On the Kuznetsov-Bruggeman Formula for a Hilbert Modular Surface Having One Cusp

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Contents

Abstract

Let $K = \mathbb{Q}(\sqrt{D})$, $D \equiv 2,3 (\mathrm{mod}\ 4)$ square–free, be a real quadratic extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K = \mathbb{Z} + \sqrt{D}\mathbb{Z}$ and let $\Gamma = PSL(2, \mathcal{O}_K)$. This paper [1] is devoted to a derivation (using an unpublished method of Don Zagier) of the untwisted and twisted Kuznetsov-Bruggeman formulas for $\Gamma \setminus H^2$, where H denotes the complex upper half plane.

1. Introduction

Let $K = \mathbb{Q}(\sqrt{D})$, $D \equiv 2,3 (\mathrm{mod}\ 4)$ square–free, be a real quadratic extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K = \mathbb{Z} + \sqrt{D}\mathbb{Z}$ and let $\Gamma := PSL(2, \mathcal{O}_K)$. For the purpose of investigating the spectrum of the Laplacian, the Selberg trace formula and Kuznetsov–Bruggeman (relative trace) formula have proved useful (see [Sel], [Br1], [Br2], and [Kuz]) and, because of the lifting theory of Saito–Shintani–Langlands [Sa] involving the "twisted" version of the Selberg trace formula, there is reason to hope that the "twisted" version of Kuznetsov's formula on $\Gamma \setminus H^2$ may also prove useful (see [Ye] for an application of the adelic version of the Kuznetsov-Bruggeman formula, which H. Jacquet refers to as the "relative trace formula", to quadratic base change). Additionally, it may be of interest to extend the theory of Deshoulliers-Iwaniec [DI] to the real quadratic case; this paper hopefully constitutes a small step in this direction, as our formula is as explicit as [Kuz] Theorems 1,2.

As the title indicates, this paper is devoted to a derivation (using an unpublished method of Don Zagier) of the untwisted and twisted Kuznetsov-Bruggeman formulas for $\Gamma \setminus H^2$. (Unfortunately, the form of the twisted formula below is not as explicit as the twisted formula since I have not been able to obtain a simple expression for the "Kloosterman sum term".) Section 1 begins with D. Zagier's method, in the classical case of $SL(2,\mathbb{Z}) \setminus H$. Since his method is unpublished, I included details in this section. The appendix to section 2, which is the only part of this paper to contain anything really new, addresses two issues: (1) it discusses a technical question relating (D. Zagier's [Z1] computation of) the function

$$Z_s k(g) = \int_{\Gamma \setminus H^2} k(gz, z) y^s \, dz \,,$$

[1] This paper was published in Math. Z. 203 in 1990 and retyped in LaTeX in 1997 with corrections included. I thank Professors Bruggemann and Miatello for pointing out a gap in the original estimates. Originally supported by an NSF fellowship.

to the "Kuznetsov transform"

$$\phi_H(x) := \frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}(x)\frac{r}{\cosh(\pi r)}\phi_J(r)\ dr$$

(see section 2 below), and (2) it verifies that a certain interchange of order of integration and summation is valid. In section 2 I recall the results leading to Selberg's spectral decomposition theorem for $\Gamma \setminus H^2$. I have included most of the details since the twisted case required a slightly more "symmetric" form that what was available in [Ef] or [Zo]. Section 3 follows D. Zagier's method for $\Gamma\setminus H^2$, assuming for simplicity that K has class number one, to derive the trace formula. In section 4, following a suggestion of Ram Murty, we turn to some of the simplest applications. This involves Poincaré series, inner product formulas, and sums of Kloosterman sums attached to a real quadratic field. A more detailed list of the topics is given in the table of contents below and the introduction to section 4.

The Selberg trace formula for $D \equiv 2,3 \pmod 4$, with K having class number one, was derived explicitly in the classical language by P. Zograf [Zo]. After this paper was written I. Efrat's book [Ef] appeared, where the more general case of a totally real extension $K/\mathbb{Q}$ was dealt with, again, classically. Presumably, the results of this paper easily extend to the case of a totally real number field of class number one, using Efrat's, rather than Zograf's, spectral decomposition of the kernel function. These results should also extend to the case of arbitrary class number, but the added complications would probably not be so simple. The adelic form of the Selberg trace formula was, of course, derived earlier by Jacquet–Langlands, but since I follow D. Zagier's method to derive the Kuznetsov formula from the classical expression for the spectral decomposition of the kernel function, the adelic form is not needed here. I believe that S. Friedberg, D. Goldfeld, I. Piatetski–Shapiro, and P. Sarnak have independently derived versions of this formula for $GL(n)$ (for S.F. and D.G.) and $GL(2)$ over a number field (for I.P.–S. and P.S.).

Acknowledgement: I want to thank Professor Don Zagier for showing me his method as well as guiding me into modular forms. I also want to thank Professor Roelof Bruggeman for encouragement, detailed criticisms, and suggestions. He has proven [Br3] a version of the Kuznetsov-Bruggeman formula in the real quadratic case using his method in [Br2]. Finally, I thank the generous referee for correctly calculating the δ_{mn}-term and for making many detailed suggestions which have substantially improved the manuscript. This paper was written at the Institute for Advanced Study while the author was supported by an NSF fellowship.

2. The case of $SL(2,\mathbb{Z}) \setminus H$

It will be far more convenient for the reader who follows the Hilbert modular case of Zagier's method to have a clear idea about what happens in the relatively simpler case of $SL(2,\mathbb{Z})\setminus H$ (all the crucial ideas already occur in this setting, they just get more complicated in the case of $\Gamma \setminus H^2$). The results of this subsection are due to D. Zagier.

Let me change notation a little, in this subsection only. Namely, let $G := PSL(2)$, $\Gamma := G(\mathbb{Z})$, $k \in C_c^\infty(\mathbb{R})$, and

$$\tilde{k}(z,z') := k\left(\frac{|z-z'|^2}{yy'}\right),$$

where $z = x+iy,\ z' = x'+iy'$. The invariant integral operator

$$Kf(z) := \int_H \tilde{k}(z,z')f(z')\ dz'$$

satisfies $Kf = h(r)f$, for all eigenfunctions f of the Laplacian $\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, with eigenvalue $1/4 + r^2$. Here h is the Selberg transform. We have

$$Q(w) := \int_w^\infty \frac{h(t)}{\sqrt{t-w}}dt, \quad k(t) := \frac{1}{\pi}\int_t^\infty \frac{Q'(w)}{\sqrt{w-t}}dw,$$

and

$$h(r) := \int_{-\infty}^\infty Q(e^u + e^{-u} - 2)e^{iru}du, \quad Q(e^u+e^{-u}-2) := \frac{1}{2\pi}\int_{-\infty}^\infty h(r)e^{iru}dr.$$

One has the following fact about the functions $k,\ h,\ g$, and Q:

Lemma 2.1. *([Z1], p. 320) The conditions*

- $k(x) <<_\epsilon x^{-\frac{1+\theta}{2}+\epsilon}$, *for each* $\epsilon > 0$,
- $Q(w) <<_\epsilon w^{-\theta/2+\epsilon}$, *for each* $\epsilon > 0$,
- $h(r)$ *is holomorphic in* $\mathrm{Im}\, r| < \theta/2$,

are all equivalent.

