carry over to finite central extensions of split reductive groups over a p-adic field. Such results reduce the determination of the unitary dual of $\overline{G}$ down to classifying the supercuspidal representations (done in [M1], [M2], [J3] when $gcd(p,n) = 1$ and [J2] for any p,n) and the constituents of the induced representations (see below).

If (π, V) denotes a $\overline{G}$-module and if $P = MN$ denotes a standard parabolic of G, let $V_{\overline{N}}$ denote the Jacquet module

$$V_{\overline{N}} = V/\{\pi(n)v - v \mid v \in V,\ n \in \overline{N}\}.$$

This sends admissible representations to admissible representations (this is due to Jacquet, see [BZ1], sections 3.16-3.17 for the $GL(r)$ case).

Lemma 15 *(Jacquet [Ca], Theorem 5.2.1) If $V_{\overline{N}} = 0$ for all proper standard parabolics P of G then (π, V) is supercuspidal.*

Let $\mathcal{L}(G)$ denote the set of all standard Levi's (with respect to A), up to associates. Therefore, $\mathcal{L}(G) = \{A, G\}$.

Proposition 16 *(Jacquet [BZ1], section 3.19) If $\pi \in \Pi(\overline{G})$ then there is a Levi $M \in \mathcal{L}(G)$ and a supercuspidal $\sigma \in \Pi(\overline{M})$ such that π is a constituent of $I_{\overline{M}}(\sigma)$.*

In particular, every $\pi \in \Pi(\overline{G})$ which is not a constituent of an induced representation of a supercuspidal representation of a proper Levi occurs discretely in the decomposition of the right regular representation of $\overline{G}$ on $L^2(\overline{G})$.

4.4 Principal series

We can investigate whether or not two induced representations are equivalent by using the Jacquet functor.

Let $\chi, \chi' \in \Pi(\overline{A})$ and let $\overline{w} = (w, 1)$, for $w \in W$. If $\chi^{\overline{w}} \neq \chi$ for all $w \in W - \{1\}$ then we call χ **regular**. We say that χ, χ' are W**-conjugate** if $\chi' = \chi^{\overline{w}}$ for some $w \in W$. The following two results were proven for metaplectic covers of $GL(r, F)$ in [KP], however they hold in the present case as well.

Lemma 17 *([BZ2], Corollary 2.13, Theorem 2.9(b))*

(a) Let $\chi \in \Pi(\overline{A})$. The Jordan-Holder series of $I_{\overline{A}}(\chi)_{\overline{N}_0}$ has as its composition factors

$$Ind_{\overline{C}}^{\overline{A}}(\chi^{\overline{w}} \cdot \delta_A^{1/2}), \quad w \in W.$$

If χ is regular then

$$I_{\overline{A}}(\chi)_{\overline{N}_0} = \oplus_{w \in W} Ind_{\overline{C}}^{\overline{A}}(\chi^{\overline{w}} \cdot \delta_A^{1/2}).$$

(b) Let $\chi, \chi' \in \Pi(\overline{A})$. If χ is regular then

$$dim\ Hom_{\overline{G}}(I_{\overline{A}}(\chi), I_{\overline{A}}(\chi')) \leq 1,$$

with equality if and only if χ', χ are W-conjugate.

In other words, distinct W-conjugacy classes of $\chi \in \Pi(\overline{A})$ yield inequivalent representations.

Suppose that $\pi \in \Pi_u(\overline{G})$. We call π a (unitary) **principal series** representation if $\pi = I_{\overline{A}}(\chi)$ for some $\chi \in \Pi_u(\overline{A})$. These representations are tempered. In case $I_{\overline{A}}(\chi)$ is reducible and $\chi \in \Pi_u(\overline{A})$, we call the irreducible constituents **reducible principal series** (or, more precisely, **reducible principal series constituents**).

Let $\chi \in \Pi(\overline{A})$. The induced representation $I_{\overline{A}}(\chi)$ is in general not irreducible. However, we do have the following

Proposition 18 *(Moen [Mo2])*

(a) If n is even and $gcd(p, n) = 1$ then $I_{\overline{A}}(\chi)$ is irreducible and unitary for all $\chi \in \Pi_u(\overline{A})$.

(b) If n is odd and $gcd(p, n) = 1$ then $I_{\overline{A}}(\chi)$ is irreducible and unitary for all $\chi \in \Pi_u(\overline{A})$ such that (a) $\chi = 1$ or (b) $\chi^{\overline{w_0}} \neq \chi$ where $\overline{w_0} = (w_0, 1)$. If $\chi^{\overline{w_0}} = \chi$ and $\chi \neq 1$ then $I_{\overline{A}}(\chi)$ is reducible and has two irreducible constituents.

