Notes on splitting 2-fold covers of some "odd" unitary groups over a p-adic field

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11-29-95

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Abstract

This paper collects some remarks relating to the papers [Mo], [Ka], [Ku], and [Li] for 2-fold metaplectic covers quasi-split unitary groups in an odd number of variables over a p-adic field. We also address a question raised in [G]. It is known that the metaplectic $\mathbb{C}^1$-cover of these groups splits [MWV].

1. Notation

Let p be an odd prime, Let F be a p-adic field, $(\ ,\ ) = (\ ,\ )_2$ the quadratic Hilbert symbol on F, $\mathcal{O}_F$ its ring of integers, π a uniformizer, q the cardinality of the residue field $\mathbb{F} = \mathcal{O}_F/\pi\mathcal{O}_F$, and $|..| = |..|_F$ the metric on F normalized so that $|\pi| = q^{-1}$. Let $\nu : F^\times \to \mathbb{Z}$ denote the valuation defined by $|x| = q^{-\nu(x)}$.

Let E/F denote a quadratic extension, so $E = F(\sqrt{\epsilon})$ for some $\epsilon \in F^\times/F^{\times 2}$. We may assume that $q^{-1} \leq |\epsilon| \leq 1$. Let σ denote the non-trivial element of $Gal(E/F)$.

Let $\mu_n(F)$ denote the group of nth roots of unity in F and let μ_n denote the abstract group of all n^{th} roots of unity.

I thank J. Adams, C. Moen, and A. Bluher for their comments on these notes.

2. Generalities

Let

$$1 \to \mu_2 \to \overline{Sp(2n, F)} \to Sp(2n, F) \to 1$$

be the 2-fold metaplectic extension of "the" symplectic group $Sp(2n, F)$ defined by Rao's normalized cocycle, denoted β [Rao]. (A small point: since the metaplectic groups are not algebraic, it is no longer sufficient to specify a group, such as $Sp(2n)$, up to isomorphism, say, in the category of groups over the algebraic closure $\overline{F}$ or in the category of group schemes over F. We must specify which symplectic form this group is defined with respect to and hence specify its group of F-rational points $Sp(2n, F)$. Similarly, for subgroups of $Sp(2n)$ we must specify an embedding into $Sp(2n, F)$.)

Assume that V is a σ-hermitian space over E with hermitian form h, $\dim_E(V) = n$, and let

$$U(V, F) = \{g \in GL_n(E) \mid h(gv, gw) = h(v, w), \quad \forall v, w \in V\}$$

denote the associated unitary group. Assume that we have an embedding $U(V, F) \subset Sp_{2n}(F)$ and let $\overline{U(V, F)}$ denote the 2-fold cover defined by restricting β on $Sp_{2n}(F)$ to $U(V, F)$.

The question we want to address here concerns the splitting of the cocycle when restricted to a unitary group embedded into $Sp(2n, F)$. Whether or not this splits is sometimes less important in practice than knowing the relationship, as

provided by the following proposition, between the representations of the unitary group and its (possibly split) 2-fold cover.

Proposition 2.1. *Either*

(a) $\beta|_{U(V,F)}$ *splits, or*

(b) the restricted cover $\overline{U(V,F)}$ *is non-trivial but there is a genuine character* $\gamma : \overline{U(V,F)} \to \mathbb{C}^\times$.

This is essentially a result of J.-S. Li (see Lemma 2.2 in [Li]). Since it is not proven there and since it was asserted in [Li] that the cover does not split - which may be a little misleading - this is proven below. First, we state a

Corollary 2.2. *There is a natural bijection between the set* $\Pi(U(V,F))$ *of equivalence classes of irreducible admissible representations of* $U(V,F)$ *and the set* $\Pi(\overline{U(V,F)})$ *of equivalence classes of genuine irreducible admissible representations of* $\overline{U(V,F)}$. *This bijection preserves unitarity. It is given explicitly by*

$$\begin{aligned} \Pi(U(V,F)) &\to \Pi(\overline{U(V,F)}) \\ \pi &\longmapsto \pi \otimes \chi, \end{aligned}$$

where χ *is a fixed character (described in the proof below).*

proof of proposition: Let $\widetilde{U(V,F)}$ denote the metaplectic $\mathbb{C}^1$-cover of $U(V,F)$ defined by a cocycle β' [Rao]. It is known that this cover splits [MWV], ch 3. Define

$$\gamma' : \widetilde{U(V,F)} \to \mathbb{C}^1$$

by

$$(g,z) \longmapsto c(g) \cdot z,$$

where $c : U(V,F) \to \widetilde{U(V,F)}$ is a splitting (so that $\beta'(g,g) = c(g)c(g')/c(gg')$). Let χ denote the restriction of γ' to $\overline{U(V,F)}$. $\square$

Let

$$SU(V,F) = \{g \in SL_n(E) \mid h(gv,gw) = h(v,w), \quad \forall v,w \in V\}$$

denote the special unitary group associated to (V,h). The following result is due to Kazhdan ([Ka], Lemma 7).

