Notes on toric varieties

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Abstract

These notes survey some basic results in toric varieties over a field F, with examples and applications.

A computer algebra package (written by the author) is described which deals with both affine and projective toric varieties in any number of dimensions (written in both MAGMA [MAGMA] and GAP [GAP]). Among other things, the package implements the desingularization procedure, constructs some error-correcting codes associated with toric varieties, and computes the Riemann-Roch space of a divisor on a toric variety.

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The main reference for these notes is [F].
Throughout, let F denote a field.

1 Introduction

Let $R = F[x_1, ..., x_n]$ be a ring in n variables. A **binomial relation** [1] in R is one of the form

$$x_1^{k_1} ... x_n^{k_n} = x_1^{\ell_1} ... x_n^{\ell_n},$$

where $k_i \geq 0$, $\ell_j \geq 0$ are integers. A **binomial variety** [2] is a subvariety of complex affine n-space $\mathbb{A}_F^n$ defined by a finite set of binomial equations. Such varieties arise frequently "in nature". A typical "toric variety" (defined more precisely later) is binomial, though they will be introduced via an *a priori* independent construction [3].

1.1 Motivation

Toric geometry has several interesting facets.

- It gives a way of constructing an ambient space (a toric variety) in which algebraic varieties live, in the same way that projective space, and weighted projective space are primarily considered as ambient spaces.
  - Toric varieties are all rational.
  - Toric varieties are not all projective, (i.e., they cannot all be embedded in $\mathbb{P}^n$ for some n), so provide a wider class of algebraic varieties to work with.
  - Even if a subvariety of a toric variety is projective, it generally is embedded in a toric variety with affine pieces of much lower dimension than the projective space.
- In the case of a toric variety, certain algebraic geometry computations can be reformulated into simpler combinatorial problems. For example, there is a "simple" method of resolving certain singularities.
- Toric geometry is used to solve certain compactification problems.

[1] Also called a **monomial equation** [ES].

[2] Also called a monomial variety, binomial scheme, torus embedding, ... Such a variety need not be normal.

[3] In fact, we shall give two constructions of toric varieties. The basic idea of the first one is to replace each such binomial equation as above by a relation in a semigroup contained in a lattice and replace R by the "group algebra" of this semigroup. More details below. By the way, a toric variety is always normal (see for example, [F], page 29).

- Toric geometry is a source of many examples in various areas, e.g., Batyrev's construction of pairs of mirror Calabi-Yau threefolds from "reflexive polytopes".

- It has many other applications, e.g., to combinatorical geometry [E], error-correcting codes [H], and connections with germs of p-adic orbital integrals (unpublished) and non-regular continued fractions [F].

1.2 Computer programs

Toric geometry very much lends itself to a computational approach. There are a few packages available for working with toric geometry, such as

- - TiGERS
  - c program
  - by Birkett Huber and Rekha Thomas
  - http://www.math.washington.edu/~thomas/programs.html
  - Computes the Gröbner fan of a toric ideal
  - Also some maple available from same site: http://www.math.washington.edu/~thomas/program

- - part of the *singular* computer algebra system
  - http://www.singular.uni-kl.de/Manual/2-0-0/sing_376.htm
  - http://www.singular.uni-kl.de/Manual/2-0-0/sing_378.htm
  - Computes the lattice basis using LLL algorithm, and algorithms by Conti and Traverso; Pottier; Hosten and Sturmfels; Di Biase and Urbanke; Bigatti, La Scala and Robbiano, for computing the saturation.

- - "CY/4d"
  - Kreuzer and Skarke
  - Data base of the classification of all 473,800,776 reflexive polytopes in 4 dimensions. (3 dimensional version also available.)
  - http://hep.itp.tuwien.ac.at/~kreuzer/CY/CYcy.html

  - this gives a method of constructing different examples of mirror pairs of families of Calabi-Yau threefolds, especially of interest to physicists.

- - "toric" (`toric.g` for GAP 4.3 and `toric.mag` for MAGMA 2.8)
  - Joyner
  - GAP and MAGMA programs for toric varieties.
  - [J] Examples are given in this paper and at the end of each of the files in [J] to explain the syntax.

Note that all but the last two of the above programs are restricted to the case of affine toric varieties, (the first two being more from the point of commutative algebra), and the third one is a fairly specialized situation, and only available as a data base. There is much more to toric geometry that could be programmed but so far seems not to have been.

2 Cones and semigroups

Let $V = \mathbb{Q}^n$ having basis $f_1 = (1, 0, ..., 0)$, ..., $f_n = (0, ..., 0, 1)$. Let L be a lattice in V (i.e., a free rank n $\mathbb{Z}$-module in V). We identify V and $L \otimes_{\mathbb{Z}} \mathbb{Q}$. We use $\cdot$ or $\langle\ ,\ \rangle$ to denote the (standard) inner product on V. If $e_1, ..., e_n$ is a basis for L, let e_1^*, ..., e_n^* denote the dual basis for L^*. Let

$$L^* = \mathrm{Hom}(L, \mathbb{Z}) = \{v \in V \mid \langle v, w\rangle \in \mathbb{Z},\ \forall w \in L\}$$

denote the **dual lattice**, so (since the basis of L has been choosen to be the standard one) L^* may be identified with $\mathbb{Z}^n$.

Example 2.1. *If L is the sublattice of $\mathbb{Z}^2$ generated by $e_1 + 2e_2$ and $2e_1 + e_2$ then the dual is not contained in $\mathbb{Z}^2$. However, it may be rescaled (multiplied by an integer) so that it is. MAGMA's* `Dual` *command compute's this rescaled version.*

```
> L := LatticeWithBasis(2, [1,2, 2,1]);
> L;
Lattice of rank 2 and degree 2
Basis:
(1 2)
(2 1)
```

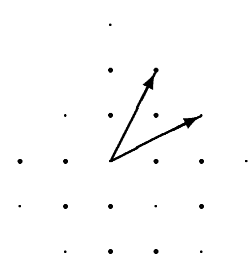
```
> Lperp:=Dual(L);
> Lperp;
Lattice of rank 2 and degree 2
Basis:
( 1  1)
( 2 -1)
> a,b,c:=DualQuotient(L);
> b;
Lattice of rank 2 and degree 2
Basis:
( 1  1)
( 1 -2)
Basis Denominator: 3
> Basis(b);
[
    (1/3 1/3),
    ( 1/3 -2/3)
]
```

This means that `Lperp`$= \{a(e_1^* + e_2^*) + b(2e_1^* - e_2^*) \mid a, b \in \mathbb{Z}\}$ *is an integral basis belonging to the dual* L^**, denoted* `b` *in the above MAGMA session. In fact,*

$$\begin{aligned} L^* &= \{a\tfrac{e_1^*+e_2^*}{3} + b\tfrac{e_1^*-2e_2^*}{3} \mid m, n \in \mathbb{Z}\} \\ &= \{(a+\eta)e_1^* + (b+\eta)e_2^* \mid a, b \in \mathbb{Z},\ \eta \in \{0, \tfrac{1}{3}, \tfrac{2}{3}\}\}, \end{aligned}$$

so `Lperp` *is "3 times* L^**".*

The lattice L *may be visualized as follows.*



A plain vanilla [4] **cone** in V is a σ of the form

$$\sigma = \{a_1 v_1 + ... + a_m v_m \mid a_i \geq 0\},$$

where $v_1, ..., v_m \in V$ is a given collection of vectors (so $0 \leq m \leq n$), called a **basis** (or **generators**) of σ. This cone is also denoted

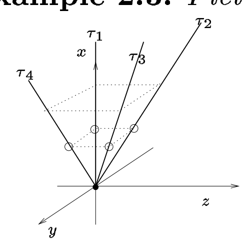
$$\sigma = \mathbb{Q}_{\geq 0}[v_1, ..., v_m].$$

A **rational cone** is one where $v_1, ..., v_m \in L$. A **strongly convex** cone is one which contains no lines through the origin. A cone generated by linearly independent vectors is necessarily strongly convex. The **dimension of** σ, denoted $\dim(\sigma)$, is the dimension of the subspace $\sigma + (-\sigma)$ of V.

Example 2.2. *Let R be a root system of V, endowed with the usual Euclidean inner product (as in Ch III of Humphreys [H], for example), and R_+ a choice of positive roots. Let L be the root lattice [5] of R and L^* its dual lattice (this is not the lattice of "dual roots"). Let σ be the cone defined by the positive Weyl chamber associated to R_+. This is a strongly convex rational cone.*

For further examples, see §§2-3 in Verrill [V].

Example 2.3. *Picture of a three dimensional cone and its faces:*



Cone σ generated by $(1, 0, 0), (1, 1, 0), (1, 1, 1)$ and $(1, 0, 1)$ (circled). The zero dimensional face is 0, and the one dimensional faces are $\tau_1 = \mathbb{R}_{\geq 0}(1, 0, 0)$, $\tau_2 = \mathbb{R}_{\geq 0}(1, 1, 0)$, $\tau_3 = \mathbb{R}_{\geq 0}(1, 1, 1)$ and $\tau_4 = \mathbb{R}_{\geq 0}(1, 0, 1)$. Two dimensional faces are given by $\langle \tau_1, \tau_2 \rangle$, $\langle \tau_2, \tau_3 \rangle$, $\langle \tau_3, \tau_4 \rangle$ and $\langle \tau_4, \tau_1 \rangle$.

The collection of all these cones forms a fan.

Definition 2.4. *A* **cone of** L *is a strongly convex rational cone.*

If σ is a cone then the **dual cone** *is defined by*

$$\sigma^* = \{w \in L^* \otimes \mathbb{Q} \mid \langle v, w \rangle \geq 0, \ \forall v \in \sigma\},$$

where L^* is the dual lattice.

[4] Sorry, couldn't resist the pun. The point is that though this is the correct and standard definition of a cone, later we shall actually reserve the word "cone" for a special type of cone.

[5] One may replace "root lattice" by "character lattice" or "weight lattice" as well.

MAGMA/GAP Example 2.5. *Let $n = 2$, $L = L^* = \mathbb{Z}^2$, and suppose $\sigma = \mathbb{Q}_{\geq 0}[e_1, 3e_1 + 4e_2]$. To check if $e_1^* - 7e_2^*$ or if $4e_1^* - 3e_2^*$ belongs to σ^* in MAGMA* [6]*, type*

```
load "/home/wdj/magmafiles/toric.mag";
//replace /home/wdj/magmafiles by your path to toric.mag
in_dual_cone([1,-7],[[1,0],[3,4]]);
in_dual_cone([4,-3],[[1,0],[3,4]]);
```

In the first case, MAGMA returns `false` *and in the second case,* `true`.

If $\sigma = \mathbb{Q}_{\geq 0}[v_1, ..., v_m]$ is a cone of $L = \mathbb{Z}^n$ and if there are vectors $v_{m+1}, ..., v_n$ such that $\det(v_1, ..., v_n) = \pm 1$*, then we say that σ is* **regular**.

Suppose σ, σ' are cones of $L = \mathbb{Z}^n$. If $dim(\sigma) = dim(\sigma')$ and if there is a $g \in GL_n(\mathbb{Z})$ for which $\sigma' = g\sigma$ then we say that σ is **isomorphic** *to σ' and write $\sigma \cong \sigma'$.*

2.1 Affine toric varieties

Associate to the dual cone σ^* the semigroup

$$S_\sigma = \sigma^* \cap L^* = \{w \in L^* \mid \langle v, w \rangle \geq 0, \ \forall v \in \sigma\}.$$

Though L^* has n generators *as a lattice*, typically S_σ will have more than n generators *as a semigroup*. If $u_1, \ldots u_t \in L^*$ are semigroup generators of S_σ then we write

$$S_\sigma = \mathbb{Z}_{\geq 0}[u_1, ..., u_t],$$

for brevity.

Remark 2.6. *The following question arises: Given a lattice $L = \mathbb{Z}[v_1, ..., v_m]$ and a cone $\sigma = \mathbb{Q}_{\geq 0}[w_1, ..., w_m]$, how do you find $u_1, ..., u_n \in L^*$ such that $S_\sigma = \mathbb{Z}_{\geq 0}[u_1, ..., u_t]$?*

First, find a basis for the dual lattice, L^*, say $L^* = \mathbb{Z}[v_1^*, ..., v_m^*]$. Next, find generators in L^* of the dual cone, say $\sigma^* = \mathbb{Q}_{\geq 0}[w_1^*, ..., w_m^*]$, Let the list of potential generators be

$$G = \{w \in L^* \mid |w| \leq \max_{1 \leq i \leq m} |v_i^*|, \ w \in \sigma^*\}.$$

It is conjectured that G is a semigroup basis for S_σ.

[6] The MAGMA 2.8 code used is available on the internet at [J].

Another related question arises: If $\sigma \subset \mathbb{Q}^n$ is a (not necessarily convex) cone, and $V = \{v_1, ..., v_m\} \subset L^* \subset \mathbb{Q}^n$, when is V a set of semigroup generators for S_σ?

It is conjectured that the following two conditions are sufficient:

- *$V \subset \sigma$,*
- *V generates each element in*

$$\{w \in S_\sigma \mid |w| \leq 2 \cdot \max_{v \in V} |v|\}.$$

MAGMA/GAP Example 2.7. *Let $n = 2$, $L = L^* = \mathbb{Z}^2$, and suppose $\sigma = \mathbb{Q}_{\geq 0}[e_1, 3e_1 + 4e_2]$. To show that*

$$S_\sigma = \mathbb{Z}_{\geq 0}[e_1, e_2, 2e_1 - e_2, 3e_1 - 2e_2, 4e_1 - 3e_2],$$

type

```
load "/home/wdj/magmafiles/toric.mag";
//replace /home/wdj/magmafiles by your path to toric.mag
D := LatticeDatabase();
Lat := Lattice(D, 2, 16);
dual_semigp_gens([[1,0],[3,4]],Lat);
```

Let

$$R_\sigma = F[S_\sigma]$$

denote the "group algebra" of this semigroup. It is a finitely generated commutative F-algebra. It is in fact integrally closed ([F], page 29).

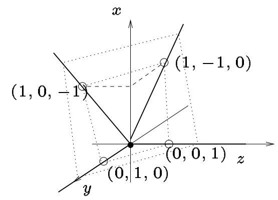
We may interprete R_σ as a subring of $R = F[x_1, ..., x_n]$ as follows: First, identify each e_i^* with the variable x_i. If S_σ is generated as a semigroup by vectors of the form $\ell_1 e_1^* + ... + \ell_n e_n^*$, where $\ell_i \in \mathbb{Z}$, then its image in R is generated by monomials of the form $x_1^{\ell_1} \dots x_n^{\ell_n}$.

Let

$$U_\sigma = \text{Spec } R_\sigma.$$

This defines an **affine toric variety** (associated to σ).

Example 2.8. *For the cone in Example 2.3, we compute the dual cone by taking the intersection of the duals of all the one dimensional cones, and* *obtain that* $\sigma^\vee$ *is spanned by* $(0,0,1),(1,-1,0),(1,0,-1)$ *and* $(0,1,0)$



So U_σ *is given by* $\mathrm{Spec}(F[y,z,x/y,x/z]) \cong \mathrm{Spec}(F[X,Y,Z,W]/(XZ-WY))$, *where the isomorphism of rings is given by*

$$\begin{array}{rcl} X & \mapsto & y \\ Y & \mapsto & z \\ Z & \mapsto & x/y \\ W & \mapsto & x/z \end{array}$$

So U_σ *is an affine threefold in* $\mathbb{A}^4$ *which is a cone over the projective quadric surface* $XZ = WY$. *There is a singular point at the origin.*

Remark 2.9. *(For those less familiar with algebraic geometry:) Here the spectrum of a ring* R, $\mathrm{Spec}\, R$, *is the set of prime ideals in* R *(with a certain topology). A basic result we use is that if* $f_1, \dots f_m$ *are a collection of polynomials in variables* $x_1, \dots, x_d$ *with coefficients in* $\mathbb{C}$, *then*

$$\mathrm{Spec}(\mathbb{C}[x_1, \dots, x_d]/(f_1, f_2, \dots, f_m))$$

is the affine variety in d *dimensional space* $\mathbb{A}^d = F^d$ *defined by the set of points where* $f_1 = f_2 = \dots = f_m = 0$. *Generally, we will want to find a way to write an* $\mathbb{C}$*-algebra in this form, so that we can understand its Spec geometrically.*

Roughly speaking, an algebraic variety is given by a collection of affine pieces $U_1, U_2, \dots, U_d$ *which "glue" together. The affine pieces are given by the zero sets of polynomial equations in some affine spaces* $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$, *and the gluings are given by maps*

$$\phi_{i,j} : U_i \to U_j$$

which are defined by ratios of polynomials on open subsets of the U_i.

A major advantage of the toric geometry description is that the relationships between the affine pieces U_σ *are simply described by the relationships between the corresponding cones. For a toric variety this gives an easy way to keep track of the "gluing" data, as we will see in §3.*

Equivalently, a toric variety is a normal variety X over F which contains a torus $T = (F^\times)^n$ as an open dense (in the Zariski topology) [7] subset:

$$T \hookrightarrow X$$

and such that the natural action of T on T extends to an action on X. In other words, there is a map:

$$T \times X \to X,$$

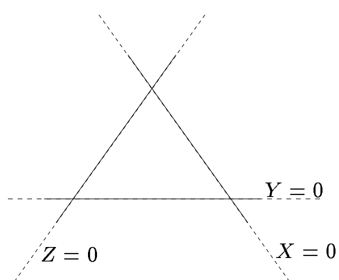
which restricts to the above map $T \times T \to T \subset X$.

Example 2.10. *Projective space $\mathbb{P}^n$ over F is a toric variety, containing $(F^\times)^n$:*

$$\begin{array}{rcl} (F^\times)^n & \hookrightarrow & \mathbb{P}^n \\ (a_1, a_2, \ldots a_n) & \mapsto & (1 : a_1 : a_2 : \ldots : a_n) \end{array}$$

The action of $T = (F^\times)^n$ on $\mathbb{P}^n$ is given by

$$(a_1, a_2, \ldots a_n) \cdot (b_0 : b_1 : b_2 : \ldots : b_n) = (b_0 : a_1 b_1 : a_2 b_2 : \ldots : a_n b_n).$$



In $\mathbb{P}^2(\mathbb{R})$ with projective coordinates X, Y, Z, we have that $\mathbb{P}^2(\mathbb{R}) \setminus (\mathbb{R}^\times)^2$ is given by the three lines $X = 0, Y = 0$ and $Z = 0$.

If $(\mathbb{R}^\times)^2$ is embedded by $(a, b) \mapsto (1, a, b)$, then the torus action on these lines is given by

$$\begin{array}{l} (a,b) \cdot (0 : Y : Z) = (0 : aY : bZ) \\ (a,b) \cdot (X : 0 : Y) = (X : 0 : bZ) \\ (a,b) \cdot (X : Y : 0) = (X : aY : 0) \end{array}$$

Lemma 2.11. *Let $\sigma \subset V = \mathbb{Q}^n$ be a cone of $L = \mathbb{Z}^n$.*

- *(Fulton [F], page 29, or Ewald [E], ch VI, §3) U_σ is smooth if and only if σ is a regular cone.*
- *(Ewald [E], Theorem 2.11, page 222) $U_\sigma \cong U_{\sigma'}$ (as algebraic varieties) if and only if $\sigma \cong \sigma'$ (as cones).*

[7]The **Zariski topology** on an algebraic variety is given by taking closed sets to be defined by the zero sets of polynomials.