Let

$$k_\Gamma(z,z') := \sum_{\gamma\in\Gamma} \tilde{k}(z,\gamma z'),$$

be the kernel function for K on $L^2(\Gamma\backslash H)$. The **Selberg spectral decomposition** states that (see, e.g., [Z1] (2.31), [Kub], or [DI])

$$\begin{array}{rl} k_\Gamma(z,z') = & \sum_{j=1}^{\infty} h(r_j)u_j(z)\overline{u_j(z')} + \frac{3}{\pi}h(i/2) \\ & +\frac{1}{4\pi}\int_{-\infty}^{\infty} E(z,\frac{1}{2}+it)E(z',\frac{1}{2}-it)h(t)\ dt, \end{array} \tag{2.1}$$

where $\{u_j\}_1^\infty$ form an orthonormal basis for the Maass forms of weight zero with r_j defined by $\Delta u_j = (1/4+ir_j^2)u_j$, and where

$$E(z,s) := \frac{y^s}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} |cz+d|^{-2s}$$

is the Eisenstein series. The $u_j(z)$ have a Fourier expansion of the form

$$u_j(z) = \sqrt{y}\sum_{n\neq 0} \rho_j(n)K_{ir_j}(2\pi|n|y)e^{2\pi inx}, \tag{2.2}$$

whereas the Fourier expansion of $E(z,s)$ can be read off from (see, e.g., [Z1] (2.6))

$$\begin{array}{rl} E^*(z,s) = & \zeta^*(2s)y^s + \zeta^*(2s-1)y^{1-s} \\ & +2\sqrt{y}\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)K_{s-1/2}(2\pi ny)e^{2\pi nx}, \end{array} \tag{2.3}$$

where $E^*(z,s) := \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s)$, $\zeta^*(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \zeta^*(1-s)$, $\sigma_s(n) := \sum_{d|n} d^s$, and

$$K_z(t) := \int_0^\infty e^{-t\cosh u}\cosh(zu)\ du, \quad t>0.$$

Now substitute these Fourier expansions into the above identity for $k_\Gamma(z,z')$ and comparing Fourier coefficients of $e^{2\pi i(nx-mx')}$, $m>0$, $n>0$, we find that

$$\begin{array}{l} \frac{1}{\sqrt{yy'}}\int_0^1\int_0^1 e^{-2\pi i(nx-mx')}k_\Gamma(x+iy,x'+iy')\ dxdx' \\ = \sum_{j=1}^{\infty} h(r_j)\rho_j(n)\overline{\rho_j(m)}K_{ir_j}(2\pi ny)K_{ir_j}(2\pi my') \\ +\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(n/m)^{ir}\sigma_{-2ir}(n)\sigma_{2ir}(m)}{|\zeta(1+2ir)|^2}h(r)\frac{K_{ir}(2\pi ny)K_{ir}(2\pi my')}{(\cosh(\pi r))^{-1}}\ dr, \end{array} \tag{2.4}$$

since $|\Gamma(1/2+it)|^2 = \pi/\cosh(\pi t)$. Let $\gamma_{c,d}$ be any representative of the form

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,$$

for $c \neq 0$. By the Bruhat decomposition, replacing x by $x - \frac{a}{c}$ (using the Γ-invariance of $\tilde{k}$) and x' by $x' + \frac{d}{c}$ (using the translation invariance of dx), the left–hand side is, neglecting the "c=0 term", given by

$$\begin{aligned} &\tfrac{1}{\sqrt{yy'}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(nx-mx')} \sum_{c=1}^{\infty} \sum_{\substack{d (\bmod c) \\ (c,d)=1}} \tilde{k}(x+iy, \gamma_{c,d}(x'+iy'))\ dx dx' \\ = &\tfrac{1}{\sqrt{yy'}} \sum_{c=1}^{\infty} \Big(\sum_{\substack{d (\bmod c) \\ ad\equiv 1 (\bmod c)}} e^{2\pi i(\frac{na+md}{c})} \Big) \int_{-\infty}^{\infty} e^{-2\pi i(nx-mx')} \tilde{k}(x+iy, \tfrac{-c^{-2}}{x'+iy'})\ dx dx'. \end{aligned} \tag{2.5}$$

The "c=0 term" of $k_\Gamma(z,z')$ is

$$\begin{aligned} \sum_{\gamma\in\Gamma_0} \tilde{k}(z,\gamma z') &= \sum_{b\in\mathbb{Z}} k(\tfrac{(x-x'+b)^2+(y-y')^2}{2yy'}) \\ &= \sum_{\ell} c_\ell(y,y') e^{2\pi i\ell(x-x')}, \end{aligned}$$

where

$$c_\ell(y,y') := \int_{-\infty}^{\infty} e^{2\pi i\ell u} k(\frac{u^2+(y-y')^2}{yy'})\ du.$$

In the expression (2.5), set $\lambda := ny$, $\lambda' := my'$, and replace x, x' by $\frac{1}{c}\cdot\sqrt{\frac{m}{n}}\cdot x$, $\frac{1}{c}\cdot\sqrt{\frac{n}{m}}\cdot x'$ respectively, so that (as vectors in $\mathbb{R}^2$) a simple computation shows that

$$(x+iy, \frac{-c^{-2}}{x'+iy'}) \longmapsto \frac{1}{c}\cdot\sqrt{\frac{m}{n}}\cdot(x+\frac{ic\lambda}{\sqrt{mn}}, \frac{-1}{x'+\frac{ic\lambda'}{\sqrt{mn}}}).$$