Lemma 2.3. *$\overline{SU(V,F)}$ splits.*

remark on the proof: This is proven in [Ka]. The proof there uses implicitly the following argument: We may use either a result of Deodhar (Theorem 5.9 in [PR]) or a result of Deligne (Theorem 5.11 in [PR]) to conclude that $\overline{SU(V,F)}$ splits if (and, by Deligne, only if) $\overline{SL(2,F)}$ splits, where $SL(2,F) \subset U(V,F)$ is the subgroup generated by the sl_2-triple associated to the long (non-reduced, divisible) simple root. (The fact that $\overline{SL(2,F)}$ splits is a known result of Tanaka as referenced in [Ka], so the lemma follows.) $\square$

3. A lemma

We need a simple lemma later.

Lemma 3.1. *Let E/F be a quadratic extension of p-adic fields. Let $T = E^1$ denote the kernel of the norm map $N : E \to F$. Assume n is an odd positive integer. If $\mu_3 \subset F$ or if E/F is ramified or if $(3,n) = 1$ then $T = T^n$.*

proof: Let Φ_d denote the dth cyclotomic polynomial, so $\deg(\Phi_d) = \phi(d)$ is the number of positive integers relatively prime to d and

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Consider the map $T \to T^n$, $x \longmapsto x^n$. If $\zeta \in T \subset E$ is a non-trivial element of the kernel of this map then we have

$$\zeta \cdot \overline{\zeta} = 1, \ \zeta^n = 1, \ \zeta \neq 1, \ n \text{ is odd}, \tag{3.1}$$

where the overline denotes the conjugation of the Galois action.

Suppose first, $(6,n) = 1$ and $d|n$, so $\phi(d) > 2$. But $\Phi_d(\zeta) = 0$ implies that the degree of ζ over F must be greater than 2. This contradicts the assumption that E/F is quadratic. Thus $(3,n) = 1$ implies $T = T^n$.

Now assume E/F is a ramified quadratic extension. By (3.1), we must have $\zeta \neq \pm 1$ and so $E = F(\zeta)$. Since ζ must be a unit in the ring of integers, this implies E/F is unramified. This is a contradition, proving that E/F ramified implies $T = T^n$.

Finally, assume $\mu_3 \subset F$. (This is the assumption in [J], chap 7, where this result is stated without proof.) We have already shown that the assumptions (3.1) imply $\zeta \neq \overline{\zeta}$. Since $\Phi_d(\zeta) = 0$, for some $d|n$, and since $\deg(\zeta) = 2$, we must have (because n is odd) $d = 3$. This implies $\Phi_3(\zeta) = \zeta^2 + \zeta + 1 = 0$. Since $\mu_3 \subset F$, we have then $\zeta \in F$. But this contradicts $\zeta \neq \overline{\zeta}$.

This proves the lemma in every case. $\square$

4. Compact subgroups

We shall make some general comments on compact subgroups.

The following result is due to C. Moore ([M], Ch III, Lemma 11.3).

Lemma 4.1. *Let G be a simply connected F-split almost simple algebraic group and let $K = G(\mathcal{O}_F)$. If $(p, n) = 1$ then K has no non-trivial n-fold topological central extension. In other words, $H^2(K, \mu_n) = 0$.*

As a corollary to this, we see that if $\overline{Sp(2n, F)}$ is a 2-fold metaplectic extension

$$1 \to \mu_2 \to \overline{Sp(2n, F)} \to Sp(2n, F) \to 1$$

then its restriction to $Sp(2n, \mathcal{O}_F)$ splits.

4.1. Bluher's trick

Let $K \subset G$ be a closed subgroup of a p-adic reductive group. Let $\overline{G}$ denote a topological central extension of G,

$$1 \to \mathbb{C}^\times \xrightarrow{\iota} \overline{G} \xrightarrow{p} G \to 1,$$

and let $\overline{K}$ denote the inverse image of K under the map $\overline{G} \to G$.