In fact, C. Moen [Mo1] explicitly computes the intertwining operators as matrices using the Kirillov model when $gcd(p, n) = 1$.

Proposition 19 *$I_{\overline{A}}(\chi)$ is irreducible and unitary for all $\chi \in \Pi_u(\overline{A})$ such that $\chi^{\overline{w_0}} \neq \chi$ where $\overline{w_0} = (w_0, 1)$.*

The above result has a direct proof, based on Bruhat theory, but the following argument deduces the this result from results in [FK].

proof: The "irreducibility if and only if $\chi^{\overline{w_0}} \neq \chi$" follows from the above Lemma 17(b).

We claim that each representation $I_{\overline{A}}(\mu)$ as in the proposition extends to a representation $I'_{\overline{A}}(\mu)$ of $\overline{GL(2, F)}$ (where $\overline{GL(2, F)}$ is defined by any cocycle of $GL(2, F)$ extending that of $\overline{SL(2, F)}$ used above). This claim follows from "Clifford-Mackey theory" and its proof is omitted (but see Schultz [Sch] and Gelbart and Piatetski-Shapiro [GPS] for a special case).

We claim that each such representation $I'_{\overline{A}}(\mu)$ of $\overline{GL(2, F)}$ is unitary. This claim follows from [FK], §§26-27 (which does not assume that p and n are relatively prime). Let V_μ denote the vector space (having a positive-definite Hermitian inner product) on which the representation $I'_{\overline{A}}(\mu)$ of $\overline{GL(2, F)}$ acts.

It follows from these claims that $I_{\overline{A}}(\mu)$ acts on a vector space (namely, V_μ) having a positive-definite Hermitian inner product on it. #

4.5 Complementary series

In this subsection, we shall briefly review some of the results of Ariturk [Ar] and use some results of Flicker and Kazhdan [FK] to generalize them to the n-fold cover ([Ar] assumed $n = 3$ and $p > 3$).

We call an irreducible unitary representation π a **complementary series** representation if $\pi = I_{\overline{A}}(\chi)$ for some $\chi \in \Pi(\overline{A}) - \Pi_u(\overline{A})$. These representations are not tempered.

Let $\mu \in \Pi(\overline{C})$, $\chi = \chi_\mu = Ind_{\overline{C}}^{\overline{A}}\mu \in \Pi(\overline{A})$. If $\mu(x) = \mu_0(x)|x|^s$, for some character μ_0 of finite order and some $s \in \mathbb{C}$ then we write $s = s(\mu) = s(\chi)$.

Let $K(\mu)$ denote the space of locally constant functions $f : F \times \overline{A} \to \mathbb{C}$ such that

(i) $f(x, a_1a_2) = \mu(a_1)f(x, a_2)$, $a_1 \in \overline{C}$, $a_2 \in \overline{A}$,

(ii) $|x|\chi(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, 1)f(x, a)$ is constant for $|x|$ large.

Let $R \subset \overline{A}$ denote a complete set of representatives of $\overline{A}/\overline{C}$, and let r denote the cardinality of R. The elements $f \in K(\mu)$ may be identified with the r-tuple $(f(x, a))_{a \in R}$.

Let $V(\mu)$ denote the space of all locally constant functions $\varphi : \overline{G} \times \overline{A} \to \mathbb{C}$ such that

(i) $\varphi(g, a_1a_2) = \mu(a_1)\varphi(g, a_2)$, $a_1 \in \overline{C}$, $a_2 \in \overline{A}$,

(ii) $\varphi(a_1ng, a_2) = \delta(a_1)\varphi(g, a_2a_1)$, where $a_1 \in \overline{A}$, $a_2 \in \overline{A}$, $n \in N_0$. Here δ denotes the usual modulus function (extended to $\overline{A}$ via the obvious pull-back).

For $\varphi \in V(\mu)$, define the map T by

$$T\varphi(g, a) = \int_F \varphi(w \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot g, waw^{-1})dx, \quad Re(s(\mu)) > 0 .$$

Lemma 20 *(Ariturk) T intertwines $I_{\overline{A}}(\mu)$ and $I_{\overline{A}}(\mu^w)$.*

This lemma does not require us to assume $gcd(p, n) = 1$.