Taking $\lambda = \lambda'$, we have

$$\begin{aligned} &\sum_{j=1}^{\infty} h(r_j)\rho_j(n)\overline{\rho_j(m)} K_{ir_j}(2\pi\lambda)^2 \\ &\quad + \tfrac{1}{\pi}\int_{-\infty}^{\infty} \tfrac{(n/m)^{ir}\sigma_{-2ir}(n)\sigma_{2ir}(m)}{|\zeta(1+2ir)|^2} h(r) \tfrac{K_{ir}(2\pi\lambda)^2}{(\cosh(\pi r))^{-1}}\ dr \\ = &\sqrt{\tfrac{mn}{\lambda^2}} \sum_{c=1}^{\infty} \tfrac{S(n,m;c)}{c^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\frac{\sqrt{mn}}{c}(x-x')} \tilde{k}\Big(x+\tfrac{ic\lambda}{\sqrt{mn}}, \tfrac{-1}{x'+\frac{ic\lambda}{\sqrt{mn}}}\Big)\ dx dx' \\ &\quad + \delta_{mn} \int_{-\infty}^{\infty} k(u^2) e^{2\pi i\lambda u} du. \end{aligned} \tag{2.6}$$

Notice that the extra factor of y in the last ("c=0") term of (2.6) has been cancelled by the $1/\sqrt{yy'}$ factor in (2.4). Here

$$S(m,n;c) := \sum_{a \pmod{c}),\ a\overline{a}\equiv 1 (\bmod c)} e^{2\pi i(am+\overline{a}n)/c}.$$

Integrate both sides of (2.6) from $\lambda = 0$ to $\lambda = \infty$. Let me assume that the interchange of integrals with respect to λ and with respect to $x,\ x'$ is valid. This assumption will be verified later, see the appendix. Using

$$\int_0^\infty K_{ir}(2\pi\lambda)^2\ d\lambda = \frac{\pi}{8\cosh(\pi r)},$$

we find that

$$\begin{array}{c} \sum_{j=1}^\infty \frac{h(r_j)}{\cosh(\pi r_j)}\rho_j(n)\overline{\rho_j(m)} + \frac{1}{\pi}\int_{-\infty}^\infty \frac{(n/m)^{it}\sigma_{-2it}(n)\sigma_{2it}(m)}{|\zeta(1+2it)|^2}h(t)\,dt \\ = \frac{8}{\pi}\sum_{c=1}^\infty \frac{\sqrt{mn}}{c^2}S(n,m;c)\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^\infty e^{-2\pi i\frac{\sqrt{mn}}{c}t}\tilde{k}\left(x+iy+t, \frac{-1}{x+iy}\right)\,dx\frac{dy}{y}dt \\ +\frac{4}{\pi}\delta_{mn}k(0), \end{array} \tag{2.7}$$

using the Fourier inversion formula for the $k(0)$ term. Writing

$$k^*(x) := \frac{2x}{\pi^2}\int_{-\infty}^\infty e^{ixt/2}(Zk)(t)\ dt,$$

where $Zk(t) := \int_H \tilde{k}(z+t,-1/z)y\ dz$, the first term on the right hand side of (2.7) may be expressed as

$$\sum_{c=1}^\infty \frac{1}{c}S(n,m;c)k^*(4\pi\frac{\sqrt{mn}}{c}).$$

We can express the function $k^*(x)$ and the number $k(0)$ directly in terms of $h(r)$ as follows: By [Z1] (4.12), $Zk(t)$ is given by

$$Zk(t) = \frac{1}{4}\int_{-\infty}^\infty u(t,r)h(r)dr,$$

with

$$u(t,r) := \begin{cases} \cos(\alpha r), & \text{if } 2 \le |t| = \cosh(\alpha/2), \\ \frac{\cosh(\alpha r)}{\cosh(\pi r)}, & \text{if } 2 > |t| = \sin(\alpha/2),\ 0 \le \alpha \le \pi. \end{cases}$$

On the other hand, from [GR], §6.69, we have

$$u(t,r)=\frac{ir}{\cosh(\pi r)}\int_0^\infty [J_{2ir}(x)-J_{-2ir}(x)]\cos(\frac{tx}{2})\frac{dx}{x},$$

so by the Fourier inversion formula we find that

$$k^*(x)=\frac{i}{\pi}\int_{-\infty}^\infty [J_{2ir}(x)-J_{-2ir}(x)]\frac{rh(r)}{\cosh(\pi r)}dr$$

(see the appendix for more details). For $k(0)$ we have the formula (standard in the classical derivation of the Selberg trace formula, see [Z1], (2.24))

$$k(0)=\frac{1}{4\pi}\int_{-\infty}^\infty rh(r)\tanh(\pi r)dr.$$

Summarizing, we have proved:

Theorem 2.2. *Let $h(r)$ be even and holomorphic in $|Im\, r|\le 1/2+\epsilon$ with $|h(r)|<<_\epsilon |r|^{-2-\epsilon}$. We have*

$$\begin{aligned}\textstyle\sum_{j=1}^\infty \frac{h(r_j)}{\cosh(\pi r_j)}\rho_j(n)\overline{\rho_j(m)}+\frac{1}{\pi}\int_{-\infty}^\infty \frac{(n/m)^{it}\sigma_{-2it}(n)\sigma_{2it}(m)}{|\zeta(1+2it)|^2}h(t)\,dt\\ =\textstyle\sum_{c=0}^\infty \frac{S(n,m;c)}{c}k^*(4\pi\frac{\sqrt{mn}}{c})+\frac{4}{\pi}\delta_{mn}k(0),\end{aligned}\tag{2.8}$$

where $h(r)$, $k(0)$, $k^*(x)$ are related as above.

Remark 1. *In practice, one sometimes is given k^* and calculates or estimates h from that. For examples, see [Kuz], subsection 5.5 below, [DI] §§4-7, or [Z1].*

On the other hand, **Kuznetsov's formula** [Kuz] states that

$$\begin{aligned}\textstyle\sum_{j=1}^\infty \frac{\phi_J(r_j)}{\cosh(\pi r_j)}\rho_j(n)\overline{\rho_j(m)}+\frac{1}{\pi}\int_{-\infty}^\infty \frac{(n/m)^{it}\sigma_{-2it}(n)\sigma_{2it}(m)}{|\zeta(1+2it)|^2}\phi_J(t)\;dt\\ =\textstyle\sum_{c=1}^\infty \frac{1}{c}S(n,m;c)\phi_H(4\pi\frac{\sqrt{mn}}{c})+\frac{1}{2\pi}\delta_{mn}\int_0^\infty \phi(u)J_0(u)\;du,\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}\phi_J(r)&:=\tfrac{i\pi}{2\sinh(\pi r)}\textstyle\int_0^\infty (J_{2ir}(x)-J_{-2ir}(x))\phi(x)\frac{dx}{x},\\ \phi_H(x)&:=\tfrac{2i}{\pi}\textstyle\int_{-\infty}^\infty J_{2ir}(x)\frac{r}{\cosh(\pi r)}\phi_J(r)\;dr,\end{aligned}\tag{2.10}$$

and $J_z(x)$ denotes the usual J–Bessel function.