We say that a character χ of $\overline{G}$ is **genuine** if $\chi(\iota(\zeta)) = \zeta$ and we say that a representation π of $\overline{G}$ is **genuine** if its central character is genuine. Let (π, V) denote a genuine irreducible admissible representation of $\overline{G}$.

The following idea was communicated to me by A. Bluher.

Lemma 4.2. *If $V^{\overline{K}}$ is one-dimensional then $\overline{K}$ splits.*

Remark 1. *As J. Adams has pointed out, this lemma is a special case of a more general principle, namely, that if a $\mathbb{C}^\times$-cover $\overline{K}$ has a genuine character then it splits.*

proof: Pick a non-zero $\phi \in V^{\overline{K}}$. There is a function $\lambda : \overline{K} \to \mathbb{C}^\times$ such that

$$\pi(k)\phi = \lambda(k)\phi, \qquad \forall k \in \overline{K}.$$

It can be checked that λ is a homomorphism. The map $k \longmapsto (p(k), \lambda(k))$ defines a splitting $K \to \overline{K}$. $\square$

5. $U(1)$

Consider the group $U(1)$ defined over F whose F-rational points are given by $U(1,F) = E^1$. We can embed this into $SL(2,F)$ by sending $x + \sqrt{\epsilon}y$ to

$$\xi(x + \sqrt{\epsilon}y) = \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \in SL(2,F).$$

Lemma 5.1.

$$\xi(U(1,F)) = \{\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \mid x, y \in F,\ x^2 - \epsilon y^2 = 1\} \subset SL(2, \mathcal{O}_F).$$

proof: We shall first show that $x \in \mathcal{O}_F$.

If $x \notin \mathcal{O}_F$ then $x = u\pi^k$, for some $k < 0$ and $u \in \mathcal{O}_F^\times$, so $|x| = q^{-k} > 1$. Since $x^2 - \epsilon y^2 = 1$, we must also have $y = u'\pi^k$, for some $u' \in \mathcal{O}_F^\times$, so $u^2 - \epsilon(u')^2 = \pi^{-k}$. This implies that there is a $u'' = u/u'$ such that $|(u'')^2 - \epsilon| = q^k < 1$. In words, ϵ is close to a square. We shall now show that this implies that this forces ϵ itself to be a square, which is the desired contradiction. We have $\epsilon = (u'')^2 + \eta$, for some $\eta \in F$ with $|\eta| < 1$. If we can solve for an $a \in F$ such that $\eta = 2u''r + r^2$ then this will imply ϵ is a square. The quadratic formula gives

$$r = \frac{-2u'' \pm \sqrt{4(u'')^2 + 4\eta}}{2} = -u'' \pm \sqrt{(u'')^2 + \eta} = -u'' \pm u''\sqrt{1 + \eta/(u'')^2}.$$

It is known that the function $f(\zeta) = \sqrt{1+\zeta}$ converges (in F) for $|\zeta| < 1$ (see [Ko], ch 1, §3, for example), so $r \in F$. This contradiction implies $x \in \mathcal{O}_F$.

Similarly, $y \in \mathcal{O}_F$. Since we have assumed (without loss of generality) $\epsilon \in \mathcal{O}_F$, the result follows. $\square$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define

$$x(g) = \begin{cases} c, & c \neq 0, \\ d, & c = 0, \end{cases}$$

For $g, g' \in SL(2, F)$, define the cocycle β by

$$\beta(g, g') = (\frac{x(gg')}{x(g)}, \frac{x(gg')}{x(g')}).$$

This defines a 2-fold metaplectic cover $\overline{SL(2, F)}$ of $SL(2, F)$ [Rao]. Let $\overline{\xi(U(1, F))}$ denote the restriction of the 2-fold cover $\overline{SL(2, F)}$ to the image of $U(1, F)$. The previous lemma, together with Lemma 4.1, implies the following:

Lemma 5.2. *The cover $\overline{\xi(U(1, F))}$ splits.*

6. $U(3)$

Let E/F denote an quadratic extension of F, $E = F(\sqrt{\epsilon})$, let $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and let

$$U(3)(F) := \{g \in GL_3(E) \mid g \cdot J \cdot {}^t g^\sigma = J\},$$

where σ denotes the Galois action of E/F. This group is named ${}^2A_{2,1}^{(1)}$ in [T1] and ${}^2A_1'$ in [T2].