Let $L(\overline{G}, \overline{B})$ denote the space of all locally constant functions φ on $\overline{G}$ such that

$$\varphi((\begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix}, \varsigma) \cdot g) = |a|^2\varphi(g) .$$

For $\varphi_1 \in V(\mu)$, $\varphi_2 \in V(\mu^w)$, the function

$$g \longmapsto \int_{\overline{A}/\overline{C}} \varphi_1(g, a)\varphi_2(g, a)da$$

belongs to $L(\overline{G}, \overline{B})$. Therefore,

$$\begin{aligned} < \varphi_1, \varphi_2 > &= \int_{\overline{B}\backslash\overline{G}} \int_{\overline{A}/\overline{C}} \varphi_1(g, a)\varphi_2(g, a)dadg \\ &= \int_F \int_{\overline{A}/\overline{C}} \varphi_1(w^{-1} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a)\varphi_2(w^{-1} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a)dad^\times x \end{aligned}$$

gives a non-degenerate bilinear form on $V(\mu) \times V(\mu^{\overline{w}})$.

Lemma 21 *(Ariturk) $I_{\overline{A}}(\mu^{\overline{w}})$ is the contragredient of $I_{\overline{A}}(\mu)$.*

This lemma does not require us to assume $gcd(p, n) = 1$.

For $f \in K(\mu)$, define the Fourier transform of f by

$$f^\wedge(x, a) = \int_{F^\times} f(y, a)\psi(xy)dy,$$

where ψ is a fixed additive character of F.

Lemma 22 *(Ariturk) Assume $gcd(p, n) = 1$. For φ_1, $\varphi_2 \in V(\mu)$, $\mu(x) = |x|^s$, we have*

$$< \varphi_1, T\varphi_2 > = \int_F \int_{\overline{A}/\overline{C}} f_1^\wedge(x, a)\overline{(Jf_2^\wedge)(x, a)}dadx,$$

where $J = J_\mu$ is a linear transformation on $K(\mu)^\wedge$ and

$$f_i(x, a) = \varphi_i(w^{-1} \cdot \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, a), \quad i = 1, 2.$$

We may identify the map $J = J_\mu$ defined in the above lemma with an $r \times r$ matrix which we still denote by J.

Lemma 23 *(Langlands, Ariturk) Assume $gcd(p, 2n) = 1$. If $0 \leq Re(s(\mu)) \leq 1/n$ and $|Im(s(\mu))| \leq \pi/n \ln(q)$ then the image of J_μ is an irreducible representation of $\overline{G}$.*

Proposition 24 *(Flicker-Kazhdan) If $0 < s(\mu) < 1/n$ then $I_{\overline{A}}(\mu)$ is a unitarizable representation of $\overline{G}$.*

proof: Each such representation $I_{\overline{A}}(\mu)$ extends to a representation $I'_{\overline{A}}(\mu)$ of $\overline{GL(2, F)}$ (where $\overline{GL(2, F)}$ is defined by any cocycle of $GL(2, F)$ extending that of $\overline{SL(2, F)}$ used above). We claim that each such representation $I'_{\overline{A}}(\mu)$ of $\overline{GL(2, F)}$ is unitary. This claim follows from [FK], §§26-27 (which does not assume that p and n are relatively prime). Let V_μ denote the vector space (having a positive-definite Hermitian inner product) on which the representation $I'_{\overline{A}}(\mu)$ of $\overline{GL(2, F)}$ acts.

It follows from these claims that $I_{\overline{A}}(\mu)$ acts on a vector space (namely, V_μ) having a positive-definite Hermitian inner product on it. #

Corollary 25 *If $0 < s < 1/n$ and $\mu(x) = |x|^s$ then $< \varphi_1, T\varphi_2 >$ is a positive definite form. In particular, $I_{\overline{A}}(\mu)$ is unitary in this range. If, in addition, $gcd(p, 2n) = 1$ then J_μ is a positive definite matrix.*

Remark 3 *In the case $n = 2$, this follows from [GS], Proposition 1 following the argument of [Ar]. See also [Mo1] if $p > 2$ and [G], §5.4. In the case $n = 3$, $p > 3$, this was proven in [Ar].*

Though this corollary is not needed for the character relations in §5, it does indicate that some of the results of Ariturk [Ar] generalize to the n-fold cover case without any condition on the prime p.

proof: First, recall the following form due to Jacquet, but see [C], page 118) of Schur's lemma holds: If (π, V) is an irreducible smooth representation of $\overline{G}$ then every intertwining map $T : V \to V$ is a scalar.