- *(Fulton [F], page 18, Ewald [E], Theorem 6.1, page 243) Let $\phi : L' \to L$ be a homomorphism of lattices which maps a cone σ' of L to a cone σ of L (i.e., $\phi(\sigma') = \sigma$). Then the dual $\phi_L^* : L^* \to (L')^*$ gives rise to a map $\phi_\sigma^* : S_\sigma \to S_{\sigma'}$. This determines a map $\phi_* : R_\sigma \to R_{\sigma'}$, and hence a morphism $\phi^* : U_{\sigma'} \to U_\sigma$.*

A morphism $\phi^* : U_{\sigma'} \to U_\sigma$ arising as in the above lemma from a homomorphism $\phi : L' \to L$ be a homomorphism of lattices will be called an **affine toric morphism**.

Summarizing the construction, we have the following sequence to determine an affine toric variety. Fix a lattice L in V.

$$\begin{array}{ccccccc} \{\text{rational cones}\} & \to & \{\text{comm. semigroups}\} & \to & \{\text{semigroup algebras}\} & \to & \{\text{affine schemes}\} \\ \sigma & \longmapsto & S_\sigma = \sigma^* \cap L^* & \longmapsto & F[S_\sigma] & \longmapsto & U_\sigma = \text{Spec } F[S_\sigma]. \end{array}$$

Using the notion of a "fan", later we shall see how to, in some cases, patch these together into a projective version.

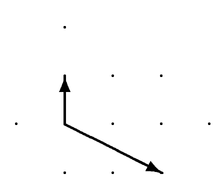
Example 2.12. *This is really a non-example.*

Let $n = 1$, so $L = \mathbb{Z}$, and consider the semigroup S generated by $2, 3$. This corresponds to the "coordinate ring" $F[S] \cong F[x_1^2, x_1^3]$. This is associated to the curve $x^3 = y^2$, which has a cusp at the origin. The element x_1 in the field of fractions $F(x_1)$ of $F[S]$ is integral over $F[S]$ but is not in $F[S]$, so $F[S]$ is not integrally closed. In fact, the curve Spec $F[S]$ *is not normal and is not a toric variety (although it is "binomial").*

Example 2.13. *Let $n = 2$, $L = \mathbb{Z}^2 = L^*$, and*

$$\sigma = \{ae_2 + b(2e_1 - e_2) \mid a, b \geq 0\}.$$

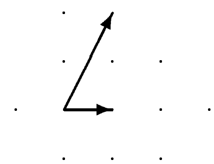
This may be visualized as follows:



In this case, σ^* is given by

$$\sigma^* = \{xe_1^* + ye_2^* \mid 2x \geq y \geq 0\}.$$

This may be visualized as follows:



Therefore the semigroup $S_\sigma = \sigma^* \cap L^*$ *is generated by* $u_1 = e_1^*, u_2 = e_1^* + e_2^*, u_3 = e_1^* + 2e_2^*$. *We associate to these generators the monomials* $x_1, x_1x_2, x_1x_2^2$, *respectively, since* $me_1^* + ne_2^* \leftrightarrow x_1^m x_2^n$. *This implies* [8]

$$F[S_\sigma] = F[x_1, x_1x_2, x_1x_2^2].$$

In fact [9] *, this ring is isomorphic to* $F[x,y,z]/(xz-y^2)$, *via the map* $x \longmapsto x_1,\ y \longmapsto x_1x_2,\ z \longmapsto x_1x_2^2$, *so*

$$U_\sigma = \mathrm{Spec}(F[S_\sigma]) = \mathrm{Spec}(F[x,y,z]/(xz-y^2)),$$

which is the surface $xz - y^2 = 0$. *This identity* $xz - y^2 = 0$ *corresponds to the vector identity* $u_1 + u_2 = 2u_3$, *which may be easily verified from the picture above.*

How does one (in general) find the equation(s) of the toric variety associated to a cone? One (algebraic) method is to use the following fact.

Lemma 2.14. *([F], page 19, Exercise) If* S_σ *is generated by* $u_1, ..., u_t$ *then*

$$F[S_\sigma] \cong F[\chi^{u_1}, ..., \chi^{u_t}] \cong F[y_1, ..., y_t]/I,$$

where $\chi^{e_i^*} = x_i$, $1 \leq i \leq t$, *and where* I *is the ideal generated by binomials of the form*

$$y_1^{a_1}...y_t^{a_t} - y_1^{b_1}...y_t^{b_t},$$

where $a_i \geq 0,\ b_j \geq 0$ *are integers satisfying*

$$a_1u_1 + ... + a_tu_t = b_1u_1 + ... + b_tu_t. \tag{1}$$

[8] See Lemma 2.14 below for a more rigorous approach.
[9] Again, see Lemma 2.14 below.

Using this lemma, the determination of I may be reduced to a linear algebra problem (over $\mathbb{Z}$). For a detailed example of this, see §5.1.2 below.

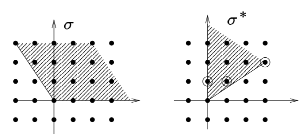
Another way to find the equation(s) of the toric variety U_σ is to find a complete (but finite) set of semigroup generators of S_σ, since each generator of I corresponds to a relation (1) for the generators of S_σ. This provides, at the very least, a nice geometric interpretation of the generators of I, as we saw at the end of Example 2.13 above.

Definition 2.15. *A* **torus** *(of dimension n, defined over the field F) is a product $T = (F^\times)^n$ of n copies of $F^\times = F \setminus \{0\}$ for a field F. There is a natural map $T \times T \to T$ given by pointwise multiplication. E.g.,*

$$(F^\times)^2 \times (F^\times)^2 \to (F^\times)^2$$

$$(a, b) \cdot (c, d) = (ac, bd).$$

Example 2.16. *For another example, take the cone σ spanned by e_1 and $-2e_1 + 3e_2$.*



Then $\sigma^* \cup L^*$ is generated by $e_2, e_1 + e_2$, and $3e_1 + 2e_2$, corresponding to monomials $X = y, Y = xy, Z = x^3y^2$, so U_σ is isomorphic to an affine variety in $\mathbb{A}^3$ given by equation

$$Y^3 = XZ.$$

Note, to verify this we just need to check that the map

$$\begin{array}{rcl} F[X, Y, Z]/(Y^3 - XZ) & \to & F[y, xy, x^3y^2] \\ X, Y, Z & \mapsto & y, xy, x^3y^2 \end{array}$$

is an isomorphism.

Example 2.17. *Let $L = L^* = \mathbb{Z}^n$ be the standard lattice.*

- *Let $\sigma = \{0\}$ be the trivial cone. Then $\sigma^* = \mathbb{Q}^n$ is "everything" and so the semigroup $S_\sigma = \sigma^* \cap L^*$ is generated by $\pm e_1^*, ..., \pm e_n^*$. This means that*
$$R_\sigma = F[S_\sigma] \cong F[x_1, x_1^{-1}, ..., x_n, x_n^{-1}].$$
This is the coordinate ring of the torus $(F^\times)^n$, so $U_\sigma = (F^\times)^n$.

- *Let $\sigma = \{a_1e_1 + .. + a_ne_n \mid a_i \geq 0\}$. The dual cone σ^* is equal to σ, so $S_\sigma = \sigma^* \cap L^*$ is generated as a semigroup by $e_1^*, ..., e_n^*$. This implies $R_\sigma \cong F[x_1, ..., x_n]$, so $U_\sigma \cong F^n$.*

Lemma 2.18. *([F], Proposition, page 29) A toric variety U_τ is smooth if and only if τ is generated by a subset of a basis for L.*

Example 2.19. *([F], page 35) Let σ be the cone generated by e_1 and $-e_1 + ne_2$. Then $U_\sigma \cong F^2/\mu_n$, where $\mu_n = \{z \in F \mid z^n = 1\}$ and where μ_n acts on F^2 by $\zeta : (z_1, z_2) \longmapsto (\zeta z_1, \zeta z_2)$.*

2.2 Faces and subvarieties

Suppose τ is a face [10] of the cone σ of L. How is the associated toric variety U_τ related to U_σ?

Lemma 2.20. *If $\tau \subset \sigma$ is a face then there is a dominant morphism $U_\tau \to U_\sigma$.*

proof: If τ is a face of σ then there is a $u \in S_\sigma = \sigma^* \cap L^*$ such that $\tau = u^\perp \cap \sigma$, where $u^\perp = \{v \in V \mid u \cdot v = 0\}$. We may regard $R_\sigma = F[S_\sigma]$ as the coordinate ring of U_σ. As such,

$$U_\tau = \{x \in U_\sigma \mid u(x) \neq 0\}.$$

It is clear that the "inclusion" map $U_\tau \to U_\sigma$ is surjective on a Zariski dense set, so it is dominant. $\square$

Example 2.21. *Let $n = 2$, $L = \mathbb{Z}^2 = L^*$, and*

$$\sigma = \{ae_2 + b(2e_1 - e_2) \mid a, b \geq 0\}$$

and let $\tau = \mathbb{Q}_{\geq 0} \cdot (2e_1 - e_2)$. In fact, if we take $u = e_1^* + 2e_2^*$ then $\tau = u^\perp \cap \sigma$. If we represent $x \in U_\sigma$ by the point $x_1e_1 + x_2e_2 = (x_1, x_2)$ then $u(x) = x_1x_2^2$ (or $x_1 + 2x_2$, but this will instead be written multiplicatively, as $x_1x_2^2$).

[10] Some, like Ewald [E], use the convention that $\emptyset, \sigma$ are (improper) faces of σ. We shall define a **face of** σ to be either σ itself or a subset of the form $H \cap \sigma$, where H is a "supporting hyperplane of σ" (i.e., a codimension 1 subspace of V for which $\sigma \cap H \neq \emptyset$ and σ is contained in exactly one of the two half-spaces determined by H).

The dual cone of τ is a half-plane,

$$\tau^* = \{xe_1^* + ye_2^* \mid 2x - y \geq 0\},$$

so $S_\tau = \tau^* \cap L^*$ is generated as a semigroup by e_1^*, $-e_2^*$, $e_1^* + 2e_2^*$, $-e_1^* - 2e_2^*$. The coordinate ring associate to τ is given by

$$R_\tau = F[e_1^*, -e_2^*, e_1^* + 2e_2^*, -e_1^* - 2e_2^*] \cong F[x_1, x_2^{-1}, x_1x_2^2, x_1^{-1}x_2^{-2}].$$

By inspection, we have an isomorphism

$$R_\tau \cong F[x, y, z, w]/(wz^2 - x, xy - z^2),$$

via $x = x_1$, $y = x_1^{-1}x_2^{-2}$, $z = x_2^{-1}$, $w = x_1x_2^2$. This implies that U_τ is the variety in F^4 defined by

$$wz^2 = x, \qquad xy = z^2.$$

Presumably, we may embed the variety U_σ given by $xy = z^2$ determined in Example 2.13 into x, y, z, w-space and regard U_τ as the dense subvariety $x \neq 0$ of this. (This condition implies $w \neq 0$.)

MAGMA/GAP Example 2.22. *Let $n = 2$, $L = L^* = \mathbb{Z}^2$, and suppose $\sigma = \mathbb{Q}_{\geq 0}[e_1, 3e_1 + 4e_2]$. To find the ideal I defining the quotient ring $R_\sigma = F[x_1, ..., x_5]/I$, type*

```
load "/home/wdj/magmafiles/toric.mag";
//replace /home/wdj/magmafiles by your path to toric.mag
D := LatticeDatabase();
Lat := Lattice(D, 2, 16);
ideal_affine_toric_variety([[1,0],[3,4]],Lat);
```

MAGMA returns

```
Ideal of Polynomial ring of rank 5 over Rational Field
Lexicographical Order
Variables: x1, x2, x3, x4, x5
Basis:
[
    x1*x5^3 - x4^4,
    x2*x5^2 - x4^3
]
```

2.3 The dense torus

Let $\sigma \subset \mathbb{Q}^n$ be a cone in L. In the case $\tau = \{0\} \subset \sigma$, the dominant morphism $U_\tau \to U_\sigma$ maps a torus $U_\tau \cong F^{\times n}$ into a dense subvariety of U_σ. We shall briefly recall another way to view this.

Let $\mathbb{G}_m$ denote the multiplicative algebraic group over F. For each integer k, the map $z \longmapsto z^k$ defines an element of $\mathrm{Hom}_{alg.gp.}(\mathbb{G}_m, \mathbb{G}_m)$. In fact, each element of $\mathrm{Hom}_{alg.gp.}(\mathbb{G}_m, \mathbb{G}_m)$ arises in this way. Given a lattice $L \subset \mathbb{Q}^n$, let $T_L = \mathrm{Hom}_{ab.gp.}(L^*, \mathbb{G}_m)$. By Lemma 2.14, we have $T_L \subset U_\sigma$. They have the same dimension, so T_L must be dense in U_σ. For an explicit example, see Example 3.18 below (where $T_L = U_{\sigma_6}$).

MAGMA/GAP Example 2.23. *To define a torus of dimension (for example) 5 in MAGMA, as a scheme over $\mathbb{Q}$, type*

```
load "/home/wdj/magmafiles/toric.mag";
//replace /home/wdj/magmafiles by your path to toric.mag
create_torus(5);
```

The variables `x1, x2, x3, x4, x6, x7, x8, x9, x10` *satisfy* $x_i x_{i+5} = 1$, *for* $1 \leq i \leq 5$.

Let $n = 2$, $L = L^* = \mathbb{Z}^2$, and suppose $\sigma = \mathbb{Q}_{\geq 0}[e_1, 3e_1 + 4e_2]$. To obtain the rational map (of schemes) $T_L \to U_\sigma$, type

```
embedding_affine_toric_variety([[1,0],[3,4]]);
```

MAGMA returns

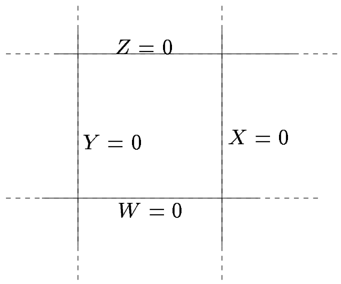
```
Mapping from: Scheme over Rational Field defined by
x1*x3 - 1
x2*x4 - 1 to Affine Space of dimension 5
Variables : $.1, $.2, $.3, $.4, $.5 %$
with equations :
x2
x1
x1^2*x4
x1^3*x4^2
x1^4*x4^3
```

This means that U_σ is 2-dimensional and the dense embedding given by

$$\begin{array}{ccc} (F^\times)^2 & \to & U_\sigma, \\ (x_1, x_2) & \longmapsto & (x_2, x_1, x_1^2 x_4, x_1^3 x_4^2, x_1^4 x_4^3) = (x_2, x_1, x_1^2 x_2^{-1}, x_1^3 x_2^{-2}, x_1^4 x_2^{-3}). \end{array}$$

Example 2.24. *Another "toric compactification" of $(F^\times)^n$ is given by taking the product of n copies of $\mathbb{P}^1$. There is a map*

$$\begin{array}{rcl} (F^\times)^n & \hookrightarrow & \overbrace{\mathbb{P}^1 \times \mathbb{P}^1 \dots \mathbb{P}^1}^{n \text{ copies}} \\ (a_1, a_2, \dots, a_n) & \mapsto & (1 : a_1) \times (1 : a_2) \times \dots \times (1 : a_n) \end{array}$$

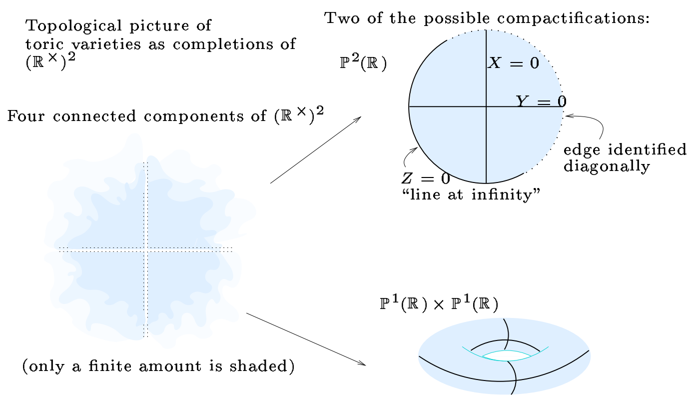


In $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ with projective coordinates X, Y and Z, W for the two components, we have that $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \setminus (\mathbb{R}^\times)^2$ is given by 4 lines, $X = 0, Y = 0, Z = 0$ and $W = 0$

If the embedding is given by $(a, b) \mapsto (1 : a) \times (1 : b)$ then the torus action on $Y = 0$ is given by

$$(a, b) \cdot ((1 : 0) \times (Z : W)) = (1 : 0) \times (Z, bW)$$

The following picture gives a topological view of the situation:



3 Fans and toric varieties

3.1 Fans

A fan is a collection of cones which "fit together" well.

Definition 3.1. *A* **fan in** L *is a set* $\Delta = \{\sigma\}$ *of rational strongly convex polyhedral cones in* $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$ *such that*

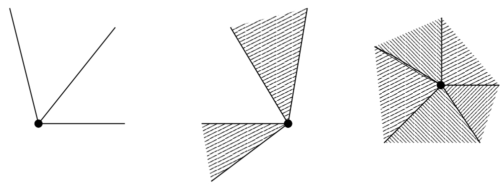
- *if* $\sigma \in \Delta$ *and* $\tau \subset \sigma$ *is a face of* σ *then* $\tau \in \Delta$,

- *if $\sigma_1, \sigma_2 \in \Delta$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 (and hence belongs to Δ by the above).*

If $V = \cup_{\sigma \in \Delta} \sigma$ then we call the fan **complete**.

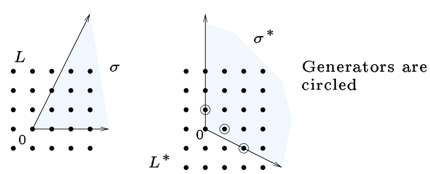
We shall assume that all fans are finite.

Example 3.2. *Some fans in $\mathbb{R}^2$:*



In the picture 2-dimensional cones are partially shaded.