6.1. The embedding into $Sp(6)$

In this subsection, we follow [Mo].

The form

$$(v, w) \longmapsto {}^t v^\sigma J w$$

is a σ-hermitian form $E^3 \times E^3 \to E$ [B]. Fix an element $\beta \in E$ such that $\beta^\sigma = -\beta$ and let

$$F(v, w) = {}^t v^\sigma \beta J w.$$

This is an $\epsilon - \sigma-$hermitian form, with $\epsilon = -1$ [B] and $F(v,w) + F(v,w)^\sigma$ is a symplectic form $E^3 \times E^3 \to F$.

Let

$$a = [x_1 + \sqrt{\epsilon} y_1, x_2 + \sqrt{\epsilon} y_2, x_3 + \sqrt{\epsilon} y_3], \quad b = [u_1 + \sqrt{\epsilon} v_1, u_2 + \sqrt{\epsilon} v_2, u_3 + \sqrt{\epsilon} v_3],$$

where $x_i, y_i, u_i, v_i \in F$. We map $\xi : E^3 \to F^6$ in such a way that these get mapped to

$$\xi(a) = [-y_1, x_1, x_2, y_2, -x_3, y_3], \quad \xi(b) = [-v_1, u_1, u_2, v_2, -u_3, v_3].$$

If we let

$$J_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & -2\,\delta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,\delta^2 \\ 0 & 0 & 0 & 2\,\delta^2 & 0 & 0 \\ 0 & 0 & -2\,\delta^2 & 0 & 0 & 0 \\ 2\,\delta^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\delta^2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then we have

$$F(a,b) + F(a,b)^\sigma = {}^t\xi(a) \cdot J_0 \cdot \xi(b).$$

(This was discovered using MAPLE - it does not agree with the matrix in [Mo], p 142, so there may be a typo there.) Thus J_0 is the matrix of this symplectic form. If we define

$$Sp(6)(F) := \{g \in GL_6(F) \mid g \cdot J. \, {}^t g = J_0\},$$

then we have an embedding

$$\xi : U(3)(F) \hookrightarrow Sp(6, F).$$

C. Moen [Mo] makes this result explicit by describing ξ on the generators of the group. From this description, we may conclude the following

Lemma 6.1. *If Z denotes the center of $U(3)(F)$ then $Z \cong U(1,F) \cong E^1$ and $\xi(Z) \subset Sp(6, \mathcal{O}_F)$.*

6.2. The splitting

Lemma 6.2. *If either $\mu_3 \subset F$ or E/F is ramified then the center Z of $U(3)(F)$ is*

$$Z = \{ \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} \mid z \in E^1 \}.$$

Furthermore, every element $g \in U(3)(F)$ may be written as $g = zh$, for some $z \in Z$ and some $h \in SU(3)(F)$.

Furthermore, the cosets $z\mu_3$ and $h\mu_3$ are uniquely determined.
proof: Let

$$z = \begin{pmatrix} \det(g)^{1/3} & 0 & 0 \\ 0 & \det(g)^{1/3} & 0 \\ 0 & 0 & \det(g)^{1/3} \end{pmatrix}.$$

By Lemma 3.1, $z \in Z$. Clearly, $g/z \in SU(3)(F)$, so the result follows. □

Conjecture 6.3. *Let G denote a reductive p-adic group, let $G' = [G, G]$ denote the commutator subgroup and let Z denote its center. The map*

$$\begin{array}{ccc} H^2(G, \mathbb{C}^\times) & \to & H^2(G', \mathbb{C}^\times) \times H^2(Z, \mathbb{C}^\times) \\ \beta & \longmapsto & \beta|_{G' \times G'} \times \beta|_{Z \times Z} \end{array}$$

is an injection.

We shall prove a special case ($G = U(3)$) of this below.
The following is our main result:

Theorem 6.4. *If $p \neq 2$ then the 2-fold cover $\overline{U(3)(F)}$ splits.*

Remark 2. *Assuming the above conjecture, the theorem follows easily from what we've already established. (We shall prove the conjecture later in this special case by actually verifying the splitting itself.)*

6.3. The inflation-restriction sequence

We shall prove the conjecture in the case of $G = U(3)$ in this subsection.