Now we claim: If (π, V) is an irreducible smooth representation of $\overline{G}$ then every there is at most one $\overline{G}$-invariant inner product on V. This is a well-known (and not hard to prove) consequence of Schur's lemma.

The previous proposition says that the representation space of $I_{\overline{A}}(\mu)$ has a $\overline{G}$-invariant, positive-definite, Hermitian inner product. It follows $< \varphi_1, T\varphi_2 >$ is positive-definite. This implies J_μ is also positive-definite, as claimed $0 < s < 1/n$ and $\mu(x) = |x|^s$. #

4.6 The special and the "trash" representations

For the origin of the term "trash" representation, see [G].

In the notation above, we have the following

Proposition 26 *Let $s = 1/n$ and $\mu(x) = |x|^s$.*

(a) The irreducible subrepresentation of $I_{\overline{A}}(\mu)$ (if $gcd(p, 2n) = 1$, the kernel of J_μ) is "the special representation" π_{sp}. It is tempered and square-integrable (hence unitary). If $gcd(p, n) = 1$ then it also contains an Iwahori fixed vector.

(b) If $n > 1$ then the irreducible quotient of $I_{\overline{A}}(\mu)$ (if $gcd(p, 2n) = 1$, image of J_μ) is an infinite-dimensional, non-tempered representation π_{nt}. If $gcd(p, n) = 1$ then it is also spherical.

Remark 4 *In the case $n = 3$ and $p > 3$, this proposition follows from [Ar]. In case $n = 2$, most of the statements are proven in [GS].*

proof: These statements follow easily from results of Flicker [F] and Flicker-Kazhdan [FK], using an inversion formula relating the character of a representation of $\overline{G}$ to values of the character of $\overline{GL(2, F)}$ which it extends to (see page 39 of [Sch]). #

Proposition 27 *(Kazhdan-Patterson) π_{nt} is unitary.*

Remark 5 *This is true if $n = 2$ ([GS], Theorem 2) or if $n = 3$, $p \neq 3$ ([Ar], Theorem 5.4).*

proof[2]: This follows from Kazhdan-Patterson [KP], Theorem II.2.1. #

5 Character relations

In this section, we use the above results to determine a "natural" correspondence between the (unramified) principal series of $\overline{G}$ and (unramified) principal series of G. We assume throughout this section that $gcd(p, 2n) = 1$, since we shall require some of the results in §4.1 above.

We show in this section that if

- the image of $\overline{f}$ under the Harish-Chandra transform
$$C_c^\infty(\overline{G})_{gen} \to C_c^\infty(\overline{A}^{n_0})^W_{gen}$$
"matches up with" (in a way we will make explicit below) the image of f under the Harish-Chandra transform
$$C_c^\infty(G) \to C_c^\infty(A)^W,$$
and

[2] I thank the referee for this proof.

- the principal series representation $\overline{\pi} = I_{\overline{A}}(\chi)$ "corresponds with" (in a way we will make explicit below) the principal series representation $\pi = I_A(\mu)$,

then we have a character relation of the form

$$\operatorname{tr}\pi(f) = \operatorname{tr}\overline{\pi}(\overline{f}).$$

We also give an analog of this result for spherical functions.

5.1 Character calculations for principal series

Let A denote the diagonal subgroup of G. Recall $W = N_G(A)/A$ denotes the Weyl group of A. If $x \in \overline{G}$ is regular then let $\overline{G}_t = Cent(t, \overline{G})$.

Lemma 28 *Let h denote an "anti-genuine" class function on $\overline{G}$ and suppose that h satisfies the property that, for all $f \in C_c^\infty(\overline{G})_{gen}$, we have*

$$\int_{\overline{G}} h(g)f(g)dg = c\int_{\overline{A}} \Delta(a)^{1/2}h(a)\int_{\overline{G}_a\backslash\overline{G}} f(g^{-1}ag)\frac{dg}{da}da$$

for some constant c. Then the following holds:

(1) $c = \frac{1}{|W_A|}$,

(2) if $g \in \overline{G}$ is regular then $h(g) = 0$ unless g is conjugate to an element of $\overline{A}$.

proof: First, note that the integral $\int_{\overline{G}_a\backslash\overline{G}} f(g^{-1}ag)\frac{dg}{da}da$ is zero unless $a \in \overline{A^{n_0}}$, in which case it is equal to $\int_{\overline{A}\backslash\overline{G}} f(g^{-1}ag)\frac{dg}{da}da$. Lemma 28 above is now a corollary of the non-metaplectic version, which is equivalent to Lemma 4. #