Example 3.3. *Consider the fan given by the two dimensional cone σ spanned by e_1 and $e_1 + 2e_2$, and its faces, where e_1, e_2 is the standard basis for the lattice $N \cong \mathbb{Z}^2$.*



Let e_i^* be the dual basis for L^*. Then we have

$$\sigma^\vee \cup L^* = \langle e_1, e_2, 2e_1 - e_2 \rangle,$$

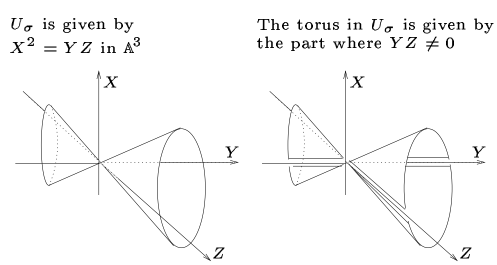
So

$$F[S_\sigma] = F[x, y, x^2/y],$$

where $x = e^{e_1}$ and $y = e^{e_2}$. If we set $X = x, Y = y$ and $Z = x^2/y$, then we have

$$F[S_\sigma] = F[X, Y, Z]/(X^2 - ZY).$$

So U_σ is a quadric cone in $\mathbb{A}^2$ given by $X^2 - ZY$.



For one 1 *dimensional face of* σ *we have*

$$\langle e_1 + 2e_2\rangle^* \cap L^* = \langle 2e_1 - e_2, -2e_1 + e_2, e_2\rangle,$$

so

$$\begin{aligned} U_{\langle e_1+2e_2\rangle} &= \operatorname{Spec}(F[y, x^2/y, y/x^2]) \\ &= \operatorname{Spec}(F[X, Y, Z, Z^{-1}]/(X^2 - ZY)) = U_\sigma \setminus \{Z = 0\} \end{aligned}$$

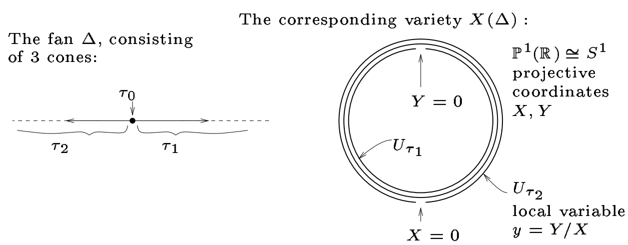
For the other 1 *dimensional cone we have*

$$U_{\langle e_1\rangle} = \operatorname{Spec}(F[x, y, 1/y]) = U_\sigma \setminus \{Y = 0\}.$$

For the zero dimensional cone, we get the embedding

$$(a, b) \mapsto (a, b, a^2/b) \in U_\sigma.$$

Example 3.4. *For the fan* Δ *in* $\mathbb{R}$ *consisting of* 3 *cones,* $\tau_0 = \{0\}$*,* $\tau_1 = \mathbb{R}_{\geq 0}$ *and* $\tau_2 = \mathbb{R}_{\leq 0}$*,*



we have

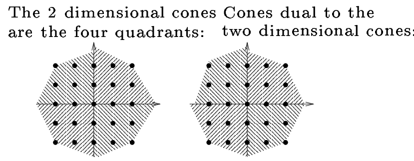
$$\begin{aligned} U_{\tau_1} &= \operatorname{Spec}(F[x]) \cong \mathbb{A}^1 \\ U_{\tau_2} &= \operatorname{Spec}(F[x^{-1}]) \cong \mathbb{A}^1 \\ U_{\tau_0} &= \operatorname{Spec}(F[x, x^{-1}]) \cong F^\times \end{aligned}$$

So $X(\Delta) \cong \mathbb{P}^1$*, with projective coordinates* X, Y*, and*

$$\begin{aligned} U_{\tau_1} &= \operatorname{Spec}(F[x]) \cong \mathbb{P}^1 \setminus Y = 0 \\ U_{\tau_2} &= \operatorname{Spec}(F[x^{-1}]) \cong \mathbb{P}^1 \setminus X = 0 \end{aligned}$$

Where the map between $\mathbb{P}^1$ *and* U_{τ_1} *is given by* $x = X/Y$*, and the map between* $\mathbb{P}^1$ *and* U_{τ_2} *is given by* $x^{-1} = Y/X$*.*

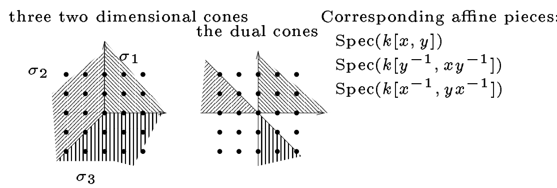
Example 3.5. *$\mathbb{P}^1 \times \mathbb{P}^1$ is given by the following fan:*



The whole situation is the "product" of the situation for $\mathbb{P}^1$ above.

More generally, for the fan Δ_n given by taking n dimensional cones spanned by $(\pm e_1, \pm e_2, \ldots, \pm e_n)$ in $\mathbb{R}^n$, the toric variety $X(\Delta_n)$ is isomorphic to the product of n copies of $\mathbb{P}_1$, i.e., $\mathbb{P}^1 \times \ldots \mathbb{P}_1$.

Example 3.6. *$\mathbb{P}^2$ is given by the following fan:*



If X, Y, Z are the projective coordinates for $\mathbb{P}^2$, then we can identify the affine pieces with the pieces with local variables

$$\sigma_1 : x = X/Z, y = Y/Z,$$

$$\sigma_2 : x^{-1} = Z/X, yx^{-1} = Y/X,$$

$$\sigma_3 : y^{-1} = Z/Y, xy^{-1} = X/Y,$$

which are the natural coordinates for the $Z \neq 0$, $X \neq 0$ and $Y \neq 0$ parts of $\mathbb{P}^2$ respectively.

More generally, $\mathbb{P}^n$ can be given as a toric variety. It is "prettier" to take the lattice to be given by the sublattice of $\mathbb{Z}^{n+1}$ given by

$$L = \{(a_0, a_1, a_2, \ldots, a_n) \in \mathbb{Z}^{n+1} | \sum a_i = 0\},$$

and to take cones σ_j spanned by $e_i - e_j$ for $j = 0, \ldots n$. So that U_{σ_j} has local coordinates x_i/x_j, where $\prod_{k=0}^n x_k = 1$. If $\mathbb{P}^n$ is a projective variety with projective coordinates X_i, then U_σ can be identified with the part where $X_j \neq 0$, via a map $x_i/x_j \mapsto X_i/X_j$.

Remark 3.7. *Some general remarks on how fans might be implemented on a computer.*

A fan is a set of cones satisfying certain conditions. Each cone σ is specified by a set [11] *of vectors $\{v_1, ..., v_k\} \subset L$:*

$$\sigma \leftrightarrow \{v_1, ..., v_k\}$$
$$\sigma = \mathbb{Q}_{\geq 0}[v_1, ..., v_k].$$

The dimension of σ is the dimension of the vector space span, $\mathrm{span}_{\mathbb{Q}}\{v_1, ..., v_k\}$.

Suppose $\sigma \leftrightarrow \{v_1, ..., v_k\}$ and $\sigma' \leftrightarrow \{v'_1, ..., v'_{k'}\}$ belong to a fan Δ. We conjecture that if the generators $v_i, v'_j \in L$ of σ are choosen to have minimum length then

$$\sigma \cap \sigma' \leftrightarrow \{v_1, ..., v_k\} \cap \{v'_1, ..., v'_{k'}\}.$$

We therefore regard a fan Δ as a set of sets of vectors V satisfying the following conditions:

- *any subset of V is in Δ,*
- *if V and V' are in Δ then $V \cap V'$ is in Δ.*

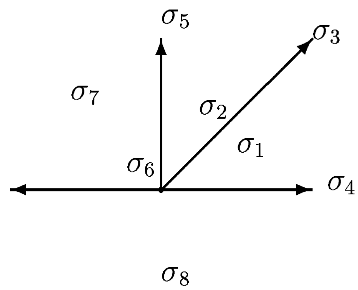
Moreover, Δ is complete if and only if $\mathbb{Q}^n = \cup_{V\in\Delta} \cup_{v\in V} \mathbb{Q}_{\geq 0}[v]$.

Example 3.8. *This is really a non-example. Let $n = 2$, $L = \mathbb{Z}^2$. Let*

$$\begin{gathered}\sigma_1 = \mathbb{Q}_{\geq 0}e_1 + \mathbb{Q}_{\geq 0}(e_1 + e_2),\\ \sigma_2 = \mathbb{Q}_{\geq 0}(e_1 + e_2) + \mathbb{Q}_{\geq 0}e_2,\\ \sigma_3 = \mathbb{Q}_{\geq 0}(e_1 + e_2),\\ \sigma_4 = \mathbb{Q}_{\geq 0}e_1,\\ \sigma_5 = \mathbb{Q}_{\geq 0}e_2,\\ \sigma_6 = \{0\},\\ \sigma_7 = \mathbb{Q}_{\geq 0}(-e_1) + \mathbb{Q}_{\geq 0}e_2,\\ \sigma_8 = \mathbb{Q}e_1 + \mathbb{Q}_{\geq 0}(-e_2).\end{gathered}$$

The set $\Delta = \{\sigma_i \mid 1 \leq i \leq 8\}$ may be visualized as follows:

[11] The empty set $\emptyset$ corresponds to the zero cone $\{0\}$.



However, it is <u>*not*</u> *a fan since σ_8 is not a strongly rational cone.*

3.2 Morphisms of fans and toric varieties

This section is related to the material in §8 below.

Definition 3.9. *Let Δ be a fan of L and Δ' a fan of L'. Given a homomorphism of lattices $\phi : L' \to L$ which maps each cone $\sigma' \in \Delta'$ to a cone $\sigma \in \Delta$, we obtain a map $\phi_* : \Delta' \to \Delta$ which we call the associated* **morphism of fans**. *A morphism of a fan to itself which is a bijection as a mapping of sets, and has an inverse which is a morphism associated to the inverse of ϕ, will be called an* **automorphism of a fan**.

Remark 3.10. *In particular, the an automorphism of a fan is associated to a automorphism of the underlying lattice. MAGMA has a command to determine the automorphism group of a lattice,* `AutomorphismGroup`.

Definition 3.11. *Let Δ_1 and Δ_2 be fans in lattices $N_1 \cong \mathbb{Z}^{r_1}$ and $N_1 \cong \mathbb{Z}^{r_2}$. Let $T_{N_i} = N_i \otimes k$ for $i = 1, 2$. Suppose there is a map of algebraic tori:*

$$\psi : T_{N_1} \to T_{N_2}.$$

A map $f : X(\Delta_1) \to X(\Delta_2)$ is called **equivariant** *(with respect to ψ and the actions of T_{N_1} and T_{N_2} if*

$$f(\lambda \cdot x) = \psi(\lambda) \cdot f(x).$$

Theorem 3.12. *Any map of fans $\phi : \Delta_1 \to \Delta_2$ gives rise to a map of the corresponding toric varieties*

$$\phi_* : X(\Delta_1) \to X(\Delta_2),$$

which is equivariant with respect to the map given by $\phi : N_1 \to N_2$. Conversely, any equivariant homomorphism between toric varieties

$$f : X(\Delta_1) \to X(\Delta_2)$$

corresponds to a unique map $\phi : N_1 \to N_2$, giving rise to a map between the fans Δ_1 and Δ_2, with

$$f = \phi_*.$$

By the above result, in order to give the equivariant automorphism group of a toric variety, we just need to consider the problem of finding the automorphisms of the corresponding fan.

Example 3.13.

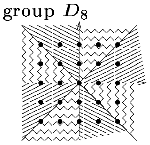
- *The fan for $\mathbb{P}^2$ has automorphism group D_3, and the corresponding automorphisms group of $\mathbb{P}^2$ is generated by*

$$(a : b : c) \mapsto (b : c : a)$$

$$(a : b : c) \mapsto (b : a : c)$$

- *Is is easy to construct two dimensional toric varieties with equivariant automorphism group the dihedral group D_n. For example, $\mathbb{P}^1 \times \mathbb{P}^1$ has automorphism group D_4, and for the fan Δ in $\mathbb{Z}^2$ with cones $\langle (0,\delta), (\epsilon,\delta) \rangle$ and $\langle (\delta, 0), (\epsilon, \delta) \rangle$, where $\epsilon, \delta = \pm 1$, the automorphism group is isomorphic to D_8. The variety $X(\Delta)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in 4 points.*

A fan with automorphism group D_8



3.3 Constructing toric varieties from a fan

We have the following recipe to construct a quasi-projective toric variety.

$$\begin{array}{ccccccc} \{\text{ fans}\} & \to & \{\text{ “fan” of comm. semigps}\} & \to & \{\text{ “fan” of semigp algs}\} & \to & \{\text{ quasi-proj schemes}\} \\ \Delta & \longmapsto & \{S_\sigma = \sigma^* \cap L^* \mid \sigma \in \Delta\} & \longmapsto & \{F[S_\sigma] \mid \sigma \in \Delta\} & \longmapsto & X(\Delta) = \{U_\sigma + \text{ gluing maps} \mid \sigma \in \Delta\}. \end{array}$$

Let Δ be a fan. If τ is a face of both $\sigma_1, \sigma_2 \in \Delta$ then Lemma 2.20 gives us maps $\phi_1 : U_\tau \to U_{\sigma_1}$ and $\phi_2 : U_\tau \to U_{\sigma_2}$. We "glue" the affine "patch" U_{σ_1} to the affine "patch" $U_{\sigma_{12}}$ the overlap using $\phi_1 \circ \phi_2^{-1}$. Define $X(\Delta)$ to be the "glued scheme" associated to these maps (as in Iitaka, §1.12, [I]):

$$X(\Delta) = (\coprod_{\sigma \in \Delta} U_\sigma)/(\text{gluing}).$$

This is the **toric variety associated to** Δ. It is known which toric varieties are complete (see the lemma below) but, in general, the classification of projective toric varieties is still an active area of research [12]. .

Lemma 3.14. *(Fulton §2.4, [F]) $X(\Delta)$ is complete (as a variety) if and only if Δ is complete (as a fan).*

Definition 3.15. *Suppose that Δ is a fan, $X = X(\Delta)$ is the associated toric variety, and that there is an injective morphism $\phi : X \to \mathbb{P}^r$ such that the torus $T = U_{\{0\}}$ in X maps to a subgroup of a torus T' which is dense in $\mathbb{P}^r$ in such a way that the action of T' on $\mathbb{P}^r$ extends that of $\phi(T)$ on $\phi(X(\Delta))$. In this case, we say that X is* **equivariantly projective**.

This basically means that there is a projective embedding which is equivariant with respect to the torus action.

Lemma 3.16. *If Δ is a 2-dimensional complete fan and if $X(\Delta)$ is smooth then it is in fact projective. In fact, $X(\Delta)$ is equivariantly projective.*

proof: These statements follow from Ewald [E], Theorem 4.7 (page 160) and Theorem 3.11 (page 277). $\square$

Example 3.17. *Let $n = 1$. Consider the fan*

$$\Delta = \{\{0\},\ \mathbb{Q}_{\geq 0},\ \mathbb{Q}_{\leq 0}\}.$$

We've seen already in Example 2.17 that

$$R_\sigma \cong \begin{cases} F[x], & \text{if } \sigma = \mathbb{Q}_{\geq 0}, \\ F[x^{-1}], & \text{if } \sigma = \mathbb{Q}_{\leq 0}, \\ F[x, x^{-1}], & \text{if } \sigma = \{0\}, \end{cases}$$

[12] See Oda [O], §2.4

and

$$U_\sigma \cong \begin{cases} F, & \text{if } \sigma = \mathbb{Q}_{\geq 0}, \\ F^\times, & \text{if } \sigma = \{0\}. \end{cases}$$

Similarly, one can show that if $\sigma = \mathbb{Q}_{\leq 0}$ *then* $U_\sigma \cong F$. *We therefore have dominant maps*

$$U_{\mathbb{Q}_{\leq 0}} \leftarrow U_{\{0\}} \rightarrow U_{\mathbb{Q}_{\geq 0}},$$

via the "obvious" embeddings

$$F[x^{-1}] \rightarrow F[x, x^{-1}] \leftarrow F[x].$$

The image of $F[x]$ *in* $F[x, x^{-1}]$ *is isomorphic to the image of* $F[x^{-1}]$ *in* $F[x, x^{-1}]$ *via the map* $x \longmapsto x^{-1}$. *This "gluing" defines the variety* $X(\Delta) = \mathbb{P}^1(F)$.

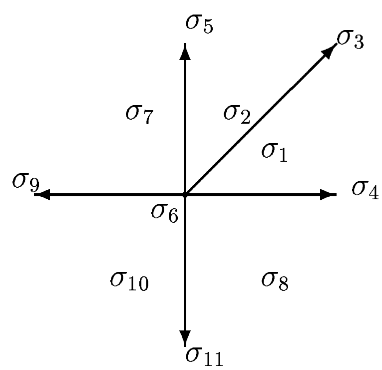
This is an example of a projective toric variety.

In general, a toric variety associated to a fan is not projective. (Recall from Lemma 3.14 that the fan is complete if and only if $X(\Delta)$ is.)

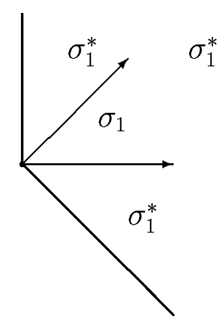
Example 3.18. *Let* $n = 2$, $L = \mathbb{Z}^2$, $L^* = \mathbb{Z}^2$. *Let*

$$\begin{gathered} \sigma_1 = \mathbb{Q}_{\geq 0} e_1 + \mathbb{Q}_{\geq 0}(e_1 + e_2), \\ \sigma_2 = \mathbb{Q}_{\geq 0}(e_1 + e_2) + \mathbb{Q}_{\geq 0} e_2, \\ \sigma_3 = \mathbb{Q}_{\geq 0}(e_1 + e_2), \\ \sigma_4 = \mathbb{Q}_{\geq 0} e_1, \\ \sigma_5 = \mathbb{Q}_{\geq 0} e_2, \\ \sigma_6 = \{0\}, \\ \sigma_7 = \mathbb{Q}_{\geq 0}(-e_1) + \mathbb{Q}_{\geq 0}(e_2), \\ \sigma_8 = \mathbb{Q}_{\geq 0} e_1 + \mathbb{Q}_{\geq 0}(-e_2), \\ \sigma_9 = \mathbb{Q}_{\geq 0}(-e_1), \\ \sigma_{10} = \mathbb{Q}_{\geq 0}(-e_1) + \mathbb{Q}_{\geq 0}(-e_2), \\ \sigma_{11} = \mathbb{Q}_{\geq 0}(-e_2). \end{gathered}$$

The set $\Delta = \{\sigma_i \mid 1 \leq i \leq 11\}$ *may be visualized as follows:*



The cones σ_7, σ_8, and σ_{10} are self-dual. The cone σ_1 and its dual σ_1^* (which contains σ_1) is pictured below.