First, we recall some results from group cohomology. Let $N < G$ be a closed normal subgroup of a topological group G and let A be a G-module. We have homomorphisms

$$\begin{aligned} \mathrm{Res} : H^q(G, A) &\to H^q(N, A), \\ \mathrm{Inf} : H^q(G/N, A) &\to H^q(G, A). \end{aligned}$$

The following result is well-known:

Lemma 6.5. *("restriction-inflation sequence", [S], Prop. 5, §6, ch VII) Given an integer $q \geq 1$, suppose that $H^i(N, A) = 0$, for $1 \leq i \leq q$. Then*

$$0 \to H^q(G/N, A) \overset{\mathrm{Inf}}{\to} H^q(G, A) \overset{\mathrm{Res}}{\to} H^q(N, A)$$

is exact and

$$H^i(G/N, A) \overset{\mathrm{Inf}}{\to} H^i(G, A), \qquad 1 \leq i \leq q-1,$$

is an isomorphism.

Here the cohomology is either continuous cohomology of topological groups or abstract cohomology (sometimes denoted H_a^* if there is danger of confusion) of abstract groups.

We use this lemma to verify that the conjecture above is true in the following special case:

$$G = U(3, F), \quad N = SU(3, F), \quad A = \mu_2.$$

($\mu_2 = \{\pm 1\}$ is a trivial G-module written multiplicatively.) Unless stated otherwise, we assume this notation for the rest of this subsection.

By Lemma 2.3, $H^2(N, A) = 0$. Since N is simply connected, a lemma of Moore (Lemma 1.1 [M]) implies $H_a^1(N^a, A) = 0$, where (as in [M]) N^a denotes N as an abstract group. The restriction-inflation sequence lemma above implies

$$H_a^2(G^a/N^a, A) \cong H_a^2(G^a, A). \tag{6.1}$$

Another lemma of Moore (Lemma 2.8, [M]) implies

$$\begin{aligned} H_a^2(G^a/N^a, A) &\cong H^2(G/N, A) \\ H_a^2(G^a, A) &\cong H^2(G, A). \end{aligned}$$

We may identify $G/N = U(3,F)/SU(3,F)$ with $U(1,F)/\mu_3(F)$ by the discussion in §6.1. Thus, we have

$$H^2(G,\mu_2) = H^2(U(1,F)/\mu_3(F),\mu_2).$$

We may apply the same reasoning to the following case:

$$G = U(1,F), \quad N = \mu_3(F), \quad A = \mu_2,$$

to get the following continuous analog of (6.1):

$$H^2(U(1,F),\mu_2) = H^2(U(1,F)/\mu_3(F),\mu_2).$$

We already know, by §5 above, that $H^2(U(1,F),\mu_2) = 0$ so $H^3(U(3,F),\mu_2) = 0$.

This proves the splitting, hence also the conjecture, in this case. □

7. Remarks

We collect some assorted remarks in this section.

7.1. On [Mo]

We have shown that the splitting of $\overline{U(3,F)}$ claimed (without proof) in [Mo] is true.

7.2. On the higher rank case

Let E/F denote an quadratic extension of F, $E = F(\sqrt{\epsilon})$, let J denote the $(2n+1)\times(2n+1)$ skew-symmetric matrix

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \vdots & 0 & 1 & 0 \\ & & \dots & & \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and let

$$U(2n+1)(F) := \{g \in GL_{2n+1}(E) \mid g \cdot J \cdot {}^t g^\sigma = J\},$$

where σ denotes the Galois action of E/F. The center Z of this group is the subgroup of scalar matrices with values in E^1.

It seems likely that the argument above can generalized. The statement below is the result:

Conjecture 7.1. *Let E/F be a quadratic extension of p-adic fields, $p \neq 2$. Let $T = E^1$ denote the kernel of the norm map $N : E \to F$. If $\mu_3 \subset F$ or if E/F is ramified or if $(3, 2n+1) = 1$ then there is an embedding $U(2n+1, F) \hookrightarrow Sp(4n+2, F)$ such that $\overline{U(2n+1, F)}$ splits.*

We finish with the following vague remark: In the case of an "even" unitary group G which is part of a dual reductive pair of a symplectic group $Sp(2N, F)$, there are cases in which the explicit splitting formulas of Kudla ([Ku], §3) along with results of Deodhar ([D], [PR], §5) implies that the 2-fold cover of G splits. This depends on whether or not there is a quadratic character of $E^\times$ which extends the character $x \longmapsto (x, \sqrt{\epsilon})_2$ of $F^\times$, where $E = F(\sqrt{\epsilon})$ is the quadratic extension of F splitting G.

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