Proposition 29 *Assume $gcd(p, 2n) = 1$. Assume $\chi = \chi_\mu = Ind_{\overline{C}}^{\overline{A}}\mu \in \Pi(\overline{A})$ and suppose that $\chi \neq \chi^w$. The character of the irreducible representation $\pi = I_{\overline{A}}(\chi)$ is given on the regular set by*

$$\Theta_\pi(x) = \begin{cases} \frac{\operatorname{tr}\chi(a)+\operatorname{tr}(\chi(a)^{-1})}{|a-a^{-1}|}, & x = a^g,\ g \in \overline{G},\ a \in \overline{A}^{n_0}_{reg}, \\ 0, & \text{if } x \in \overline{G}_{reg} - (\overline{A}^{n_0})^{\overline{G}}, \end{cases}$$

where

$$(\overline{A}^{n_0})^{\overline{G}} = \{a^g \mid a \in \overline{A}^{n_0},\ g \in \overline{G}\}.$$

Moreover, $\operatorname{tr}\chi(a) = 0$ unless $a \in \overline{A}^{n_0}$.

proof: The proof, which follows Theorem 5.14 in [G], is omitted. #

Proposition 29, Lemma 12, and Lemma 5 imply that, for $f \in C_c^\infty(\overline{G})_{gen}$,

$$\begin{aligned} \operatorname{tr}\pi(f) &= \textstyle\int_{\overline{G}} \Theta_\pi(x)f(x)dx \\ &= \tfrac{1}{2}\textstyle\int_{A^{n_0}} (\operatorname{tr}\chi(a,1) + \operatorname{tr}(\chi(a,1)^{-1}))F_f^{\overline{A}}(a,1)da \\ &= \tfrac{1}{2}\textstyle\sum_{h\in\overline{A}/\overline{C}} (F_f^{\overline{A}})^\wedge(\mu^h) + (F_f^{\overline{A}})^\wedge((\mu^{\overline{w_0}})^h) \\ &= \tfrac{1}{2}|A/C|[(F_f^{\overline{A}})^\wedge(\mu) + (F_f^{\overline{A}})^\wedge(\mu^{\overline{w_0}})]. \end{aligned} \tag{6}$$

5.2 Matching principal series characters

Note that since $F_f^{\overline{A}}$ (for $f \in C_c^\infty(\overline{G})_{gen}$) vanishes off $\overline{A}^{n_0}$, the last Fourier transform in (6), *a priori* an integral over $\overline{C}$, is in fact an integral over $\overline{A^{n_0}}$. This, with Definition 8, proves the following consequence of the above proposition.

Theorem 30 *Assume $gcd(p, 2n) = 1$. If $\chi = \chi_\mu = Ind_{\overline{C}}^{\overline{A}}\mu \in \Pi(\overline{A})$, $\chi \neq \chi^{\overline{w_0}}$, and if $\overline{\pi} = I_{\overline{A}}(\chi)$ then the character of $\overline{\pi}$ is related to that of $\pi = I_A(\mu^{n_0})$ by*

$$\operatorname{tr}\overline{\pi}(\overline{f}) = \operatorname{tr}\pi(f),$$

provided $\overline{f} \in C_c^\infty(\overline{G})_{gen}$ transfers to $f \in C_c^\infty(G)$.

We now give an analogous result for the spherical functions and the unramified principal series.

Theorem 31 *Assume $gcd(p, 2n) = 1$. Assume that the cocycle β defining $\overline{G}$ is trivial on K_0. If $\chi = \chi_\mu = Ind_{\overline{A}^{n_0}}^{\overline{A}}\mu \in \Pi(\overline{A})$ is unramified, $\chi \neq \chi^{\overline{w_0}}$, and if $\overline{\pi} = I_{\overline{A}}(\chi)$ then the character of $\overline{\pi}$ is related to that of $\pi = I_A(\mu^{n_0})$ by*

$$\operatorname{tr}\overline{\pi}(\overline{f}) = \operatorname{tr}\pi(f),$$

provided $\overline{f} \in \mathcal{H}(\overline{G}//K_0)_{gen}$ transfers to $f \in \mathcal{H}(G//K_0)$.

proof: Since $I_{\overline{A}}(\chi_\mu)$ is unramified if and only if μ is, this is a corollary of the previous theorem. #

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