In fact, we have

$$\begin{gathered}
\sigma_1^* = \{ae_1^* + be_2^* \mid a \geq 0,\ a+b \geq 0\} = \mathbb{Q}_{\geq 0}[e_2^*, e_1^* - e_2^*], \\
\sigma_2^* = \{ae_1^* + be_2^* \mid b \geq 0,\ a+b \geq 0\} = \mathbb{Q}_{\geq 0}[e_1^*, -e_1^* + e_2^*], \\
\sigma_3^* = \{ae_1^* + be_2^* \mid a+b \geq 0\} = \mathbb{Q}_{\geq 0}[e_1^* - e_2^*, e_1^* + e_2^*], \\
\sigma_4^* = \{ae_1^* + be_2^* \mid a \geq 0\} = \mathbb{Q}_{\geq 0}[e_1^*, e_2^*, -e_2^*], \\
\sigma_5^* = \{ae_1^* + be_2^* \mid b \geq 0\} = \mathbb{Q}_{\geq 0}[e_1^*, -e_1^*, e_2^*], \\
\sigma_6^* = \mathbb{Q}e_1^* + \mathbb{Q}e_2^*, \\
\sigma_7^* = \mathbb{Q}_{\geq 0}[-e_1^*, e_2^*], \\
\sigma_8^* = \mathbb{Q}_{\geq 0}[e_1^*, -e_2^*], \\
\sigma_9^* = \{ae_1^* + be_2^* \mid a \leq 0\} = \mathbb{Q}_{\geq 0}[-e_1^*, e_2^*, -e_2^*], \\
\sigma_{10}^* = \mathbb{Q}_{\geq 0}[-e_1^*, -e_2^*], \\
\sigma_{11}^* = \{ae_1^* + be_2^* \mid b \leq 0\} = \mathbb{Q}_{\geq 0}[e_1^*, -e_1^*, -e_2^*].
\end{gathered}$$

The associated semigroups are

$$\begin{gathered}
S_{\sigma_1} = \sigma_1^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, e_2^*, e_1^* - e_2^*], \\
S_{\sigma_2} = \sigma_2^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, e_2^*, -e_1^* + e_2^*], \\
S_{\sigma_3} = \sigma_3^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, e_2^*, e_1^* - e_2^*, -e_1^* + e_2^*], \\
S_{\sigma_4} = \sigma_4^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, e_2^*, -e_2^*], \\
S_{\sigma_5} = \sigma_5^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, -e_1^*, e_2^*], \\
S_{\sigma_6} = \sigma_6^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, -e_1^*, e_2^*, -e_2^*], \\
S_{\sigma_7} = \sigma_7^* \cap L^* = \mathbb{Z}_{\geq 0}[-e_1^*, e_2^*], \\
S_{\sigma_8} = \sigma_8^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, -e_2^*], \\
S_{\sigma_9} = \sigma_9^* \cap L^* = \mathbb{Z}_{\geq 0}[-e_1^*, e_2^*, -e_2^*], \\
S_{\sigma_{10}} = \sigma_{10}^* \cap L^* = \mathbb{Z}_{\geq 0}[-e_1^*, -e_2^*], \\
S_{\sigma_{11}} = \sigma_{11}^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^*, -e_1^*, -e_2^*].
\end{gathered}$$

(The element e_1^* in the set of generators of S_{σ_1} and S_{σ_3} are extraneous. However, their inclusion makes some comparisons mentioned later a little bit easier.) The associated coordinate rings are

$$\begin{gathered}
R_{\sigma_1} \cong F[x_1, x_2, x_1 x_2^{-1}] \cong F[x, y, w]/(wy - x) \\
\cong F[x_2, x_1 x_2^{-1}] \cong F[y, w], \\
R_{\sigma_2} \cong F[x_1, x_2, x_1^{-1} x_2] \cong F[x, y, z]/(zx - y), \\
R_{\sigma_3} \cong F[x_1, x_2, x_1^{-1} x_2, x_1 x_2^{-1}] \cong F[x, y, z, w]/(xz - y, zw - 1) \\
\cong F[x_2, x_1^{-1} x_2, x_1 x_2^{-1}] \cong F[y, z, w]/(zw - 1), \\
R_{\sigma_4} \cong F[x_1, x_2, x_2^{-1}] \cong F[x, y, v]/(vy - 1), \\
R_{\sigma_5} \cong F[x_1, x_1^{-1}, x_2] \cong F[x, u, y]/(ux - 1), \\
R_{\sigma_6} \cong F[x_1, x_2, x_1^{-1}, x_2^{-1}] \cong F[x, y, u, v]/(ux - 1, vy - 1), \\
R_{\sigma_7} \cong F[x_1^{-1}, x_2] \cong F[u, y], \\
R_{\sigma_8} \cong F[x_1, x_2^{-1}] \cong F[x, v], \\
R_{\sigma_9} \cong F[x_1^{-1}, x_2, x_2^{-1}] \cong F[u, y, v], \\
R_{\sigma_{10}} \cong F[x_1^{-1}, x_2^{-1}] \cong F[u, v], \\
R_{\sigma_{11}} \cong F[x_1, x_1^{-1}, x_2^{-1}] \cong F[x, u, v].
\end{gathered}$$

From these, one can read off the equations for U_{σ_i}, $1 \leq i \leq 11$. For example, U_{σ_6} is $ux - 1, vy - 1$, the torus $F^\times \times F^\times$. We have

$$U_{\sigma_1} \cong \{(x,y,w) \mid wy - x = 0\},$$
$$U_{\sigma_2} \cong \{(x,y,z) \mid zx - y = 0\},$$
$$U_{\sigma_3} \cong \{(y,z,w) \mid zw = 1\},$$
$$U_{\sigma_4} \cong F \times F^{\times},$$
$$U_{\sigma_5} \cong F \times F^{\times},$$
$$U_{\sigma_6} \cong F^{\times} \times F^{\times},$$
$$U_{\sigma_7} \cong F \times F,$$
$$U_{\sigma_8} \cong F \times F,$$
$$U_{\sigma_9} \cong F \times F^{\times},$$
$$U_{\sigma_{10}} \cong F \times F,$$
$$U_{\sigma_{11}} \cong F \times F^{\times}.$$

In particular, all these affine "patches" are non-singular.

We have $\sigma_3 \subset \sigma_1$ and $\sigma_3 \subset \sigma_2$. By Lemma 2.20, there should be corresponding morphisms between the associated affine toric varieties. (These morphisms are used to "glue" these affine pieces together.) We have U_{σ_3} is $zw-1=0$, which is the equation needed to compare U_{σ_2}, which is $xz-y=0$, with the equation for U_{σ_1}, which is $wy-x=0$, on the intersection of the two. We have maps

$$U_{\sigma_1} \leftarrow U_{\sigma_3} \rightarrow U_{\sigma_2}.$$

The map $z \longmapsto w^{-1}$ is the "gluing" map on the overlap.

As another example, we have $\sigma_6 \subset \sigma_i$, $1 \le i \le 11$. We have birational maps

$$U_{\sigma_6} \rightarrow U_{\sigma_i}, \quad 1 \le i \le 11.$$

This gives the dense embedding of the torus in $X(\Delta)$.

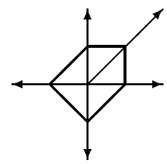
It is known that any non-singular complete surface is projective (see for example, Theorem 8.9 in Iitaka [I]), so

$$X(\Delta) = (\coprod_{\sigma \in \Delta} U_\sigma)/(\text{gluing}),$$

is projective (it is complete since the fan Δ is a complete fan).

It is in fact, equivariantly projective since the fan Δ is strongly polytopal [13] *with supporting polytope pictured below.*

[13] In the sense of Ewald [E], Definition 43 in §V4, page 159, and Theorem 3.11, §VII, page 277. In two dimensions, this basically means that the points in each wall of the polytope generate (as a cone) the cones in the fan and the vertices of the polytope generate (as a cone) the separating rays between the cones of the fan.



Let $\phi : L' \to L$ be a homomorphism of lattices and let $\phi_{\mathbb{Q}}$ denote its extension to $V' \to V$ by linearity. Let Δ be a fan of L and Δ' a fan of L' such that for each cone $\sigma' \in \Delta'$ there is a cone $\sigma \in \Delta$ such that $\phi_{\mathbb{Q}}(\sigma') \subset \sigma$. Recall from Definition 3.9 that such a $\phi_{\mathbb{Q}}$ is called a morphism of fans.

The next result recalls a generalization of Lemma 2.11

Lemma 3.19. *(Fulton [F], page 23 and Ewald [E], Theorem 6.1, page 243) Let $\phi_{\mathbb{Q}}$ be a mapping of fans as above. Then the morphisms*

$$\phi^* : \mathrm{Spec}(F[S_\sigma]) \to \mathrm{Spec}(F[S_{\sigma'}]),$$

of Lemma 2.11 piece together to give a morphism $\phi^* : X(\Delta) \to X(\Delta')$.

A morphism $\phi^* : X(\Delta) \to X(\Delta')$ arising as in the above lemma from a homomorphism $\phi : L' \to L$ be a homomorphism of lattices will be called a **toric morphism**.

Let T be a torus which embeds as a dense subvariety of $X = X(\Delta)$. Identify T with its image in X. It turns out that a toric morphism $\psi : X \to X'$ is T-equivariant [14].

3.4 The complement of the torus—stars

As mentioned earlier, all toric varieties over F of dimension n contain $(F^\times)^n$ as a dense open subset, so it is the the complement $X(\Delta) \setminus (F^\times)^n$ that distinguishes them (and how it "glues" to $(F^\times)^n$).

Definition 3.20. *The* **orbit** *of a point p in an n dimensional toric variety $X(\Delta)$ is the set of points $a \cdot x$ where $a \in (F^\times)^n$.*

A toric variety decomposes into a disjoint set of orbits. One of the orbits is $(F^\times)^n$ as a subset of $X(\Delta)$, and the other orbits form the complement.

[14] In the sense that $\psi(t)\psi(x) = \psi(\mu(t)x)$, where $\mu : T \to \mathrm{Aut}(X)$ is the natural action of T on X. See Ewald [E], Theorem 6.4, for a proof.

For example, for $\mathbb{P}^2$ we have the decomposition

$$\mathbb{P}^2 = \begin{cases} \{(a:b:c)|abc \neq 0\} \cong (F^\times)^2 \\ \cup\{(0:b:c)|bc \neq 0\} \cup \{(a:0:c)|ac \neq 0\} \cup \{(a:b:0)|ab \neq 0\} \\ \cup\{(1:0:0)\} \cup \{(0:1:0)\} \cup \{(0:0:1)\} \end{cases} .$$

The orbits are in one to one correspondence with the cones. For each d dimensional cone there is a corresponding $n-d$ dimensional orbit. If F is algebraically closed, the (Zariski) closures of the orbits are themselves toric varieties (see Lemma 3.23 below).

Definition 3.21. *For a d dimensional cone σ in a fan Δ in $N \cong \mathbb{Z}^n$, the* **star** *of σ is defined as follows. Take any projection π along σ to a lattice N_σ of dimension $n-d$. In other words, there is a linear map*

$$\pi : N \rightarrow N_\sigma,$$

with $\ker \pi = \mathbb{Z}\sigma$. This map extends to a map from $N_\mathbb{Q}$ to $(N_\sigma)_\mathbb{Q}$ which I also denote by π. Then define [15] *the fan Δ_σ in N_σ to be the set of images of all cones τ with $\sigma \prec \tau$ under the projection. In other words, we have*

$$\Delta_\sigma = \{\pi(\tau) | \tau \in \Delta,\ \sigma \prec \tau\}.$$

MAGMA/GAP Example 3.22. *To compute the star of a cone σ in a fan Δ, use the* `star` *command in the* **toric** *package.*

Consider the fan Δ determined by the cones

$$\{\mathbb{Q}_{\geq 0}[(2,-1),(1,0)], \mathbb{Q}_{\geq 0}[(1,0),(1,1)], \mathbb{Q}_{\geq 0}[(1,1),(2,0)]\}$$

and let $\sigma = \mathbb{Q}_{\geq 0}[(2,-1),(1,0)]$. The star of σ is the set $\{\sigma\}$, since it is a cone of maximal dimension in Δ. This, as well as some other examples, are computed below using GAP.

```
gap> RequirePackage("guava");
true
gap> Read("c:/gap/gapfiles/toric.g");
 toric.g for GAP 4.3, wdj, version 7-11-2002
available functions: dual_semigp_gens, cart_prod_lists,
```

[15] Note, though the concept of fan is standard, the notation Δ_σ used here may not be.

```
 in_dual_cone, max_vectors, inner_product, toric_points,
 ideal_affine_toric_variety, embedding_affine_toric_variety,
 toric_code, toric_codewords, divisor_polytope,
divisor_polytope_lattice_points, riemann_roch, flatten,
faces, in_cone, normal_to_hyperplane, number_of_cones_dim ,
subcones_of_fan, star, betti_number, cardinality_of_X,
euler_characteristic, ...
gap>
gap> Cones2:=[[[2,-1],[1,0]],[[1,0],[1,1]],[[1,1],[2,0]]];
[ [ [ 2, -1 ], [ 1, 0 ] ], [ [ 1, 0 ], [ 1, 1 ] ], [ [ 1, 1 ], [ 2, 0 ] ] ]
gap> star([[1,0]],Cones2);
[ [ [ 1, 0 ] ], [ [ 2, -1 ], [ 1, 0 ] ], [ [ 1, 0 ], [ 1, 1 ] ] ]
gap> star([[1,0],[2,-1]],Cones2);
[ [ [ 2, -1 ], [ 1, 0 ] ] ]
gap>
gap> Cones3:=[ [ [2,0,0],[0,2,0],[0,0,2] ], [ [2,0,0],[0,2,0],[1,1,-2] ] ];
[ [ [ 2, 0, 0 ], [ 0, 2, 0 ], [ 0, 0, 2 ] ],
  [ [ 2, 0, 0 ], [ 0, 2, 0 ], [ 1, 1, -2 ] ] ]
gap> star([[2,0,0]],Cones3);
[ [ [ 2, 0, 0 ] ], [ [ 0, 0, 2 ], [ 2, 0, 0 ] ], [ [ 0, 2, 0 ], [ 2, 0, 0 ] ],
  [ [ 1, 1, -2 ], [ 2, 0, 0 ] ], [ [ 2, 0, 0 ], [ 0, 2, 0 ], [ 0, 0, 2 ] ],
  [ [ 2, 0, 0 ], [ 0, 2, 0 ], [ 1, 1, -2 ] ] ]
gap> star([[2,0,0],[0,2,0]],Cones3);
[ [ [ 0, 2, 0 ], [ 2, 0, 0 ] ], [ [ 2, 0, 0 ], [ 0, 2, 0 ], [ 0, 0, 2 ] ],
  [ [ 2, 0, 0 ], [ 0, 2, 0 ], [ 1, 1, -2 ] ] ]
gap>
```

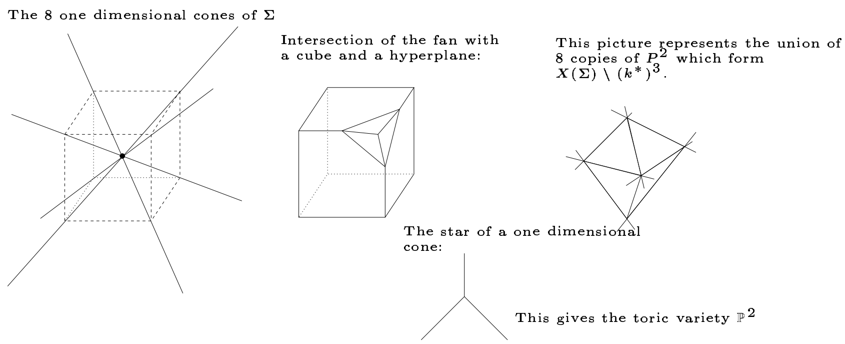
The following results may be found in §3.1 in [F].

Proposition 3.23. *Assume F is algebraically closed. The closures of the orbits of points in $X(\Delta)$ are in correspondence with the cones of Δ, and are given by $X(\Delta_\sigma)$ for $\sigma \in \Delta$.*

Corollary 3.24. *Assume F is algebraically closed. The closures of the components of the complement of $(F^\times)^n$ in a complete toric variety $X(\Delta)$ are isomorphic to the toric varieties $X(\Delta_\sigma)$ where σ runs through all the one dimensional cones in Δ.*

Example 3.25. *Consider the fan Δ in $\mathbb{Z}^3$ with one dimensional cones are generated by $(\pm 1, \pm 1, \pm 1)$. These points can be pictured as the vertices of a cube with center at the origin. For each face of the cube we define a three-dimensional cone generated by the vectors at the vertices of the face. So there are 6 three-dimensional faces, and $X(\Delta)$ is complete, and has 6 singularities.*

The star of each one dimensional cone contains 3 *two-dimensional cones, and gives a toric variety isomorphic to* $\mathbb{P}^2$.



The picture shows how these copies of $\mathbb{P}^2$ *intersect. The intersections correspond to the* 2 *dimensional cones of* Δ.

4 Polyhedra and support functions

In this section we describe a useful way of constructing toric varieties using polyhedra.

4.1 Toric varieties from polyhedra

An alternative way to define a toric variety is, instead of starting with a fan in a lattice N, to start with a polyhedron in the dual lattice M. This method has limitations, in that it can only produce complete projective toric varieties, but nevertheless, it is worth considering. In particular there are some nice results about certain subvarieties of toric varieties, which come down to counting points in faces of the corresponding polyhedra.

Definition 4.1. *Let N be a lattice of rank n, and P a convex polyhedron in $N_{\mathbb{Q}}$ with vertices at points in N.*

Then if F is a face of P of dimension d, we define a cone $\widehat{\sigma}(F)$ in L^* by

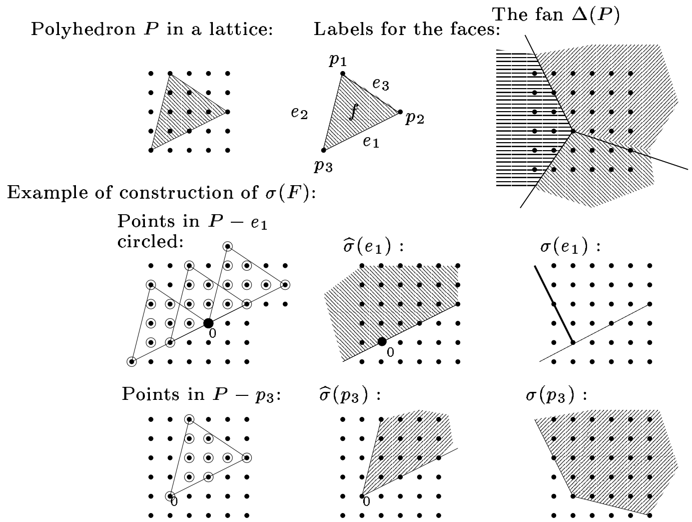
$$\widehat{\sigma}(F) = \{\lambda(x-y) \mid x \in P, y \in F, \lambda \in \mathbb{Q}\}$$

Then we define a cone $\sigma(F)$ in $N_{\mathbb{Q}}$ by taking the dual:

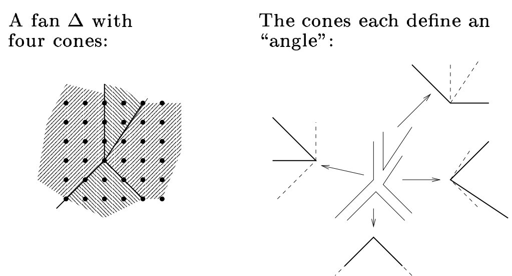
$$\sigma(F) = (\widehat{\sigma}(F))^*.$$

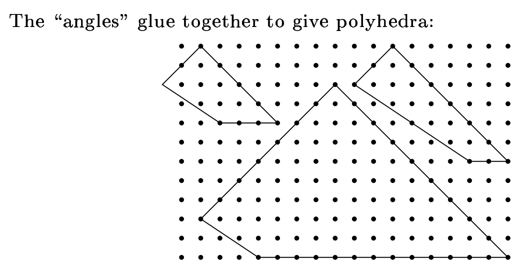
The fan $\Delta(P)$ *is given by*

$$\Delta(P) = \{\sigma(F) : F \text{ a face of } P\}.$$



Conversely, for a fan we can find a corresponding polyhedron. However, this is not unique, as shown in the following examples:





Three polyhedra Δ with $\Delta(P) = \Delta$

A polyhedra gives more structure, and can be used to define a "polarization" of the toric variety. This corresponds to taking a support function on a fan, which we will see later. A fan with support function gives rise to a unique polyhedra.