We show in the appendix, for all ϕ such that $\phi_J(r) = h(r)$, the identity $k^*(x) = \phi_H(x)$ holds.

To complete the proof that Theorem 2.2 and (2.9) are really the same formula, one must verify that

$$8k(0) = \int_0^\infty \phi(x) J_0(x) dx,$$

for ϕ as above satisfying $\phi_H(r) = h(r)$. However, this identity follows immediately from [Kuz] (6.33) and the equation for $k(0)$ just above Theorem 2.2.

2.1. Appendix to §2 : The Zagier transform

2.1.1. An identity

The first object of this appendix is to show that, for suitably "nice" functions (such as those obtained from the conditions imposed in §3), if ϕ is such that $h = \phi_J$ then

$$k^*(x) = \frac{2x}{\pi^2}(Zk)^\wedge(x) = \phi_H(x). \tag{2.11}$$

Here the notation is as above. This follows almost immediately from a result of Zagier [Z1] (4.12), expressing the Zagier transform Zk in terms of the Selberg transform h. One must also use a formula for the Fourier transform of $J_z(x)/x$, from [GR]. The result (2.11) generalizes to n–dimensions, in the obvious way, and it is the 2–dimensional version which will be applied in §4. The hypotheses on h, k are as in Theorem 2.2. In particular, we will use the fact that $h(r) = h(-r) = h(\overline{r})$, for $|\mathrm{Im}\ r| < \theta$.

Lemma 2.3. *([Z1], (4.12)) We have*

$$Zk(t) = \begin{cases} \frac{1}{4}\int_{-\infty}^{\infty} h(r)\cos(\alpha r)\ dr, & |t| = 2\cosh(\alpha/2) \geq 2, \\ \frac{1}{4}\int_{-\infty}^{\infty} h(r)\frac{\cosh(\alpha r)}{\cosh(\pi r)}\ dr, & |t| \leq 2,\ t = 2\sin(\alpha/2),\ 0 \leq \alpha \leq \pi. \end{cases}$$

From [GR] §6.69, we obtain

Lemma 2.4. *If $\epsilon > 0$ then*

$$\int_0^\infty J_{2ir+\epsilon}(x)\cos(xa/2)\frac{dx}{x} = \begin{cases} \frac{\cos((2ir+\epsilon)\sin^{-1}(\alpha/2))}{2ir+\epsilon}, & a \leq 2, \\ \frac{\cos(ir\pi + \frac{\epsilon\pi}{2})}{(2ir+\epsilon)(a/2+\sqrt{(a/2)^2-1})^{2ir+\epsilon}}, & a \geq 2. \end{cases}$$

Recall, $\frac{2}{a+\sqrt{a^2-4}} = e^b$, *where* $b = \cosh^{-1}(a/2)$.

Let

$$\phi_{H,\epsilon}(x) := \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2ir+\epsilon}(x) \frac{r}{\cosh(\pi r)} \phi_J(r)\ dr, \tag{2.12}$$

where ϕ_J is as in (2.9). This $\epsilon > 0$ is introduced in order that we may, after taking the Fourier transform of $\phi_{H,\epsilon}(x)/x$, interchange the order of integration and apply Lemma 2.3. One may alternatively use the fact that, by Cauchy's residue theorem,

$$\phi_H(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2ir+\epsilon}(x) \frac{r + i\epsilon/2}{\cosh[\pi(r + i\epsilon/2)]} \phi_J(r + i\epsilon/2)\ dr,$$

and, since $h(r + i\epsilon/2) = h(r - i\epsilon/2)$,

$$Zk(t) = \begin{cases} \frac{1}{4}\int_{-\infty}^{\infty} h(r+i\epsilon/2)\cos[(2ir+\epsilon)\cosh^{-1}(|t|/2)]\ dr, & |t| \geq 2, \\ \frac{1}{4}\int_{-\infty}^{\infty} h(r+i\epsilon/2)\frac{\cos[(2ir+\epsilon)\sin^{-1}(t/2)]}{\cosh[\pi(r+i\epsilon/2)]}\ dr, & |t| \leq 2. \end{cases}$$

However, using (2.12), note that

$$\lim_{\epsilon \to 0+} \phi_{H,\epsilon}(x) = \phi_H(x),$$

since dominated convergence allows us to take the limit under the integral. If $h(r) = \phi_J(r)$ then, by definition,

$$\phi_{H,\epsilon}(x) := \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2ir+\epsilon}(x) \frac{r}{\cosh(\pi r)} h(r)\ dr.$$

To show that (2.11) holds, it is enough to show that $(Zk)^\wedge(x)$ is $\pi^2/2$ times the limit, as $\epsilon \to 0$, of the inverse Fourier transform of $\phi_{H,\epsilon}(x)/x$. This is because one has (by dominated convergence)

$$\lim_{\epsilon \to 0} \int_0^{\infty} e^{itx/2} \phi_{H,\epsilon}(x) \frac{dx}{x} = \int_0^{\infty} e^{itx/2} \lim_{\epsilon \to 0} \phi_{H,\epsilon}(x) \frac{dx}{x}. \tag{2.13}$$

On the other hand, (2.13) can be computed from Lemma 2.3, from which the desired result follows.

2.1.2. Interchanging of orders of integration

Here we want to verify that

$$\begin{aligned}&\int_0^\infty \sum_{c\geq 1} \tfrac{S(n,m,c)}{c^2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-2\pi i\frac{\sqrt{mn}}{c}(u-u')}\tilde{k}\big(u+i\tfrac{c\lambda}{\sqrt{mn}}, \tfrac{-1}{u+i\frac{c\lambda}{\sqrt{mn}}}\big)\ dudu'\tfrac{d\lambda}{\lambda}\\ &= \sum_{c\geq 1} \tfrac{S(n,m,c)}{c^2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-2\pi i\frac{\sqrt{mn}}{c}(u-u')} \int_0^\infty \tilde{k}\big(u+i\tfrac{c\lambda}{\sqrt{mn}}, \tfrac{-1}{u+i\frac{c\lambda}{\sqrt{mn}}}\big)\ \tfrac{d\lambda}{\lambda}dudu'.\end{aligned} \tag{2.14}$$

This, in fact, converges absolutely for k such that $k \geq 0$ and $\phi_H^{(n)}(0) = 0$ for $n = 0,1,2,3$ (here $\phi^{(n)}$ denotes the n^{th} derivative of ϕ). This set of k's span a dense subspace of the space of k's in Lemma 2.1.

Let me first verify that

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-2\pi i\frac{\sqrt{mn}}{c}(u-u')}\tilde{k}\big(u+i\tfrac{c\lambda}{\sqrt{mn}}, \tfrac{-1}{u+i\frac{c\lambda}{\sqrt{mn}}}\big)\ dudu'\tfrac{d\lambda}{\lambda}\\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-2\pi i\frac{\sqrt{mn}}{c}(u-u')} \int_0^\infty \tilde{k}\big(u+i\tfrac{c\lambda}{\sqrt{mn}}, \tfrac{-1}{u+i\frac{c\lambda}{\sqrt{mn}}}\big)\ \tfrac{d\lambda}{\lambda}dudu',\end{aligned} \tag{2.15}$$

assuming k is as above. By the positivity of the kernel $\tilde{k}$, we have

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty \tilde{k}\big(u+t+iv, \tfrac{-1}{u+iv}\big)\ du\tfrac{dv}{v}\\ &= \int_{-\infty}^\infty \int_0^\infty \tilde{k}\big(u+t+iv, \tfrac{-1}{u+iv}\big)\ \tfrac{dv}{v}du\\ &= Zk(t).\end{aligned} \tag{2.16}$$

By our assumption on ϕ_H, $\hat{Zk}$ is smooth, so (integrating by parts, as usual) $Zk(t) << t^{-2}$ as t goes to infinity. In particular, the following integrals are absolutely convergent:

$$\int_{-\infty}^\infty Zk(t)e^{ixt}\ dt = \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \tilde{k}\big(u+t+iv, \frac{-1}{u+iv}\big)\ dt\frac{dv}{v}du.$$

This verifies (2.15). Weil's bound for S(n,m,c) along with (2.15), (2.11), and Lemma 2.1 yields (2.14).

It should be remarked that, by using truncation operators, Ye [Ye] verifies interchange orders of integration. However, his "relative trace formula" appears to be somewhat different than ours.

3. The case of $SL(2,\mathcal{O}_K) \setminus H^2$

Let me review some well–known material.

Some notation:

$$\begin{aligned}
K &:= \text{totally real number field of degree } 2\,,\\
\mathcal{O}_K &:= \text{ring of integers of } K\,,\\
\mathcal{O}_K^\times &:= \text{group of units}(\cong \mathbb{Z}\times \text{finite group})\,,\\
\epsilon &= \text{fundamental unit of } \mathcal{O}_K, \epsilon > 1\,,\\
\overline{a} = \tau(a) &:= \text{conjugate of } a \in K\\
N(a) &:= a\overline{a}(= \text{the norm of} a)\,,\\
\Gamma &:= PSL(2,\mathcal{O}_K)\,,\\
\Gamma_\gamma &:= \mathrm{Cent}(\gamma,\Gamma)(\text{the centralizer})\,,\\
\Gamma_z &:= \{\gamma\in\Gamma \mid \gamma z = z\}(\text{the stabilizer})\,.
\end{aligned}$$

Recall that $\gamma\in\Gamma$ acts on $z=(z_1,z_2)\in H^2$ by

$$\gamma z = \Big(\frac{az_1+b}{cz_1+d},\dots,\frac{\tau(a)z_d+\tau(b)}{\tau(c)z_d+\tau(d)}\Big)\,, \tag{3.1}$$

via the embedding $\Gamma\hookrightarrow PSL(2,\mathbb{R})^2$ given by

$$\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \longmapsto \left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \begin{pmatrix} \tau(a) & \tau(b)\\ \tau(c) & \tau(d)\end{pmatrix}\right).$$

The norm of $a\in K$ will be extended to a "norm" of $\mathbb{C}^d$ as follows: $Nz := z_1z_2$, for $z=(z_1,z_2)$.

Recall that a matrix $A\in SL(2,\mathbb{R})$, $A\neq\pm 1$, is **hyperbolic, elliptic,** or **parabolic** if $|\mathrm{tr}\, A|>2, |\mathrm{tr}\, A|<2$, or$|\mathrm{tr}\, A|=2$, respectively. This notion makes sense on the quotient $PSL(2,\mathbb{R})$. A matrix $\gamma\in\Gamma$ will be called **(totally) hyperbolic, elliptic,** or **parabolic** if both γ and $\tau\gamma$ is. Otherwise, γ is called **mixed**.

Let $\hat{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$. The following is well–known (see [Sh]).

Lemma 3.1.
- *An elliptic γ has exactly one fixed point in H^d, none in $\hat{\mathbb{R}}^d$.*
- *A parabolic γ has exactly one fixed point in $\hat{\mathbb{R}}^d$, none in H^d.*
- *A hyperbolic γ has exactly 2^n fixed points in $\hat{\mathbb{R}}^d$, none in H^d.*

Lemma 3.2. *(Selberg, see [Sh], footnote p.45) For $\gamma\in\Gamma$ mixed, $\tau_j\gamma$ is never parabolic.*

For the purposes of parameterizing conjugacy classes, one may separate the hyperbolic elements into two classes: let's say a hyperbolic element $\gamma \in \Gamma$ is of **compact type** if it has the property that none of its fixed points are cusps. Recall that a **cusp** is a fixed point in $\hat{\mathbb{R}}^d$ of a parabolic element. If $\gamma \in \Gamma$ is hyperbolic and not of compact type, then it is said to be of **non–compact type**. This terminology is suggested by the following fact:

Lemma 3.3. *(Shimizu [Sh]): If $\gamma \neq \pm 1$ is an element of Γ which is hyperbolic and <u>not</u> of non–compact type then $G(\mathbb{R})_\gamma/\Gamma_\gamma$ is compact where*

$$G := PSL(2)\,,$$

and $G(\mathbb{R})_\gamma :=$ $\mathrm{Cent}(\gamma, G(\mathbb{R}))$. If γ is hyperbolic and of non–compact type then $G(\mathbb{R})_\gamma/\Gamma_\gamma$ is non–compact.

As for the cusps themselves, there is the following basic result:

Lemma 3.4. *(Maass) There are h (parabolic) cusps,*

$$\begin{aligned}
\zeta_1 &= (i\infty, \ldots, i\infty)\,,\\
\zeta_2 &= (\zeta_2^{(1)}, \ldots, \zeta_2^{(d)})\,,\\
&\;\;\vdots\\
\zeta_h &= (\zeta_h^{(1)}, \ldots, \zeta_h^{(d)})
\end{aligned}$$

such that no two are Γ–equivalent and any other cusp is Γ–equivalent to exactly one of these.