Remark 4.2. *We do not have to define a fan to get a toric variety from a polyhedron in a lattice. We can go directly from a polyhedron to a toric variety as follows:*

Definition 4.3. *For a polyhedron P in a lattice L^*, define a cone C_P on P by*

$$C_P = \{0\} \cup \{(u, \boldsymbol{v}) \in \mathbb{Q} \oplus L^* \mid u > 0, (\boldsymbol{v}/u) \in P\}.$$

Then let

$$S_P = F[C_P]$$

be the F-algebra generated by C_P. As before, we use variables y_i when we consider elements of L^* multiplicatively, writing

$$e^{a_1 \boldsymbol{e}_1 + \ldots + a_n \boldsymbol{e}_n} = y_1^{a_1} y_2^{a_2} \ldots y_n^{a_n}.$$

The algebra S_P has a grading inherited from the grading on $k[y_1, y_1^{-1}, \ldots y_n, y_n^{-1}]$. We define the toric variety associated to P by

$$X(P) = \operatorname{Proj}(F[S_P]),$$

where Proj *of a ring means the collection of homogenous prime ideals in the ring.*

To go from a fan to a polyhedron, we need to define support functions.

4.2 Support functions

This section is based on Chapter 2 of Oda's book [O].

Definition 4.4. *Let Δ be a fan in a lattice L. A* **Δ-linear support function** *is a real valued function*

$$h : |\Delta| \rightarrow \mathbb{R},$$

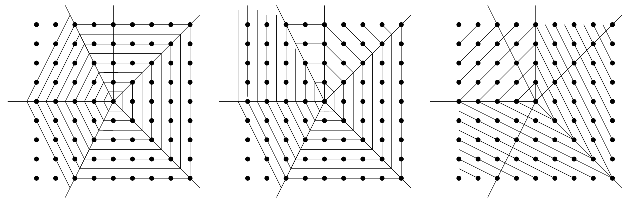
such that

$$h : |\Delta| \cap N \rightarrow \mathbb{Z},$$

and h is linear on all cones $\sigma \in \Delta$.

This means that if $\sigma, \tau \in \Delta$, then $h|_\sigma$ and $h|_\tau$ are linear functions, which are equal on $\sigma \cap \tau$. To illustrate a support function on a fan Δ we can draw lines where the fan has constant value 0, 1, 2, and so on, as in the examples in the following pictures:

Lines of constant integer values for 3 different support functions for the same fan



In the above examples the value of the support function is always positive, but this is not necessary. Any element of the dual, L^* gives a support function on any fan Δ in L.

A support function can be defined by specifying for each cone σ in Δ an element $h_\sigma \in L^*$ such that

$$h(n) = \langle n, h_\sigma \rangle \tag{2}$$

for all $n \in \sigma$, and such that $\langle n, h_\sigma \rangle = \langle n, h_\tau \rangle$ for $n \in \tau \prec \sigma$.

Definition 4.5. *For a fan Δ, the set $SF(L, \Delta)$ is the set of all Δ-linear support functions.*

A support function is determined by its values on the one dimensional cones of a fan. If we let

$$\Delta(1) = \{\sigma \in \Delta | \dim \sigma = 1\},$$

then we have

$$SF(\Delta) \cong \mathbb{Z}^{\Delta(1)}. \tag{3}$$

(See §2.1 in [O].)

4.3 From a fan with support function to a polyhedron

Definition 4.6. *Suppose Δ is a complete fan in $L^* \cong \mathbb{Z}^n$, with support function h. Let $\Delta(n)$ be the set of n dimensional cones in Δ. Then we define a polyhedron $P(\Delta, h)$ by*

$$P(\Delta, h) = \{m \in L^*_{\mathbb{Q}} \mid \langle m, n \rangle \geq -h(n),\ \forall n \in L_{\mathbb{Q}}\} = \bigcap_{\sigma \in \Delta(n)} (-h_\sigma + \sigma^\vee),$$

where h_σ satisfies (2).

To see the second equality above, note that if $m = -h_\sigma + s$ for some $s \in \sigma^*$, we have for $n \in \sigma$ that

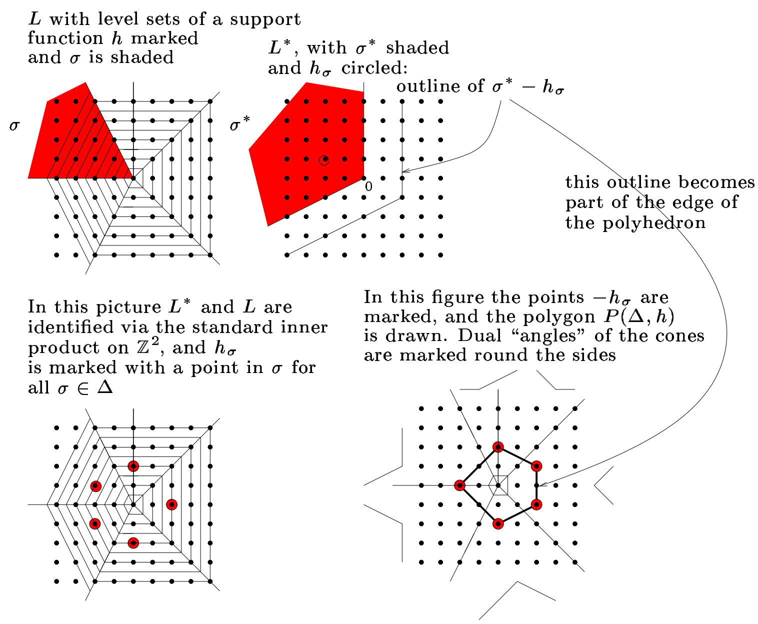
$$\langle m, n\rangle = \langle -h_\sigma + s, n\rangle = \langle -h_\sigma, n\rangle + \langle s, n\rangle \geq \langle h_\sigma, n\rangle = h(n),$$

So $\bigcap_{\sigma\in\Delta(n)}(-h_\sigma + \sigma^\vee) \subset P(\Delta, h)$. Conversely, if $\langle m, n\rangle \geq -h(n)$ for all $n \in \sigma \in \Delta$, then for all $n \in \sigma$ we have

$$\langle m + h_\sigma, n\rangle = \langle m, n\rangle + h(n) \geq 0$$

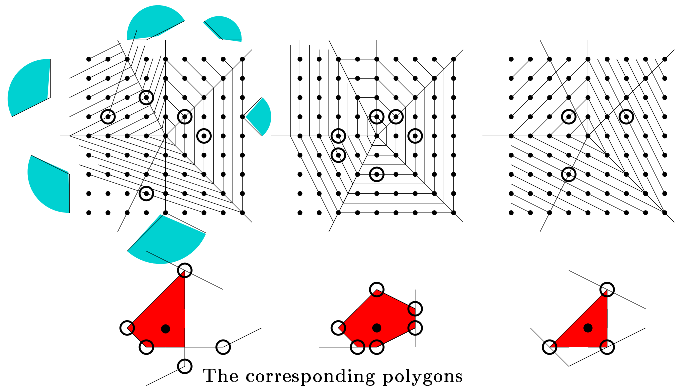
so $m + h_\sigma \in \sigma^\vee$, i.e., $m \in -h_\sigma + \sigma^\vee$, which gives the reverse inclusion.

Example 4.7. *In this diagram we show how to construct the polyhedron $P(\Delta, h)$ for a fan with support.*



Here are 3 more examples:

Lines of constant integer values for 3 different support functions for the same fan
Angles of dual cones indicated, and points h_σ marked (L and L^* are identified).



The corresponding polygons

In the above four examples we see that in only one case of h do we get a polyhedron which will satisfy $\Delta(P(\Delta, h)) = \Delta$. In a theorem below we will give the condition we need on h to have this property.

4.4 From polyhedra to fans with support function

We saw how to get from a polyhedron to a fan. Here we try to reverse that direction.

Definition 4.8. *A support function h on $N \cong \mathbb{Z}^n$ is* **upper convex** *if*

$$h(n_1 + n_2) \leq h(n_1) + h(n_2).$$

If in addition

$$h_{\sigma_1} \neq h_{\sigma_2},$$

for σ_1, σ_2 two n dimensional cones in Δ, then h is called **strictly upper convex**.

Fulton [F] uses the same terminology but omits the adjective "upper".

A polyhedron P in L^* also gives a strictly upper convex support function on the fan $\Delta(P)$ in L, which is defined on L by:

$$h_P(n) = -\inf\{\langle m, n\rangle \mid m \in P\}.$$

Theorem 4.9. *For a support function h on a fan Δ, we have*

$$\Delta(P(\Delta, h)) = \Delta$$

if and only if h is strictly upper convex.

For further details on the above result, see Lemma 2.12 and Theorem 2.22 in [O].

We also have the following important result:

Theorem 4.10. *For a (finite) complete fan Δ, the toric variety $X(\Delta)$ is projective (i.e., can be embedded by rational functions in $\mathbb{P}^m$ for some m) if and only if there is a strictly upper convex support function on Δ.*

For further details, see Theorem 2.13 and Corollary 2.16 in [O].

Note that a polyhedron P in L^* defines a finite complete fan with a strictly upper convex support function, h_P, so $X(\Delta(P))$ is projective. If we want to restrict attention to projective toric varieties, we can restrict ourselves to toric varieties given by polyhedra. To understand where the above result comes from, we need to look at linear systems (or line bundles or sheaves) on $X(\Delta)$ defined by support functions.

5 Resolution of singularities for surfaces

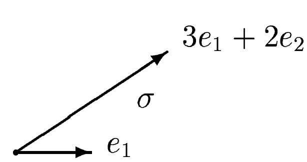
In this section, we describe in some detail how to resolve singularities of an affine toric variety.

5.1 Singular examples

First, we shall give some examples of singular toric surfaces.

5.1.1 Example 1

Consider the cone σ generated by e_1 and $3e_1 + 2e_2$. This is a "cone in L", where $L = \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z}^2$. The angle between the generating vectors is less than $\pi/2$, so we shall call this an "acute cone."



We have $\sigma = \mathbb{Q}_{\geq 0}[e_1, 3e_1 + 2e_2]$, $\sigma^* = \mathbb{Q}_{\geq 0}[2e_1^* - 3e_2^*, e_2^*]$, so

$$S_\sigma = \mathbb{Z}_{\geq 0}[e_2^*, e_1^* - e_2^*, 2e_1^* - 3e_2^*].$$

This implies that $F[S_\sigma] = F[x_2, x_1x_2^{-1}, x_1^2x_2^{-3}] \cong F[x, y, z]/(y^2 - xz)$, so U_σ is the (singular) cone $y^2 = xz$.

To resolve this, we try to choose a subcone (more precisely, a refinement of the fan associated to σ) $\tau \subset \sigma$ and we try to find a morphism $\pi : U_\tau \to U_\sigma$ such that U_τ is smooth and π is birational.

Let $\tau = \mathbb{Q}_{\geq 0}[e_1, 2e_1 + e_2]$, so $\tau^* = \mathbb{Q}_{\geq 0}[e_1^* - 2e_2^*, e_2^*]$, and

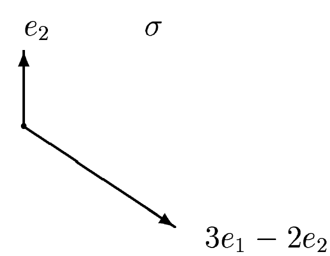
$$S_\tau = \tau^* \cap L^* = \mathbb{Z}_{\geq 0}[e_1^* - 2e_2^*, e_2^*].$$

This implies $F[S_\tau] = F[x_2, x_1x_2^{-2}]$, so $U_\tau \cong F^2$. Since $U_\sigma = F[x_2, x_1x_2^{-1}, x_1^2x_2^{-3}] \cong F[x, y, z]/(y^2 - xz)$, $(X, Y) \longmapsto (X, XY, XY^2)$ defines a morphism $U_\tau \to U_\sigma$. This is the resolution π.

5.1.2 Example 2

Consider the cone σ generated by e_2 and $3e_1 - 2e_2$. This is a "cone in L", where $L = \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z}^2$.

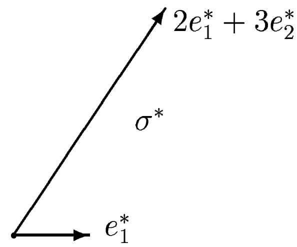
The angle between the generating vectors is greater than $\pi/2$, so we shall call this an "obtuse cone."



The dual cone σ^* is given by

$$\sigma^* = \{ae_1^* + be_2^* \mid b \geq 0,\ \frac{3}{2}a \geq b\},$$

so it is generated by e_1^* and $2e_1^* + 3e_2^*$.



The associated semigroup S_σ is given by

$$S_\sigma = \sigma^* \cap L^*,$$

and is generated (as a semigroup) by

$$u_1 = e_1^*, \quad u_2 = 2e_1^* + 3e_2^*, \quad u_3 = e_1^* + e_2^*.$$

What are the equations defining the affine variety $U_\sigma = \operatorname{Spec} F[S_\sigma]$ associated to the cone σ?

To determine these equations, we use Lemma 2.14 above.

For our example, let $c_i = a_i - b_i$, $i = 1, 2, 3$, in the notation of Lemma 2.14. In this case, (1) becomes

$$\vec{0} = c_1u_1+c_2u_2+c_3u_3 = c_1e_1^*+c_2(2e_1^*+3e_2^*)+c_3(e_1^*+e_2^*) = (c_1+2c_2+c_3, 3c_2+c_3).$$

This implies $3c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$, so (subtracting) $c_1 = c_2$. In other words,

$$a_1 - b_1 = a_2 - b_2, \quad 3(a_2 - b_2) = b_3 - a_3.$$

In particular, if $a_1 = b_1$ then we must also have $a_2 = b_2$ and $a_3 = b_3$, so $y_1^{a_1}y_2^{a_2}y_3^{a_3} - y_1^{b_1}y_2^{b_2}y_3^{b_3} = 0$. Moreover, if $a_1 > b_1$ then $a_2 > b_2$ and $a_3 < b_3$. For example, if we take

$$a_1 = 1,\ b_1 = 0,\ a_2 = 1,\ b_2 = 0,\ a_3 = 0,\ b_3 = 3,$$

then $y_1y_2 - y_3^3$ is a generator of I. Since the value of c_1 determines those of c_2 and c_3, we have

$$I = (y_1y_2 - y_3^3),$$

so

$$U_\sigma = \{(y_1, y_2, y_3) \mid y_1y_2 = y_3^3\}.$$

Moreover, in the notation of Fulton [F], §1.3, we have

$$\chi^{u_1} = x_1,\ \chi^{u_2} = x_1^2 x_3^3,\ \chi^{u_3} = x_1 x_2,$$

so the coordinate ring of U_σ is given by

$$F[S_\sigma] = F[x_1, x_1^2 x_3^3, x_1 x_2],$$

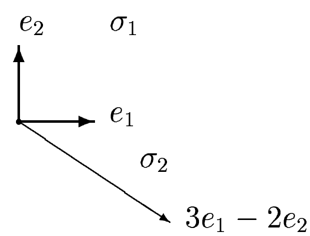
by Lemma 2.14.

From this equation for U_σ, we see that it has a singularity at $(0,0,0)$ (the Jacobian of $y_1 y_2 - y_3^3$ vanishes at the origin). How can the surface U_σ be desingularized?

To this end, consider the two subcones of σ,

$$\sigma_1 = \mathbb{Q}_{\geq 0} e_1 + \mathbb{Q}_{\geq 0} e_2,$$

$$\sigma_2 = \mathbb{Q}_{\geq 0}(3e_1 - 2e_2) + \mathbb{Q}_{\geq 0} e_1.$$



We shall see that the varieties associated to these subcones have birational morphisms satisfying

$$U_{\sigma_1} \to U_\sigma, \quad U_{\sigma_2} \to U_\sigma.$$

First, we determine the equations defining $U_{\sigma_1}, U_{\sigma_2}$ as affine varieties. To do this, we compute the dual cones and then use Lemma 2.14 as we did for U_σ above. The dual cone for σ_1 is easy

$$\sigma_1^* = \mathbb{Q}_{\geq 0} e_1^* + \mathbb{Q}_{\geq 0} e_2^*.$$

The dual cone for σ_2 is given by

$$\begin{aligned}\sigma_2^* = \{ae_1^* + be_2^* \mid (ae_1^* + be_2^*)e_1 \geq 0,\ (ae_1^* + be_2^*)(3e_1 - 2e_2) \geq 0\} \\ = \{ae_1^* + be_2^* \mid a \geq 0,\ \tfrac{3}{2}a - b \geq 0\} \\ = \mathbb{Q}_{\geq 0}(-e_1^*) + \mathbb{Q}_{\geq 0}(2e_1^* + 3e_2^*).\end{aligned}$$

Next, we determine the semigroups S_{σ_1} and S_{σ_2}. We have

$$S_{\sigma_1} = \sigma_1^* \cap L^* = \mathbb{Z}_{\geq 0} e_1^* + \mathbb{Z}_{\geq 0} e_2^*,$$

and

$$S_{\sigma_2} = \sigma_2^* \cap L^* = \mathbb{Z}_{\geq 0}(-e_2^*) + \mathbb{Z}_{\geq 0}(2e_1^* + 3e_2^*) + \mathbb{Z}_{\geq 0}(e_1^* + e_2^*).$$

In the first case,

$$F[S_{\sigma_1}] \cong F[x_1, x_2],$$

which is the coordinate ring of our variety U_{σ_1}. Thus

$$U_{\sigma_1} \cong F^2,$$

which is of course non-singular.