Here $h \geq 1$ denotes the class number of K. Each cusp $\lambda \in \hat{\mathbb{R}}^d$ of Γ can be considered as an element of $\hat{K} := K \cup \{\infty\}$, via $\hat{K} \hookrightarrow \mathbb{R}^d$. Of the h ideal classes, let us choose integral ideals $A_1, \ldots, A_h$ in such a way that each belongs to a different ideal class and each has a minimum norm among all integral ideals of its class. These A_i will be fixed once and for all.

For each cusp λ, Siegel [Si] has given a fundamental domain for the stabilizer Γ_λ: write $\lambda = a/c,\ a, c \in \mathcal{O}_K$, with $a = 1,\ c = 0$ if $\lambda = \infty = (\infty, \ldots, \infty) \in \hat{\mathbb{R}}^d$, Let $A = (a, c)$ be one of the ideals A_i above, let $\alpha_1, \ldots, \alpha_d$ be a basis for the $\mathcal{O}_K$–module A^{-2}, and fix a

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, K)\,,$$

with $b, d \in A^{-1}$. Then there are local coordinates $X_i, Y_j, 1 \le i \le d,\ 1 \le j \le d-1$, depending on $z = x + iy \in \mathbb{C}^d$, such that

$$\sum_{j=1}^{d-1} Y_j \log |\tau_k \epsilon_j| = \frac{1}{2} \log((\gamma^{-1} y)_k / N(\gamma^{-1} y)^{1/d}) \,, \tag{3.2}$$

for $1 \le k \le d-1$ (where the subscript k denotes the k^{th} coordinate) and such that

$$\sum_{i=1}^{d} X_i \tau_k(\alpha_i) = (\gamma^{-1} x)_k \,, \tag{3.3}$$

for $1 \le k \le d$. A **fundamental domain** for Γ_γ in H^d is

$$F_\lambda := \{z \in H^d \mid \quad \begin{array}{l} X_i \in [-\frac{1}{2}, \frac{1}{2})\,, \text{for } 1 \le i \le d, \text{ and} \\ Y_j \in [-\frac{1}{2}, \frac{1}{2})\,, \text{for } 1 \le j \le d-1\} \,. \end{array} \tag{3.4}$$

Furthermore,

$$\Gamma_\lambda = \gamma \cdot \left\{ \begin{pmatrix} \epsilon & \zeta\epsilon^{-1} \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \epsilon \in \mathcal{O}_K^\times \,,\ \zeta \in A^{-2} \right\} \cdot \gamma^{-1} \,, \tag{3.5}$$

where $\gamma = \gamma(\lambda)$ is as choosen above, and (in particular) if $\lambda = \infty$ then,

$$\Gamma_\infty = \left\{ \begin{pmatrix} \epsilon & \zeta\epsilon^{-1} \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \epsilon \in \mathcal{O}_K^\times,\ \zeta \in A^{-2} \right\}, \tag{3.6}$$

as in [Si] pp. 249-250. In fact, in this case, $A^{-2} = \mathcal{O}_K$.

From this the fundamental domain of Γ can be constructed, as follows. For a cusp $\lambda = a/c$ as above, with $\gamma = \gamma(\lambda)$ as above, let

$$\Delta(z, \lambda) := N(\gamma^{-1} y)^{-1/2} \,, \qquad z \in H^d \,.$$

This is the **Siegel distance** from z to λ. Observe that the distance, with respect to the usual metric

$$dz := \prod_{j=1}^{d} y_j^{-2} \cdot dx_j dy_j \,, \quad ds^2 := \sum_{j=1}^{d} y_j^{-2} (dx_j^2 + dy_j^2) \,,$$

from z to λ is infinite. The Siegel distance has the **invariance property**

$$\Delta(gz, g\lambda) = \Delta(z, \lambda) \,,$$

for $g \in \Gamma$, and the value of $\Delta(z,\lambda)$ does not depend on the above choice of $\gamma = \gamma(\lambda)$. Let

$$\Delta(z) := \inf_\lambda \Delta(z,\lambda). \tag{3.7}$$

Lemma 3.5. *(Siegel, [Si] p. 257) For each $z \in H^d$ there is a cusp $\lambda \in \hat{K}$ such that $\Delta(z) = \Delta(z,\lambda)$.*

The cusp mentioned in this lemma is usually unique (see [Si]) and is the cusp with the smallest Siegel distance to z. If $\Delta(z) = \Delta(z,\lambda)$ then we say that z is **semi–reduced with respect to** λ. Let

$$H_i := \{z \in H^d \mid \Delta(z) = \Delta(z,\zeta_i)\}, \tag{3.8}$$

in the notation of Lemma 3.4, and let F_i be a fundamental domain for ζ_i. Define z to be **reduced with respect to** Γ if

- z is semi–reduced with respect to some ζ_i, and
- $z \in \overline{F}_i$.

Then

$$\begin{aligned} F \; &:= \{z \in H^d \mid z \text{ is reduced with respect to } \Gamma\} \\ &= \bigcup_{i=1}^h (\overline{F}_i \cap H_i) \subset \bigcup_{i=1}^h \overline{F}_i \end{aligned} \tag{3.9}$$

is a **fundamental domain for** Γ.

If $K = \mathbb{Q}(\sqrt{D})$, D squarefree, $D > 1$, and $\lambda = \infty$, then equations (3.2, 3.3, 3.6) give that

$$\Gamma_\infty = \left\{ \begin{pmatrix} \epsilon & \epsilon^{-1}\zeta \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \epsilon \in \mathcal{O}_K^\times, \zeta \in \mathcal{O}_K \right\} \tag{3.10}$$

has fundamental domain

$$F_\infty = \left\{ z \in H^2 \mid \epsilon_1^{-2} \le \frac{y_1}{y_2} \le \epsilon_1^2, \; -\frac{1}{2} \le \frac{(x_1+x_2)}{2} \le \frac{1}{2}, \; -\frac{1}{2} \le \frac{(x_1-x_2)}{2\cdot\sqrt{D}} \le \frac{1}{2} \right\}, \tag{3.11}$$

where $z := (x_1 + iy_1, x_2 + iy_2)$, and ϵ_1 is a fundamental unit > 1. Incidently, this notation does not seem to be the same as [Zo].

If ζ_j denotes the j^{th} cusp, Γ_j the stabilizer for ζ_j, and $\sigma_j \in SL(2,K)$ the element such that

$$\sigma_j^{-1}\Gamma_j\sigma_j = \Gamma_\infty\,, \tag{3.12}$$

then consider the truncated fundamental domains

$$\begin{aligned} F_i^Y &:= \{z \in F_i \mid N(\mathrm{Im}(\sigma_i^{-1}z)) < Y\}\,, \\ F^Y &:= \{z \in F \mid N(\sigma_i^{-1}z)) < Y\,, \forall\, 1 \le i \le h\}\,. \end{aligned} \tag{3.13}$$

Later, for the purpose of estimating some error terms, it will be useful to know that these truncated domains satisfy the following property:

$$F - F^Y \subset \bigcup_{i=1}^{h} (\overline{F}_i - \overline{F}_i^Y)\,,$$

for $Y > 1$.

3.1. The cuspidal contribution

This material is also well–known (see Kubota [Kub] and Zograf [Zo]).

Take $K := \mathbb{Q}(\sqrt{D})$, $D > 1$ squarefree, let $\epsilon > 1$ be a fixed fundamental unit, and let $G(\mathbb{R}) := PSL(2,\mathbb{R})^2$. Consider

$$dz := dz_1 dz_2\,, \quad dz_j := \frac{dx_j dy_j}{y_j^2}\,,$$

and $u(w,w') := |w-w'|^2/(\mathrm{Im}\, w \cdot \mathrm{Im}\, w')$, $\quad (w,w' \in H)$, so that $u(gw,gw') = u(w,w')$, for all $g \in PSL(2,\mathbb{R})$. Let $\mathcal{S}(\mathbb{R}^2)$ denote the Schwartz space. Fix two $k_1, k_2 \in \mathcal{S}(\mathbb{R}^2)$, write (abusing notation)

$$k_j(z,z') := k_j(u(z_1,z_1'), u(z_2,z_2'))\,;$$

such functions are sometimes called **point–pair invariants**, since $k_j(gz,gz') = k_j(z,z')$, for all $g \in G(\mathbb{R})$. Let me assume that there is a $k_0 \in \mathcal{S}(\mathbb{R}^2)$ such that $k_0(u(z_1,z_1'), u(z_2,z_2')) = k(z,z')$, where

$$\begin{aligned} k(z,z') &:= (k_1 * k_2)(z,z') \\ &= \int_{H^2} k_1(z,z'')k_2(z'',z')\, dz''\,. \end{aligned}$$

This is a $G(\mathbb{R})$–invariant function on H^2. Let me also assume that the integral function

$$Kf(z) := \int_{H^2} k(z,z')f(z')\,dz', \quad f \in C_c^\infty(H^2)\,, \tag{3.14}$$

is positive–definite. These assumptions can be weakened considerably (see, e.g., [Zo] p. 1651).

Let F denote our fundamental domain for Γ and define $L^2(F,dz)$ to be the completion of the space $C_c^\infty(\Gamma\setminus H)$ of smooth Γ–invariant functions with respect to the inner product

$$(f_1,f_2) := \int_F f_1(z)\overline{f_2(z)}\,dz\,.$$

The positive–definite differential operators

$$D_j := -y_j^2\Big(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\Big) \quad , \quad j=1,2\,,$$

on $C_c^\infty(\Gamma\setminus H^2)$ extend to unbounded densely–defined self–adjoint operators on $L^2(F,dz)$. Furthermore, they generate the (commutative) algebra of $G(\mathbb{R})$–invariant differential operators on H^2 . Let

$$k_\Gamma(z,z') := \sum_{\gamma\in\Gamma} k(z,\gamma z')\,, \quad (z,z'\in F)\,, \tag{3.15}$$

and let

$$K_\Gamma f(z) := \int_F k_\Gamma(z,z')f(z')\,dz'\,, \tag{3.16}$$

for $f\in C_c^\infty(\Gamma\setminus H^2)$. Clearly, for all such f,

$$\begin{aligned} K_\Gamma f(z) &= Kf(z)\,,\\ D_jK_\Gamma f(z) &= K_\Gamma D_j f(z)\,, \end{aligned}$$

since

$$\int_F k_\Gamma(z,z')f(z')\,dz' = \int_{H^2} k(z,z')f(z')\,dz'\,,$$

and since D_j is self–adjoint.

Our first goal is to obtain a spectral decomposition for $k_\Gamma(z,z')$. The first step is to isolate the part of k_Γ in the sum (2.4) which arises, in a sense to be made precise later, from the cusps, i.e., the boundary of F. The following is well-known.

Proposition 3.6. *For any two cusps ζ_i, ζ_j, we have*

$$k_\Gamma(z,z') - \sum_{i=1}^{h} \sum_{\gamma\in\Gamma_i} k(z,\gamma z') = o(1)\,,$$

as $z \to \zeta_i,\ z' \to \zeta_j$ in F.

In other words, the contribution of the cusps to the kernel of k_Γ is

$$\sum_{i=1}^{h} \sum_{\gamma\in\Gamma_i} k(z,\gamma z') = \sum_{i=1}^{h} \sum_{\gamma\in\Gamma_\infty} k(\sigma_i^{-1}z, \gamma\sigma_i^{-1}z')\,, \tag{3.17}$$

since $\sigma_i^{-1}\Gamma_i\sigma_i = \Gamma_\infty$. In order to compare this expression for the cuspidal contribution with another more intricate expression introduced later, we need to derive a simple approximate formula for (3.17). This is easy, but the details will require more formulas, especially in the twisted case.

Let $\tau : a \longmapsto \overline{a}$ denote the non–trivial automorphism of $K/\mathbb{Q}$ and let

$$T_\tau(z_1,z_2) := (z_2,z_1), \quad T_\tau : H^2 \to H^2\,. \tag{3.18}$$

Clearly, $T_\tau^{-1} = T_\tau$. Note that T_τ doesn't commute with the action of $SL(2,K)$, in fact

$$T_\tau \circ \gamma = \tau(\gamma) \circ T_\tau = \overline{\gamma} \circ T_\tau\,.$$

On the other hand, T_τ sends cusp forms on $\Gamma \setminus H^2$ to cusp forms and automorphic forms to automorphic forms under

$$T_\tau : f(z) \longmapsto f(T_\tau z)\,,$$

see H. Saito [Sa]. The fact that T_τ sends automorphic forms to themselves follows immediately from (3.18) and the fact that the automorphy factors of γ and of $\overline{\gamma}$ coincide. The fact that T_τ also preserves cusp forms follows from Siegel's Taylor expansion [Si], Theorem 17 p. 275. (This action does <u>not</u> commute with the Hecke operators. Indeed, the subspace of the space of cusp forms on which T_τ <u>does</u> commute with the Hecke operators is precisely the image of a base change lift; see H. Saito [Sa].)