In the second case, S_{σ_2} is generated by $u_1 = -e_2^*$, $u_2 = 2e_1^* + 3e_2^*$, $u_3 = e_1^* + e_2^*$, so

$$F[S_{\sigma_2}] \cong F[x_2^{-1}, x_1^2 x_2^3, x_1 x_2] \cong F[y_1, y_2, y_3]/I,$$

where I is generated by

$$y_1^{a_1} y_2^{a_2} y_3^{a_3} - y_1^{b_1} y_2^{b_2} y_3^{b_3},$$

and $a_i \geq 0$, $b_j \geq 0$ satisfy

$$c_1(-e_2^*) + c_2(2e_1^* + 3e_2^*) + c_3(e_1^* + e_2^*) = \vec{0},$$

where $c_i = a_i - b_i$, $i = 1, 2, 3$. This implies $2c_2 + c_3 = 0$, $-c_1 + 3c_2 + c_3 = 0$, so (subtracting) we have $c_1 = c_2$ and $c_3 = -2c_2$, or $a_1 - b_1 = a_2 - b_2$ and $a_3 - b_3 = -2(a_2 - b_2$. If

$$a_1 = 1,\ b_1 = 0,\ a_2 = 1,\ b_2 = 0,\ a_3 = 0,\ b_3 = 2,$$

then $y_1 y_2 - y_3^2$ is a generator of I. As in the case of U_σ, we have that U_{σ_1} is given by $y_1 y_2 = y_3^2$. This is singular.

What is the relationship, if any (and there is one), between U_σ, U_{σ_1}, and U_{σ_2}?

Since σ^* is a subcone of σ_1^*, S_σ is a sub-semigroup of S_{σ_1}, so there is an inclusion homomorphism

$$F[S_\sigma] \to F[S_{\sigma_1}].$$

Indeed, the obvious map

$$F[S_\sigma] = F[x_1, x_1^2 x_2^3, x_1 x_2] \to F[S_{\sigma_1}] = F[x_1, x_2]$$

fits the bill. This implies that there is a morphism

$$U_{\sigma_1} = \mathrm{Spec}(F[S_{\sigma_1}]) \to U_\sigma = \mathrm{Spec}(F[S_\sigma]).$$

The semigroups S_{σ_1} and S_σ generate the same sub*group* of L^* (namely L^* itself). Therefore (by [F], page 18, Exercise), we see that the morphism $U_{\sigma_1} \to U_\sigma$ is birational. In fact, we can determine this morphism explicitly. To this end, recall U_σ is given by $y_1 y_2 = y_3^3$. The map

$$(y_1, y_3) \longmapsto (y_1, y_3^3/y_1, y_3),$$

sends a point of $U_{\sigma_1} - \{(y_1, y_3) \mid y_1 \neq 0\}$ to a point of $U_\sigma - \{(y_1, y_2, y_3) \in U_\sigma \mid y_1 \neq 0\}$. This is the birational morphism, $U_{\sigma_1} \to U_\sigma$.

Since σ^* is a subcone of σ_2^*, S_σ is a sub-semigroup of S_{σ_2}, so there is an inclusion homomorphism

$$F[S_\sigma] \to F[S_{\sigma_2}].$$

Indeed, $F[S_\sigma] = F[x_1, x_1^2 x_2^3, x_1 x_2]$ is contained in $F[S_{\sigma_2}] = F[x_2^{-1}, x_1^2 x_2^3, x_1 x_2]$, since $x_1 = (x_2^{-1})(x_1 x_2)$. There is a morphism

$$U_{\sigma_2} \to U_\sigma$$

which is birational (for the same reason as in the U_{σ_1} case above). In fact, we can determine this morphism explicitly. To this end, recall U_{σ_2} is given by $y_1 y_2 = y_3^2$. Send a point

$$(Y_1, Y_2, Y_3) \in U_{\sigma_2}$$

to the point

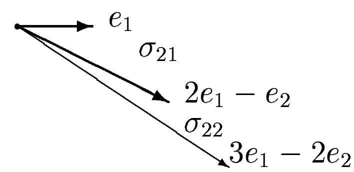
$$(y_1, y_2, y_3) = (Y_1, Y_2/Y_3, Y_3) \in U_\sigma.$$

This defines a birational morphism (defined on the subset $Y_3 \neq 0$).

Next, consider the two subcones of σ_2,

$$\sigma_{21} = \mathbb{Q}_{\geq 0} e_1 + \mathbb{Q}_{\geq 0}(2e_1 - e_2),$$

$$\sigma_{22} = \mathbb{Q}_{\geq 0}(2e_1 - e_2) + \mathbb{Q}_{\geq 0}(3e_1 - 2e_2).$$



We shall see that the varieties associated to these subcones have birational morphisms satisfying

$$U_{\sigma_{21}} \to U_{\sigma_2}, \quad U_{\sigma_{22}} \to U_{\sigma_2}.$$

To compute the equations defining $U_{\sigma_{21}}, U_{\sigma_{22}}$, we first determine the dual cones σ_{21}^*, σ_{21}^*. We have

$$\sigma_{21}^* = \{ae_1^* + be_2^* \mid a \geq 0,\ 2a \geq b\} = \mathbb{Z}_{\geq 0}(-e_2^*) + \mathbb{Z}_{\geq 0}(e_1^* + 2e_2^*)$$

and

$$\begin{aligned}\sigma_{22}^* &= \{ae_1^* + be_2^* \mid 3a \geq 2b,\ 2a \geq b\} \\ &= \mathbb{Z}_{\geq 0}(-e_1^* - 2e_2^*) + \mathbb{Z}_{\geq 0}(2e_1^* + 3e_2^*).\end{aligned}$$

These give

$$F[S_{\sigma_{21}^*}] = F[x_2^{-1}, x_1x_2^2],$$

which contains x_1, and

$$F[S_{\sigma_{22}^*}] = F[x_1^{-1}x_2^{-2}, x_1^2x_2^3].$$

Since σ_{21} is generated by e_1, $2e_1 - e_2$ (which is a basis for L), $U_{\sigma_{21}}$ is smooth, by the above lemma. In fact, $U_{\sigma_{21}} \cong F^2$. (Indeed, the ideal I associated to $U_{\sigma_{21}}$ is generated by $y_1^{a_1}y_2^{a_2}y_3^{a_3} - y_1^{b_1}y_2^{b_2}y_3^{b_3}$ where $c_1(-e_2^*) + c_2(e_1^* + 2e_2^*) = \vec{0}$, $c_i = a_i - b_i$, $i = 1, 2, 3$. These imply $c_1 = c_2 = 0$, so I is trivial.)

The ideal I for $U_{\sigma_{22}}$ is generated by the binomials $y_1^{a_1}y_2^{a_2}y_3^{a_3} - y_1^{b_1}y_2^{b_2}y_3^{b_3}$ where $c_1(-e_1^* - 2e_2^*) + c_2(2e_1^* + 3e_2^*) = \vec{0}$, $c_i = a_i - b_i$, $i = 1, 2$. This implies $c_1 = 2c_2$, $2c_1 = 3c_2$. These imply $c_1 = c_2 = 0$, so I is trivial.

What is the relationship between

$$U_{\sigma_2}, U_{\sigma_{21}}, U_{\sigma_{22}}?$$

Recall U_{σ_2} was given by $y_1 y_2 = y_3^3$. The morphism

$$(y_1, y_3) \longmapsto (y_1, y_3^3/y_1, y_3)$$

defines a birational morphism

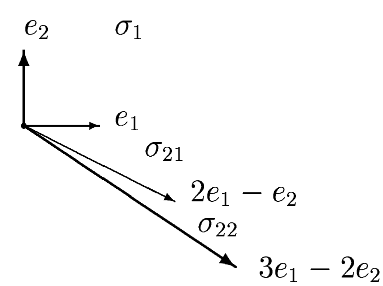
$$U_{\sigma_{21}} \to U_{\sigma_2},$$

and similarly for $U_{\sigma_{22}} \to U_{\sigma_2}$.

The composite morphism

$$U_{\sigma_{21}} \to U_{\sigma_2} \to U_\sigma$$

is a resolution of singularities. This corresponds to the refinement pictured below.



5.2 Resolution of singularities

Let Δ be a fan of $L = \mathbb{Z}^2$. A refinement Δ' of Δ yields an equivariant birational map $X(\Delta') \to X(\Delta)$. We seek here two things.

- A refinement Δ' for which $X(\Delta')$ is smooth.
- An explicit expression for the morphism $X(\Delta') \to X(\Delta)$.

To simplify matters, we shall assume that Δ is the fan associated to a cone of L. Furthermore, we shall simply replace $X(\Delta)$ by U_σ. The examples in the previous section provide a good blueprint of what to expect.

MAGMA/GAP Example 5.1. *Let $n = 2$, $L = L^* = \mathbb{Z}^2$, and suppose $\sigma = \mathbb{Q}_{\geq 0}[e_1, 3e_1 + 4e_2]$. To obtain the birational desingularization (as schemes) $F^2 \to U_\sigma$ in MAGMA, type*

```
load "/home/wdj/magmafiles/toric.mag";
//replace /home/wdj/magmafiles by your path to toric.mag
D := LatticeDatabase();
Lat := Lattice(D, 2, 16);
desing_affine_toric_variety([[1,0],[3,4]],Lat);
```

MAGMA returns

```
Mapping from: Affine Space of dimension 2
Variables : x1, x2 to Affine Space of dimension 5
Variables : $.1, $.2, $.3, $.4, $.5
with equations :
$.2
$.1
$.1^2/$.2
$.1^3/$.2^2
$.1^4/$.2^3 %$
```

Basically, this is the same as the dense embedding map, given in MAGMA 2.23, except that the notation for the variables returned by MAGMA is a little different (a MAGMA "feature", since the mapping does not belong to the coordinate ring of functions). This means that U_σ is 2-dimensional and the mapping is given by given by

$$\begin{array}{ccc} F^2 & \rightarrow & U_\sigma, \\ (x_1, x_2) & \longmapsto & (x_2, x_1, x_1^2 x_4, x_1^3 x_4^2, x_1^4 x_4^3) = (x_2, x_1, x_1^2 x_2^{-1}, x_1^3 x_2^{-2}, x_1^4 x_2^{-3}). \end{array}$$

Let $\sigma_{a,b}$ denote the following two-dimensional cone

$$\sigma_{a,b} = \mathbb{Q}_{\geq 0}[e_1, ae_1 + be_2],$$

where $(a, b) \in \mathbb{Z}^2$, $b \neq 0$. Let $S_{a,b} = S_{\sigma_{a,b}}$ and $U_{a,b} = U_{\sigma_{a,b}}$. By Oda [O], Proposition 1.24, $U_{a,b} \cong F^2/\mu_b$ (see also Exercise 2.10 above).

Lemma 5.2. *(a) $\sigma_{a,b} \cong \sigma_{a,-b}$ and $\sigma_{a,b} \cong \sigma_{a\pm b,b}$.*
(b) $U_{a,b} \cong U_{a,-b}$ and $U_{a,b} \cong U_{a\pm b,b}$.

proof: (a) The map $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(\mathbb{Z})$ defines an isomorphism $\sigma_{a,b} \to \sigma_{a,-b}$ and $A = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$ defines an isomorphism $\sigma_{a,b} \to \sigma_{a\pm b,b}$.

(b) This follows from Lemma 2.11 using (a). $\square$

The following result shows that, in dimension 2, "most" affine toric varieties are of the above form (up to isomorphism).

Lemma 5.3. *Let $a, b, c, d \in \mathbb{Z}$ with $gcd(a, b) = 1$ (in particular, $a \neq 0$, $b \neq 0$), and let $\sigma = \mathbb{Q}_{\geq 0}[(a, b), (c, d)]$ be a 2-dimensional cone. There are $e, f \in \mathbb{Z}$ such that $\sigma \cong \sigma_{e,f}$, hence $U_\sigma \cong U_{e,f}$.*

proof: First, if $gcd(a, b) = r$ then we may replace a by a/r and b by b/r, since $\mathbb{Q}_{\geq 0}[(a, b), (c, d)] = \mathbb{Q}_{\geq 0}[(a/r, b/r), (c, d)]$ (they are equal, not just isomorphic). So, we may assume without loss of generality that $gcd(a, b) = 1$. In this case (by a standard result in elementary number theory), there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & -b \\ y & a \end{pmatrix} = \begin{pmatrix} 1 & b' \\ c' & d' \end{pmatrix},$$

for some $b', c', d' \in \mathbb{Z}$. Note $\begin{pmatrix} x & -b \\ y & a \end{pmatrix} \in GL_2(\mathbb{Z})$. Finally, note

$$\begin{pmatrix} 1 & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & b' \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c' & d'' \end{pmatrix},$$

for some $d'' \in \mathbb{Z}$. Let $e = c'$, $f = d''$. $\Box$

Because of the following lemma, we may dispose of the case $b = \pm 1$.

Lemma 5.4. *If $\sigma = \sigma_{a,1}$ or $\sigma = \sigma_{a,-1}$ then $U_\sigma \cong F^2$. (In particular, U_σ is smooth.*

proof: Exercise (using Lemma 2.14). $\Box$

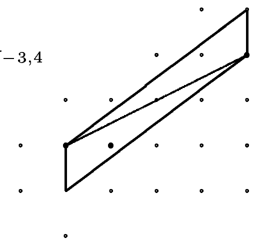
Therefore, we may assume $|n| > 1$.

Let σ_0 denote the cone generated by e_1, e_2. Let $\sigma = \sigma_{m,n}$, where $m \neq 0$ and $n \neq 0$. In this case, $U_{\sigma_0} \cong F^2$ is smooth and there is a birational resolution $\pi : U_{\sigma_0} \to U_\sigma$.

We explicitly (as is possible) determine π.

It suffices to consider the case $n > 0$ and $gcd(|m|, n) = 1$. (By Lemma 5.2 above, we may assume we are in this case.) Note $\sigma^* = \mathbb{Q}_{\geq 0}[e_2^*, ne_1^* - me_2^*]$.

Lemma 5.5. *S_σ is generated by e_2^*, $ne_1^* - me_2^*$, and elements of the form $ae_1^* + be_2^*$, $(a, b) \in J$, where J is a subset of lattice points in L contained in the parallelogram from $(0, 0)$ to $(0, 1)$ to $(n, 1 - m)$ to $(n, -m)$.*

generators for $S_{\sigma_{-3,4}}$ 

Before proving this lemma, note it implies

$$F[S_\sigma] = F[x_2, x_2^n x_2^{-m}, \{x_1^a x_2^b\}_{(a,b)\in J}].$$

If we identify U_{σ_0} with F^2, hence $F[S_{\sigma_0}]$ with $F[x_1, x_2]$, then the map

$$(x_1, x_2) \longmapsto (x_2, x_2^n x_2^{-m}, \{x_1^a x_2^b\}_{(a,b)\in J}) \tag{4}$$

defines a morphism $\pi : U_{\sigma_0} \to U_\sigma$. This is the explicit resolution.

proof: The semigroup generators of S_σ must belong to $L \cap \sigma^*$. Each such generator must in fact belong to the parallelogram spanned by the vectors $(0, 1)$ and $(n, -m)$ since the parallelogram forms a fundamental domain for $\mathbb{Q}^2/\mathbb{Z}[e_2^*, ne_1^* - me_2^*]$. $\square$

Here's a stronger version of the above lemma.

Theorem 5.6. *S_σ is generated by e_2^*, $ne_1^* - me_2^*$, and elements of the form $ae_1^* + be_2^*$, $(a, b) \in J$, where J is contained in the triangle from $(0, 0)$ to $(0, 1)$ to $(n, -m)$.*

proof: By the above lemma, we know that the generators of S_σ belong to the parallelogram generated by the vectors e_2^*, $ne_1^* - me_2^*$. Let P be this parallelogram, let T be the triangle of the theorem, and let w denote the vertex $e_2^* + ne_1^* - me_2^*$. To prove the theorem, we need to show that: if $v \in P \cap L$ then there are $t_i \in T \cap L$, $i = 1, 2$, such that $v = t_1 + t_2$. Symbolically, we need to show: if $v \neq w$ then

$$T \cap L \cap (v - T \cap L) \neq \emptyset.$$

By symmetry, this is equivalent to the following *claim*: if $v \in w - T \cap L$, $v \neq w$, then

$$(w - T \cap L) \cap (w - v + T \cap L) \neq \emptyset.$$

This is geometrically "obvious". To prove it, suppose not. Assume

$$(w - T \cap L) \cap (w - v + T \cap L) = \emptyset.$$

Let v be a vector satisfying this and where $w - v$ is as small as possible (lexicographically). But

$$(w - T \cap L) \cap (v_0 + T \cap L) \neq \emptyset,$$

where $v_0 = (1, 1)$. Since v_0 is the smallest vector in $T \cap L$, our assumption must be false. $\square$

6 Riemann-Roch spaces

In this section we will assume X is non singular, so that Weil diviors and Cartier divisors are the same. (The space of Cartier divisors is the subspace of locally principal Weil divisors.)

6.1 Divisors and Linear Systems

Definition 6.1. *For an algebraic variety X the group of* **Weil divisors** *on X is (roughly) given by*

$$\mathrm{Div}(X) = \mathbb{Z}[\{\text{irreducible subvarieties of } X \text{ of codimension } 1\}].$$

Previously we have mentioned orbits of points in toric varieties. For a cone $\sigma \in \Delta$ we define

$$\mathrm{orb}(\sigma) = \{\text{unique } T - \text{orbit in } U_\sigma \text{ which is closed in } U_\sigma\},$$

where as usual closed is with respect to the Zariski topology.

For example, for $\mathbb{P}^2$, with homogenous coordinates X, Y, Z, one of the affine pieces corresponding to a cone is $X \neq 0$. In this piece the orbit $XYZ \neq 0$ is not closed, since it is not defined by the vanishing of any finite set of polynomials. But the subvariety given by the point $(1 : 0 : 0)$ is closed and T invariant. In the affine piece $XY \neq 0$, the unique T invariant divisor is given by the line $Z = 0$ (restricted to this piece).

Define a Weil divisor corresponding to σ by:

$$V(\sigma) = \text{closure of } \mathrm{orb}(\sigma).$$

Definition 6.2. *For a toric variety X with dense open torus T, a Weil divisor D is T* **invariant** *if $D = T \cdot D$. The space of T invarant Weil divisors is denoted $T\mathrm{Div}(X)$.*

Definition 6.3. *For any support function h on a fan Δ we have a corresponding T invariant divisor given by*

$$D_h = \sum_{\sigma \in \Delta(1)} h(n(\sigma))V(\sigma),$$

where $n(\sigma)$ is an element of N which generates σ. (Note, $n(\sigma)$ is only defined when σ is one dimensional.)

Theorem 6.4. *We have*

$$T\mathrm{Div}(X) = \sum_{\sigma \in \Delta(1)} \mathbb{Z}V(\sigma) \cong SF(\Delta),$$

where SF is as in Definition 4.5.

For further details, see §4.2 above, §3.3 of [F], or §2.1 in [O].

Definition 6.5. *For any rational function f on a variety X (i.e., defined by polynomials), there is a corresponding valuations on divisors given by*

$$v_f(D) = \text{order of vanishing of} f \text{ along } D.$$

If f has a pole along D then the valuation is negative.

Rather than give a proper definition of "order of vanishing", we give an example below.

Definition 6.6. *If f is a rational function on a variety X (i.e., defined by polynomials), then there is a corresponding Weil divisor, given by the free abelian group*

$$(f) = \sum_{D \in \mathrm{Div}(X)} v_f(D)D.$$

A divisor of this form is called a **principal divisor***.*

Example 6.7. *For $\mathbb{P}^2$, with projective coordinates X, Y, Z, consider the function*

$$f(X,Y,Z) = \frac{X^2(Y-3Z)}{(XY-Z^2)(X-Z)}.$$

This funtion is zero on the lines $X = 0$, $Y = 3Z$. It vanishes to order 2 on $X = 0$. It has poles along the curve $XY = Z^2$ and the line $X - Z$, so the corresponding divisor is:

$$(f) = 2(X=0) + (Y=3Z) - (XY=Z^2) - (X=Z).$$

Definition 6.8. *Two Weil divisors D_1 and D_2 are* **linearly equivalent**, *written $D_1 \sim D_2$, if there is some function f with*

$$(f) = D_1 - D_2.$$

Example 6.9. *Any two lines on $\mathbb{P}^2$ are linearly equivalent, since if $l_1(X,Y,Z)$ and $l_2(X,Y,Z)$ are linear functions defining lines L_1 and L_2, then*

$$\left(\frac{l_1}{l_2}\right) = L_1 - L_2.$$

Definition 6.10. *If D is a divisor on X, then the* **complete linear system** *defined by D is given by*

$$|D| = \{D' \in \mathrm{Div}(X) | D \sim D'\}.$$

$|D|$ has the structure of a projective space, since any $D' \in |D|$ corresponds to some function f, with $(f) = D' - D$, and we also have $f \mapsto (f) - D \in |D|$. Functions can be added together, and multiplied by elements of k, but $(cf) = (f)$ for any non zero constant c, so we quotient the vector space of non-zero functions on X by $F^\times$ to obtain the desired projective space.

Linear systems are important because they can be used to define functions from varieties to projective space. If $D_1 \sim D_2 \sim D_3$ are divisors, locally defined by the vanishing of some polynomials f_1, f_2, f_3 on some affine piece of a variety, then there is a function given (locally) by

$$x \mapsto (f_1(x) : f_2(x) : f_3(x)).$$

The fact that $D_1 \sim D_2$ means that this map can be patched together globally, with the ratio $f_1(x)/f_2(x)$ giving the value of the function locally.

Theorem 6.11. *Let X be a smooth toric variety. For a support function h on a fan Δ, the linear system $|D_h|$ defines a smooth embedding of $X(\Delta)$ in projective space if and only if h is strictly upper convex.*

This is equivalent to the lemma on page 69 of Fulton [F].

6.2 An explicit basis for $L(D)$

Let Δ be a fan in a lattice L. Denote by τ_1, ..., τ_n the edges or rays of the fan and let v_i denote the first (smallest) lattice point along the ray τ_i. Let D_i denote the Weil divisor

$$D_i = \mathrm{Hom}(\tau_i^* \cap L^*, F^\times),$$

which may be regarded as the closure of the orbit of T acting on the edge τ_i. Let

$$P_D = \{x = (x_1, ..., x_n) \mid \langle x, v_i \rangle \geq -d_i, \ \forall 1 \leq i \leq k\}$$

denote the polytope associated to the Weil divisor $D = d_1 D_1 + ... + d_k D_k$, where D_i is as above. There is a fairly simple condition which determines whether or not D is Cartier - see the exercise on page 62 of [F]. Moreover, there is a fairly simple condition which tells you whether or not D is ample - see the proof of the proposition on page 68 of [F].

If F is a topological field, a **line bundle** on an F-variety X is given by a F-manifold L with a surjective map

$$\pi : L \to X,$$

such that the inverse image $\pi^{-1}(x)$ is a one dimensional vector space over the underlying field F.

For a toric variety $X(\Delta)$, containing an open dense torus $(k^\times)^n$, a line bundle $\pi : L \to X(\Delta)$ is called an **equivariant line bundle** if $(F^\times)^n$ acts on L, and for all $a \in (F^\times)^n$ and all $v \in L$ we have

$$\pi(a \cdot v) = a \cdot \pi(v).$$

We won't go into sheaves in detail, but all invertible sheaves on (a smooth) toric variety X are defined by T-invariant Weil divisors. Moreover, certain computations of the cohomology of invertible sheaves on X boils down to combinatorial computations. For example, we have results like the following.

Theorem 6.12. *For a polyhedron P in a lattice L^*, we have a support function $h = h_P$ on the toric variety $X(\Delta(P))$, and a corresponding sheaf $\mathcal{O}(D_h)$. The space of global sections of the sheaf, $H^0(X(\Delta(P)), \mathcal{O}(D_h))$ is a finite dimensional vector space with basis given by the set of lattice points in $P \cap L^*$.*

For further details, see §3.4 in [F] or Lemma 2.3 in [O].

We define the Riemann-Roch space $L(D)$ by

$$L(D) = \Gamma(X, \mathcal{O}(D)) = H^0(X, \mathcal{O}(D))$$

(see for example Griffiths and Harris [GH], page 136, for a natural isomorphism between this space and the "usual definition"). By Fulton [F], page 66, we have

$$L(D) = \oplus_{u \in P_D} F \cdot \chi^u,$$

where $\chi = (x_1, ..., x_n)$ and χ^u is the associated monomial in multi-index notation.

Example 6.13. *We continue example 8.5 above.*

Let Δ be the fan generated by

$$v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + 2e_2, \quad v_3 = -e_1 - e_2.$$

In the notation above, the divisor $D = d_1 D_1 + d_2 D_2 + d_3 D_3$ is a Cartier divisor [16] *if and only if $d_1 \equiv d_2 \equiv d_3 \pmod 3$. Let*

$$\begin{aligned} P_D &= \{(x, y) \mid \langle (x, y), v_i \rangle \geq -d_i, \ \forall i\} \\ &= \{(x, y) \mid 2x - y \geq -d_1, -x + 2y \geq -d_2, -x - y \geq -d_3\} \end{aligned}$$

denote the polytope associated to D. The `divisor_polytope` *command in* `toric.mag` *implements an algorithm which determines the inequalities describing P_D in general (at the end of the file* `toric.mag` *[J] there are examples of this).*

If $d_1 = d_2 = 6$ and $d_3 = 0$ then P_D is a triangle in the plane with vertices at $(-6, -6)$, $(-2, 2)$, and $(2, -2)$. Note that it remains invariant under the action of G. Moreover, the G action on $P_D \cap L^*$ has 7 singleton orbits (the lattice points $(-i, -i)$, where $0 \leq i \leq 6$) and 12 orbits of size 2. A basis for the Riemann-Roch space $L(D)$ is returned by the `toric.mag` *commands:*

[16] This is an Exercise on page 65 of [F], the solution of which is an easy calculation using the Exercise on page 62, which is in turn, basically solved in the back of the book.

```
> load "/home/wdj/magmafiles/toric.mag";
Loading "/home/wdj/magmafiles/toric.mag"

toric.mag for MAGMA 2.8, wdj, version 3-2-2002
available functions: dual_semigp_gens, cart_prod_lists,
in_dual_cone, max_vectors, create_affine_space, toric_points,
ideal_affine_toric_variety, affine_toric_variety,
embedding_affine_toric_variety, desing_affine_toric_variety,
divisor_polytope, divisor_polytope_lattice_points, riemann_roch,
toric_code, toric_codewords, ...

>
> DB := LatticeDatabase();
> Lat := Lattice(DB, 2, 16);Lat;
Standard Lattice of rank 2 and degree 2
> Cones:=[[[2,-1],[-1,2]],[[-1,2],[-1,-1]],[[-1,-1],[2,-1]]];
> Div:=[6,6,0];
> RR:=riemann_roch(Div,Cones,Lat);
> RR;
[
    1/(x1^6*x2^6),
    1/(x1^5*x2^5),
    1/(x1^5*x2^4),
    1/(x1^4*x2^5),
    1/(x1^4*x2^4),
    1/(x1^4*x2^3),
    1/(x1^4*x2^2),
    1/(x1^3*x2^4),
    1/(x1^3*x2^3),
    1/(x1^3*x2^2),
    1/(x1^3*x2),
    1/x1^3,
    1/(x1^2*x2^4),
    1/(x1^2*x2^3),
    1/(x1^2*x2^2),
    1/(x1^2*x2),
    1/x1^2,
    x2/x1^2,
    x2^2/x1^2,
    1/(x1*x2^3),
    1/(x1*x2^2),
    1/(x1*x2),
    1/x1,
    x2/x1,
    1/x2^3,
```

```
    1/x2^2,
    1/x2,
    1,
    x1/x2^2,
    x1/x2,
    x1^2/x2^2
]
```

7 Application to error-correcting codes

Error-correcting codes associated to a toric variety were introduced by J. Hansen [H] in the case of surfaces. In many cases, he found good estimates for the parameters n (the length), k (the dimension), and d (the minimum distance) of the code. The estimates on d were based on more general techniques by his student S. Hansen [Han2] (in fact, this paper was originally part of his PhD thesis [Han1]).

The next section briefly recalls his J. Hansen's construction.

The section after that gives a construction which is a little more general than that in [H], though it still falls in the framework of the general class of codes constructed in [Han2].

7.1 Hansen codes

We recall briefly some codes associated to a toric surface, constructed by J. Hansen [H].

Let $F = \mathbb{F}_q$ be a finite field with q elements and let $\overline{F}$ denote a separable algebraic closure. Let L be a lattice in $\mathbb{Q}^2$ generated by $v_1, v_2 \in \mathbb{Z}^2$, P a polytope in $\mathbb{Q}^2$, and $X(P)$ the associated toric surface. Let $P_L = P \cap \mathbb{Z}^2$.

There is a dense embedding of $GL(1)\times GL(1)$ into $X(P)$ given as follows. Let $T_L = Hom_{\mathbb{Z}}(L, GL(1))$ (which is $\cong GL(1)\times GL(1)$ by sending $t = (t_1, t_2)$ to $m_1 v_1 + m_2 v_2 \longmapsto e(\ell)(t) = t_1^{m_1} t_2^{m_2}$) and let $e(\ell) : T_L \to \overline{F}$ be defined by $e(\ell)(t) = t(\ell)$, $t \in T_L$.

Impose an ordering on the set $T_L(F)$ (changing the ordering leads to an equivalent code). Define the code $C = C_P \subset F^n$ to be the linear code generated by the vectors

$$B = \{(e(\ell)(t))_{t\in T_L(F)} \mid \ell \in L \cap P_L\}, \tag{5}$$

where $n = (q-1)^2$. In some special cases, the dimension if C is known and an estimate of its minimum distance can be given (see Theorem 7.2 below).

Example 7.1. *The case $q = 2$ is trivial.*

The first non-trivial example is for $q = 3$. Let $L = \mathbb{Z}^2$, so

$$T_L(F) = \{(1,1),(1,2),(2,1),(2,2)\},$$

which we use as our ordering. Let P_L be the polytope with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

In this case,

t	*(1,1)*	*(1,2)*	*(2,1)*	*(2,2)*
$e(0,0)(t)$	*1*	*1*	*1*	*1*
$e(1,0)(t)$	*1*	*1*	*2*	*2*
$e(0,1)(t)$	*1*	*2*	*1*	*2*

Thus,

$$\begin{aligned} C &= \mathrm{span}_F\{(1,1,1,1),(1,1,2,2),(1,2,1,2)\} \\ &= \{(0,0,0,0),(1,1,1,1),(2,2,2,2),(0,1,0,1), \\ &\quad (1,2,1,2),(0,2,0,2),(2,1,2,1),(1,0,1,0), \\ &\quad (2,1,2,1),(2,0,2,0),(1,2,1,2),(0,0,1,1), \\ &\quad (0,0,2,2),(1,1,2,2),(1,1,0,0),(2,2,0,0), \\ &\quad (2,2,1,1),(2,0,0,1),(1,0,0,2),(0,2,1,0), \\ &\quad (0,1,2,0),(0,2,2,1),(0,1,1,2),(1,2,2,0), \\ &\quad (2,1,1,0),(2,1,0,2),(1,2,0,1)\}. \end{aligned}$$

It's minimum distance is 2.

Hansen gives lower bounds on the minimum distance d of such codes in the cases:

(a) P is an isoceles triangle with vertices $(0,0)$,(a,a),$(0,2a)$,

(b) P is an isoceles triangle with vertices $(0,0)$,$(a,0)$,$(0,a)$, or

(c) P is a rectangle with vertices $(0,0)$,$(a,0)$,$(0,b)$,(a,b),

provided q is "sufficient large". His precise result is recalled below.

Theorem 7.2. *(Hansen [H]) Let a, b be positive integers. Let P be the polytope defined in (a)-(c) above.*

(a) Assume $q > 2a+1$. The code $C = C_P$ has

$$n = (q-1)^2, \quad k = (a+1)^2, \quad d \geq n - 2a(q-1).$$

(b) *Assume $q > a+1$. The code $C = C_P$ has*

$$n = (q-1)^2, \quad k = (a+1)(a+2)/2, \quad d \geq n - a(q-1).$$

(c) *Assume $q > \max(a,b)+1$. The code $C = C_P$ has*

$$n = (q-1)^2, \quad k = (a+1)(b+1), \quad d \geq n - a(q-1) - b(q-1) + ab.$$

MAGMA/GAP Example 7.3. *We examine an example of part (b) of Hansen's theorem stated above. First load* `toric.mag` *as follows (replace* `/home/wdj/magmafiles` *by your path to* `toric.mag`*).*

```
> load "/home/wdj/magmafiles/toric.mag";
Loading "/home/wdj/magmafiles/toric.mag"

toric.mag for MAGMA 2.8, wdj, version 8-28-2001
available functions: dual_semigp_gens, cart_prod_lists,
in_dual_cone, max_vectors, create_affine_space, toric_points,
ideal_affine_toric_variety, affine_toric_variety,
embedding_affine_toric_variety, desing_affine_toric_variety,
toric_code, toric_codewords, ...

> Polyb:=[[0,0],[0,1],[1,0],[1,1],[0,2],[1,2],[2,0],[2,1],[0,3],[3,0]];
> C:=toric_code(Polyb,GF(3),Lat);C;
[4, 4, 1] Linear Code over GF(3)
```

The command `toric_codewords(Polyb,GF(3),Lat);` *returns the list of all codewords:*

$$\begin{gathered}
\{(0,0,0,0),(1,0,0,0),(2,0,0,0),(2,1,0,0),(0,1,0,0),\\
(1,1,0,0),(1,2,0,0),(2,2,0,0),(0,2,0,0),(0,2,1,0),(1,2,1,0),\\
(2,2,1,0),(2,0,1,0),(0,2,0,0),(0,2,1,0),(1,2,1,0),(2,2,1,0),\\
(2,0,1,0),(0,0,1,0),(1,0,1,0),(1,1,1,0),(2,1,1,0),(0,1,1,0),\\
(0,1,2,0),(1,1,2,0),(2,1,2,0),(2,2,2,0),(0,2,2,0),(1,2,2,0),(1,0,2,0),\\
(2,0,2,0),(0,0,2,0),(0,0,2,1),(1,0,2,1),(2,0,2,1),(2,1,2,1),(0,1,2,1),\\
(1,1,2,1),(1,2,2,1),(2,2,2,1),(0,2,2,1),(0,2,0,1),(1,2,0,1),(2,2,0,1),\\
(2,0,0,1),(0,0,0,1),(1,0,0,1),(1,1,0,1),(2,1,0,1),(0,1,0,1),(0,1,1,1), \quad c\\
(1,1,1,1),(2,1,1,1),(2,2,1,1),(0,2,1,1),(1,2,1,1),(1,0,1,1),(2,0,1,1),\\
(0,0,1,1),(0,0,1,2),(1,0,1,2),(2,0,1,2),(2,1,1,2),(0,1,1,2),\\
(1,1,1,2),(1,2,1,2),(2,2,1,2),(0,2,1,2),(0,2,2,2),(1,2,2,2),(2,2,2,2),\\
(2,0,2,2),(0,0,2,2),(1,0,2,2),(1,1,2,2),(2,1,2,2),(0,1,2,2),\\
(0,1,0,2),(1,1,0,2),(2,1,0,2),(2,2,0,2),(0,2,0,2),\\
(1,2,0,2),(1,0,0,2),(2,0,0,2),(0,0,0,2)\}.
\end{gathered}$$

Other examples (we invite the reader to experiment her/himself) support the following conjecture.

Conjecture 7.4. *Hansen's theorem holds if one replaces the symbol $\geq$ in the lower estimate for d by $=$.*

7.2 Other toric codes

Though the results in [F] apply to toric varieties over $\mathbb{C}$, we shall work over a finite field $\mathbb{F}_q$ having $q = p^k$ elements, where p is a prime and $k \geq 1$. We assume that the results of [F] have analogs over $\mathbb{F}_q$.

Let $M \cong \mathbb{Z}^n$ be a lattice in $V = \mathbb{R}^n$ and let $N \cong \mathbb{Z}^n$ denote its dual. Let Δ be a fan (of rational cones, with respect to M) in V and let $X = X(\Delta)$ denote the toric variety associated to Δ. Let T denote a dense torus in X.

Let $D = P_1 + ... + P_n$ be a positive 1-cycle on X, where the points $P_i \in X(\mathbb{F}_q)$ are distinct. Let G be a T-invariant divisor on X which does not "meet" D, in the sense that no element of the support of D intersects any element in the support of G. We write this as

$$\mathrm{supp}(G) \cap \mathrm{supp}(D) = \emptyset.$$

Some additional assumptions on G and D shall be made later. Let

$$L(G) = \{0\} \cup \{f \in \mathbb{F}_q(X)^\times \mid div(f) + G \geq 0\}$$

denote the Riemann-Roch space associated to G. According to [F], §3.4, there is a polytope P_G in V such that $L(G)$ is spanned by the monomials x^a (in multi-index notation), for $a \in P_G \cap N$. Let $C_L = C_L(D, G)$ denote the code define by

$$C_L = \{(f(P_1), ..., f(P_n)) \mid f \in L(G)\}.$$

This is the **Goppa code associated to X, D, and G**. The dual code is denoted

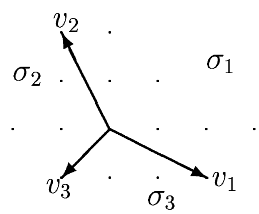
$$C = \{(c_1, ..., c_n) \in \mathbb{F}_q^n \mid \sum_{i=1}^{n} c_i f(P_i) = 0, \forall f \in L(G)\}.$$

Example 7.5. *Let Δ be the fan generated by*

$$v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + 2e_2, \quad v_3 = -e_1 - e_2.$$

Let X be the toric variety associated to Δ.

In the notation above, the divisor $D = d_1 D_1 + d_2 D_2 + d_3 D_3$ is a Cartier divisor if and only if $d_1 \equiv d_2 \equiv d_3 \pmod{3}$.



Let

$$P_D = \{(x, y) \mid \langle (x, y), v_i \rangle \geq -d_i, \ \forall i\}$$
$$= \{(x, y) \mid 2x - y \geq -d_1, -x + 2y \geq -d_2, -x - y \geq -d_3\}$$

denote the polytope associated to the Weil divisor $D = d_1 D_1 + d_2 D_2 + d_3 D_3$, where D_i is as above.

Let

$$G = 10 D_3, \quad D = D_1 + D_2 + D_3.$$

Then P_G is a triangle in the plane with vertices at $(0,0)$, $(-10/3, 20/3)$, and $(20/3, 10/3)$, see Figure 1. It's area is $50/3$.

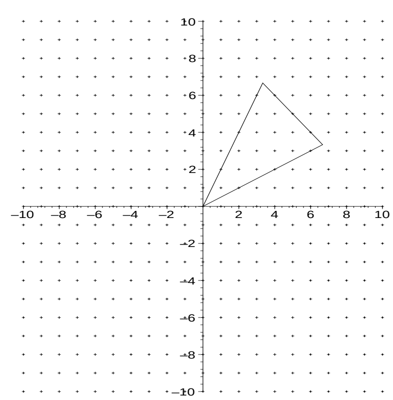


Figure 1: The polytope associated to the divisor G.

Let $T \subset X$ denote the dense torus of X. In this example, the patch U_{σ_1} is an affine variety with coordinates x_1, x_2, x_3 given by $x_1^3 - x_2 x_3 = 0$.

The torus embedding $T \hookrightarrow U_{\sigma_1}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1t_2, t_1t_2^2, t_1^2t_2)$. The patch U_{σ_2} is an affine variety with coordinates y_1, y_2, y_3 given by $y_2^2 - y_1y_3 = 0$. The torus embedding $T \hookrightarrow U_{\sigma_2}$ is given by sending (t_1, t_2) to $(y_1, y_2, y_3) = (t_1^{-2}t_2^{-1}, t_1^{-1}, t_1^{-1}t_2)$. The patch U_{σ_3} is an affine variety with coordinates z_1, z_2, z_3 given by $z_2^2 - z_1z_3 = 0$. The torus embedding $T \hookrightarrow U_{\sigma_3}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1^{-1}t_2^{-2}, t_2^{-1}, t_1t_2^{-1})$.

In the local coordinates of U_{σ_1}, the space $L(G)$ has as a basis,

$$\{h_i\} = \{1, x_1x_2, x_1^2x_2, x_1x_2^2, x_1^2x_2^2, x_1^3x_2^2, x_1^4x_2^2, x_1^2x_2^3, x_1^3x_2^3, \\ x_1^4x_2^3, x_1^5x_2^3, x_1^6x_2^3, x_1^2x_2^4, x_1^3x_2^4, x_1^4x_2^4, x_1^5x_2^4, x_1^6x_2^4, \\ x_1^3x_2^5, x_1^4x_2^5, x_1^5x_2^5, x_1^3x_2^6, x_1^4x_2^6\}.$$

In particular, it is 22-dimensional.

Choose points $P_1, ..., P_n \in X(\mathbb{F}_q)$ and let

$$C = \{(c_1, ..., c_n) \in \mathbb{F}_q^n \mid \sum_{i=1}^n c_i f(P_i) = 0, \forall f \in L(G)\}.$$

For example, we know

$$|X(\mathbb{F}_2)| = 7, \quad |X(\mathbb{F}_3)| = 13, \quad |X(\mathbb{F}_4)| = 21, \\ |X(\mathbb{F}_5)| = 31, \quad |X(\mathbb{F}_7)| = 57, \quad |X(\mathbb{F}_8)| = 73, ...$$

One can (though we do not do so here) use [Han2] to estimate the parameters n, k, d for $C_L = C_L(G, D, X)$. The result, roughly speaking, says that if X is a "nice" toric variety of dimension N, if G is a "nice" divisor on X, and if D is a 1-cycle of "sufficiently large degree", then

$$n = \deg(D),$$

$$k = \dim H^0(X, \mathcal{O}(G)) = |P_G \cap M|,$$

and d is bounded from below (roughly) by $n - N! \cdot k$.

8 The toric automorphism group

Since a (dense) torus T acts on $X = X(\Delta)$, T forms a subgroup of the automorphism group of X. From Lemma 3.19, we see that another source of automorphisms comes from the automorphisms of the fan Δ.

Let Δ be a fan in $V = L \otimes \mathbb{Q}$, $T_L = \mathrm{Hom}_{ab.gp.}(L^*, \mathbb{G}_m)$, and let Δ' be a fan in $V' = L' \otimes \mathbb{Q}$, $T_{L'} = \mathrm{Hom}_{ab.gp.}((L')^*, \mathbb{G}_m)$.

Lemma 8.1. *(Theorem 1.13, Oda [O]) Let* $f : T_{L'} \to T_{L'}$ *be a homomorphism. If* $\psi : X(\Delta) \to X(\Delta')$ *is a morphism which is equivariant with respect to* f *then there is a mapping of fans* $\phi_{\mathbb{Q}} : \Delta' \to \Delta$ *such that* $\psi = \phi^*$ *(in the sense of Lemma 3.19).*

In particular, an equivariant automorphism of a toric variety is a toric automorphism.

Example 8.2. *Consider the fan*

$$\Delta = \{\{0\},\ \mathbb{Q}_{\geq 0},\ \mathbb{Q}_{\leq 0}\}$$

in Example 3.17 above. This corresponds to the toric variety $X(\Delta) = \mathbb{P}^1$. *The auomorphism* ϕ *of* Δ *which swaps* $\mathbb{Q}_{\geq 0}$ *with* $\mathbb{Q}_{\leq 0}$ *and leaves* $\{0\}$ *alone induces a toric morphism* ϕ^* *on* $X(\Delta)$. *Locally, this morphism* ϕ^* *is simply the "gluing" map discussed in Example 3.17.*

Remark 8.3. *We observe a connection with the previous section 7 on codes.*

Note that if G *is a subgroup of the toric automorphism group of* $X(\Delta)$ *and* G *induces an automorphism of the lattice* L *which leaves the polytope* P_L *invariant then* G *induces an automorphism on the code* C *defined in (5).*

Remark 8.4. *Here's an algorithm to find the automorphism group of a fan (hence the toric automorphism group of a toric variety):*

Recall all the cones in a fan are polyhedral. It follows from this that the automorphism group of a fan [17]*, is contained in the automorphism group of its set of rays (all the one-dimensional cones in the fan)* [18]*. To find the automorphism group of the rays, we*

1. *write each ray as* $\tau_i = \mathbb{Q}_{\geq 0} \cdot v_i$, *for some* $v_i \in L$,
2. *for each* $g \in \mathrm{Aut}(L) \subset \mathrm{Aut}(\mathbb{Q}^n)$ *do:*
   (a) *for each* i, *compute* $g(v_i)$, *and ask: is* $g(v_i) \in \tau_j$, *for some* j*?*
   (b) *if no for some* i, *go to the next* g*; if yes for all* i, *add* g *to* $\mathrm{Aut}(Rays)$.
   (c) *Go to the next* g, *if one exists. If no more* g *exist, then return* $\mathrm{Aut}(Rays)$.

[17] Recall Definition 3.9.

[18] The rays do not uniquely determine the fan, nor even the cones themselves.

3. *Now that* Aut$(Rays)$ *has been determined, perform the following double for loop. For each g in* Aut$(Rays)$ *do: for each cone $\sigma \in \Delta$ do: does g send σ to some cone $\sigma' \in \Delta$. If the answer is "yes", for all σ then add g to* Aut(Δ). *Go to the next g, if one exists. If no more g exist, then return* Aut(Δ).

Example 8.5. *Let Δ be the fan generated by*

$$v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + 2e_2, \quad v_3 = -e_1 - e_2.$$

There is a picture of this in Example 7.5 above.

Let G denote the automorphism group of Δ generated by the map g sending (x, y) to (y, x), swapping v_1 and v_2 and leaving v_3 fixed. By Theorem 1.13 in Oda [O], this corresponds to a T-equivariant automorphism of $X(\Delta)$.

In this example, the patch U_{σ_1} is an affine variety with coordinates x_1, x_2, x_3 given by $x^3 - x_2x_3 = 0$. The automorphism g acts on U_{σ_1} sending (x_1, x_2, x_3) to (x_1, x_3, x_2). The torus embedding $T \hookrightarrow U_{\sigma_1}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1t_2, t_1t_2^2, t_1^2t_2)$.

The patch U_{σ_2} is an affine variety with coordinates y_1, y_2, y_3 given by $y^2 - y_1y_3 = 0$. The automorphism g does not act on U_{σ_2}. The torus embedding $T \hookrightarrow U_{\sigma_2}$ is given by sending (t_1, t_2) to $(y_1, y_2, y_3) = (t_1^{-2}t_2^{-1}, t_1^{-1}, t_1^{-1}t_2)$.

The patch U_{σ_3} is an affine variety with coordinates z_1, z_2, z_3 given by $z^2 - z_1z_3 = 0$. The automorphism g does not act on U_{σ_3}. The torus embedding $T \hookrightarrow U_{\sigma_3}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1^{-1}t_2^{-2}, t_2^{-1}, t_1t_2^{-1})$. The automorphism g sends U_{σ_2} to U_{σ_3} by sending (y_1, y_2, y_3) to (z_1, z_2, z_3).

9 Betti numbers

Let $X = X(\Delta)$ be an n-dimensional smooth toric variety associated to a fan Δ. Let d_k denote the number of distinct k-dimensional cones in Δ. The k^{th} **Betti number** of X is the rank of $H^k(X, \mathbb{Z})$. Let b_k denote the the k^{th} Betti number of X. It is known (see Fulton, §4.5) that

$$b_{2k} = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} d_{n-j}.$$

The `toric` package can compute this.

Example 9.1. *Let Δ be the fan whose cones of maximal dimension are defined by*

$$\sigma_1 = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1), \qquad \sigma_2 = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(-1,0),$$

$$\sigma_3 = \mathbb{R}_{\geq 0}(-1,0) + \mathbb{R}_{\geq 0}(0,-1), \qquad \sigma_4 = \mathbb{R}_{\geq 0}(0,-1) + \mathbb{R}_{\geq 0}(1,0).$$

The toric variety associated to this fan is $X = \mathbb{P}^1 \times \mathbb{P}^1$. The `toric` *package commands to compute the Betti numbers b_{2k} are as follows* [19].

```
gap> RequirePackage("guava");
gap> Read("c:/gap/gapfiles/toric.g");
gap> Cones:=[[[1,0],[0,1]],[[0,1],[-1,0]],[[-1,0],[0,-1]],[[0,-1],[1,0]]];
gap> betti_number(Cones,1);
gap> betti_number(Cones,2);
gap> euler_characteristic(Cones);
4
```

GAP returns $b_1 = 0$ to the first command, `betti_number(Cones,1);`*, and $b_2 = 2$ to the second command. The last command tells us that the Euler characteristic of X is* 4*.*

10 Counting points over a finite field

Let $X = X(\Delta)$ be a smooth toric variety associated to a fan Δ. Let q be a prime power and $\mathbb{F} = GF(q)$ denote a field with q elements. It is known (see Fulton, §4.5) that

$$|X(\mathbb{F})| = \sum_{k=0}^{n} (q-1)^k d_{n-k}.$$

The `toric` package can compute this.

Example 10.1. *Let Δ be the fan whose cones of maximal dimension are defined by*

$$\sigma_1 = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1), \qquad \sigma_2 = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(-1,0),$$

$$\sigma_3 = \mathbb{R}_{\geq 0}(-1,0) + \mathbb{R}_{\geq 0}(0,-1), \qquad \sigma_4 = \mathbb{R}_{\geq 0}(0,-1) + \mathbb{R}_{\geq 0}(1,0),$$

so $X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1$. The `toric` *package commands to compute the number of points mod p, $|X(\mathbb{F}_p)|$ are as follows* [20].

[19] We will use the GAP version to compute this example.
[20] We will use the GAP version to compute this example.

```
gap> RequirePackage("guava");
gap> Read("c:/gap/gapfiles/toric.g");
gap> Cones:=[[[1,0],[0,1]],[[0,1],[-1,0]],[[-1,0],[0,-1]],[[0,-1],[1,0]]];
gap> cardinality_of_X(Cones,2);
gap> cardinality_of_X(Cones,3);
gap> cardinality_of_X(Cones,5);
```

GAP returns $|X(\mathbb{F}_2)| = 9$ *to the first command,* `cardinality_of_X(Cones,2);`, $|X(\mathbb{F}_3)| = 16$ *to the second command,* $|X(\mathbb{F}_5)| = 36$ *to the last command.*

11 The compactification of a universal elliptic curve

In this section, we discuss the compactification of the universal family of elliptic curves with level N structure.

An elliptic curve is given by $\mathbb{C}/L$ for some lattice L in $\mathbb{C}$. Two elliptic curves given by lattices L_1 and L_2 are **homothetic** if there is some $\alpha \in \mathbb{C}$ with $L_1 = \alpha L_2$. We have

$$L_1, L_2 \text{ are homothetic} \iff E_1, E_2 \text{ are isomorphic.}$$

By suitable scaling, an elliptic curve is isomorphic to

$$E_\tau = \mathbb{C}/\langle 1, \tau \rangle.$$

The subgroup of points of order N on E_τ is given by

$$E[N] = \{\overline{(n + m\tau)/N} | n, m \in \mathbb{Z}\} \cong (\mathbb{Z}/N\mathbb{Z})^2$$

An **elliptic curve with full level N structure** is an elliptic curve together with two points spanning $E[N]$. E.g., $(E_\tau, 1/N, \tau/N)$.

Let

$$\Gamma(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \middle| \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \bmod N \right\}.$$

Any elliptic curve over $\mathbb{C}$ can be described as the quotient of the complex plane by a lattice generated by 1 and $\tau \in \mathcal{H} = \{x \in \mathbb{C} | \mathrm{Im}(x) > 0\}$. We have

$$E_\tau = \mathbb{C}/\langle 1, \tau \rangle.$$

The Weierstrass $\wp$ function gives the algebraic equations for E_τ [Har].

We can show that

$$E_\tau \cong E_{\tau'} \iff \tau' = g\tau \text{ for some } g \in SL_2(\mathbb{Z}).$$

So, the quotient $SL_2(\mathbb{Z}) \setminus \mathcal{H}$ parameterises elliptic curves over $\mathbb{C}$. For a congruence subgroup $\Gamma_0(N) \subset SL_2(\mathbb{Z})$, the space

$$Y_0(N) = \Gamma_0(N) \setminus \mathcal{H}$$

parameterizes elliptic curves with level N structure, i.e., with a subgroup of order N. The problem with $Y_0(N)$ is that it is not compact. The compactification is given by

$$X_0(N) = \Gamma(N) \setminus \mathcal{H}^*$$

where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. It is a bit harder to find the compactification of $\mathcal{E}^\circ(N)$; we need to know what is the appropriate preimage of the cusps, (which are the finite number of points $\Gamma(N)\setminus(\mathbb{Q}\cup\infty)$), and how these "glue" into the rest of the space.

First of all, we can describe the whole of $\mathcal{E}^\circ$ as a quotient: Let

$$H(N) = \left\{ \begin{pmatrix} 1 & Nr & Ns \\ 0 & Na+1 & Nb \\ 0 & Nc & Nd+1 \end{pmatrix} \in SL_3(\mathbb{Z}) \middle| a,b,c,d,r,s \in \mathbb{Z} \right\}$$
$$\cong (N\mathbb{Z} \oplus N\mathbb{Z}) \rtimes \Gamma(N).$$

$H(N)$ acts on $\mathbb{C} \times \mathcal{H}$ by

$$\begin{pmatrix} 1 & Nr & Ns \\ 0 & a & b \\ 0 & c & d \end{pmatrix} (z,\tau) = \left(\frac{z + Nr\tau + Ns}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).$$

Then we can take $\mathcal{E}^\circ = H(N) \setminus \mathbb{C} \times \mathcal{H}$, and there is a map

$$\begin{tikzcd}
\mathcal{E}^\circ \arrow[r, equal] \arrow[d] & H(N) \setminus \mathbb{C} \times \mathcal{H} \arrow[d] & \supset (N\mathbb{Z}\tau \oplus N\mathbb{Z}) \setminus \mathbb{C} \arrow[d] \\
Y(N) \arrow[r, equal] & \Gamma(N) \setminus \mathcal{H} & \ni \tau
\end{tikzcd}$$

To construct the compactification of $\mathcal{E}^\circ$, we define affine pieces which patch together, and which "fill in" something over each cusp. We will just look at what happens for the cusp at infinity.

We take the quotient in two steps.

The stabilizer in $H(N)$ of the fiber over ∞ is given by

$$H_\infty = \left\{ \begin{pmatrix} 1 & Nr & Ns \\ 0 & 1 & Nb \\ 0 & 0 & 1 \end{pmatrix} \middle| b, r, s \in \mathbb{Z} \right\}.$$

This is an extension:

$$1 \longrightarrow A \longrightarrow H_\infty \longrightarrow B \longrightarrow 1 \,,$$

where

$$A = \left\{ \begin{pmatrix} 1 & 0 & Ns \\ 0 & 1 & Nb \\ 0 & 0 & 1 \end{pmatrix} \middle| s, b \in \mathbb{Z} \right\},$$

and

$$B = \left\{ \begin{pmatrix} 1 & 0 & Nr \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| r \in \mathbb{Z} \right\}.$$

We take the quotient in two steps. A acts on a neigborhood W of $\mathbb{C} \times \infty$ by

$$\begin{pmatrix} 1 & 0 & Ns \\ 0 & 1 & Nb \\ 0 & 0 & 1 \end{pmatrix} : (z, \tau) \mapsto (z + Ns, \tau + Nb),$$

and the quotient is a subset of $(\mathbb{C}^\times)^2$—a torus. This quotient map is given by

$$(z, \tau) \mapsto (w = e^{2\pi i z/N}, t = e^{2\pi i \tau/N}).$$

Then B acts on this quotient by

$$\begin{pmatrix} 1 & 0 & Nr \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; : \; (w, t) \mapsto (t^{Nr} w, t).$$

We define an infinite fan Δ in $\mathbb{Z}^2$, with cones spanned by $(n, 1), (n + 1, 1)$. Then $X(\Delta) \setminus (\mathbb{C}^\times)^2$ is given by an infinite number of copies of $\mathbb{P}^1$, and we still have an action of B on $X(\Delta)$, which extends the action of B on $\mathbb{C}^2 \subset X(\Delta)$. This action on $X(\Delta) \setminus (\mathbb{C}^\times)^2$ is given by sending a $\mathbb{P}^1$ to a $\mathbb{P}^1$ N steps further along.

The variety $X(\Delta)$ is smooth, and the action of B is fixed point free, and so the quotient $X(\Delta)/B$ is smooth, and gives a compactification of $\mathcal{E}^\circ$ in the neighborbood of ∞. The fiber over ∞ is given by N copies of $\mathbb{P}_1$.

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