Just as in the untwisted case, where there is the kernel

$$k_\Gamma(z,z') := \sum_{\gamma\in\Gamma} k(z,\gamma z')\,, \tag{3.19}$$

in the twisted case there is the **twisted kernel**

$$k_\Gamma^\tau(z,z') := \sum_{\gamma\in\Gamma} k(z,\gamma T_\tau z')\,. \tag{3.20}$$

For $f \in C^\infty(\Gamma \setminus H^2)$, we have

$$K_\Gamma^\tau f(z) := \int_F k_\Gamma^\tau(z,z') f(z')\ dz' = K^\tau f(z)\,, \tag{3.21}$$

where

$$K^\tau f(z) := \int_{H^2} k(z, T_\tau z') f(z')\, dz'.$$

As operators on $C^\infty(\Gamma \setminus H^2)$, neither k_Γ nor k_Γ^τ are invariant; however, K and K^τ <u>are</u> invariant. From

$$T_\tau D_1 f(z) = D_2 T_\tau f(z), \quad T_\tau D_2 f(z) = D_1 T_\tau f(z)\,,$$

it follows that if f is a simultaneous eigenfunction of both $D_1,\ D_2$ then $T_\tau f$ is also a simultaneous eigenfunction of the D_j. The theory invariant operators [Kub], Theorem 1.3.2, gives us the following

Lemma 3.7. *There are functions $h(r_1,r_2),\ h^\tau(r_1,r_2)$ such that if f is a smooth function on H with $D_j f = \lambda_j f,\ j=1,2$ then*

$$\begin{aligned} Kf(z) &= h(\lambda_1,\lambda_2) f(z)\,, \\ K^\tau f(z) &= h^\tau(\lambda_1,\lambda_2) f(z)\,. \end{aligned}$$

Lemma 3.8. *We have*

$$h^\tau(r_1,r_2) = h(r_2,r_1)\,,$$

and, for any simultaneous eigenfunction (of the D_j) $f \in C^\infty(\Gamma \setminus H^2)$,

$$k_\Gamma f = h(\lambda_1,\lambda_2) f\,, \quad k_\Gamma^\tau f = h^\tau(\lambda_1,\lambda_2) f\,.$$

The proof of (3.8) is omitted.

The function h is called the **Selberg** or **Harish–Chandra transform** of k_0 (or of k) in (3.14). We may regard h^τ as the **twisted Selberg/Harish–Chandra**

transform of k_0. If $k_0 \in \mathcal{S}(\mathbb{R}^2)$ then h is determined from the following integral formulas:

$$Q(w_1, w_2) := \int_{w_2}^{\infty} \int_{w_1}^{\infty} \frac{k_0(t_1,t_2)}{[(t_1-w_1)(t_2-w_2)]^{1/2}}\ dt_1 dt_2\,,$$

$$k_0(t_1, t_2) = \pi^{-2} \int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{\frac{\partial^2 Q(w_1,w_2)}{\partial w_1 \partial w_2}}{[(w_1-t_1)(w_2-t_2)]^{1/2}}\ dw_1 dw_2\,,$$

$$h(r) = \int_{\mathbb{R}^2} Q(e^{u_1} + e^{-u_1} - 2, e^{u_2} + e^{-u_2} - 2) e^{ir\cdot u}\, du\,,$$
$$Q(e^{u_1} + e^{-u_1} - 2, e^{u_2} + e^{-u_2} - 2) = (2\pi)^{-2} \int_{\mathbb{R}^2} h(r) e^{-ir\cdot u}\ dr,$$

where $u := (u_1, u_2),\ r := (r_1, r_2),\ r \cdot u := r_1 u_1 + r_2 u_2$. (See, for example, [Kub], [Ef], or [Z1]). The behavior at infinity of these functions is known, thanks to a Paley–Wiener type theorem (see Lemma 2.1 above).

Let A be any integral ideal of $\mathcal{O}_K$ and let $Diff$ denote the different of $K/\mathbb{Q}$, so that $\hat{A} = A/Diff$, where

$$\hat{A} := \{x \in K \mid Tr_{K/\mathbb{Q}}(ax) \in \mathcal{O}_K, \text{for all} a \in A\}$$

is the dual lattice of A with respect to the trace form. Also, the norm of the different is the discriminant and this is related to the volume of the torus $\mathbb{R}^2/A$ as follows:

$$\mathrm{vol}(\mathbb{R}^2/A) = N(A) \cdot \mathrm{discr}(K/\mathbb{Q})^{1/2}\,.$$

Regarding $x \in K$ as $(x, \overline{x}) \in \mathbb{R}^2$, we find that

$$\hat{A} = \{x \in K \mid a \cdot x \in \mathcal{O}_K, \forall a \in A\},$$

where $a \cdot x$ denotes the inner product of $(x, \overline{x})$ and $(a, \overline{a})$. The **Poisson summation formula** for the lattice A is, for $x \in \mathbb{R}^2$,

$$\sum_{a \in A} f(x + a) = (\mathrm{vol}(\mathbb{R}^2/A))^{-1} \sum_{b \in \hat{A}} \hat{f}(b) \exp(2\pi i b \cdot x)\,, \tag{3.22}$$

regarding b as $(b, \overline{b}) \in \mathbb{R}^2$, and

$$\hat{f}(b) := \int_{\mathbb{R}^2} f(x) e^{-2\pi i b\cdot x}\, dx.$$

As a first step towards computing the cuspidal contribution, take $A = \mathcal{O}_K$ in the summation formula (3.26) and choose

$$\begin{aligned} r_{11} &:= \tfrac{(y_1-y_1')^2}{y_1y_1'} = e^{u_{11}} + e^{-u_{11}} - 2, \quad u_{11} := \log\left(\tfrac{y_1}{y_1'}\right) \\ r_{22} &:= \tfrac{(y_2-y_2')^2}{y_2y_2'} = e^{u_{22}} + e^{-u_{22}} - 2, \quad u_{22} := \log\left(\tfrac{y_2}{y_2'}\right) \\ r_{12} &:= \tfrac{(y_1-y_2')^2}{y_1y_2'} = e^{u_{12}} + e^{-u_{12}} - 2, \quad u_{12} := \log\left(\tfrac{y_1}{y_2'}\right) \\ r_{21} &:= \tfrac{(y_2-y_1')^2}{y_2y_1'} = e^{u_{21}} + e^{-u_{21}} - 2, \quad u_{21} := \log\left(\tfrac{y_2}{y_1'}\right). \end{aligned}$$

Lemma 3.9. *As $z, z' \to i\infty$, we have*

$$\sum_{a\in\mathcal{O}_K} k\big(z, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} z'\big) = \frac{1}{2\sqrt{D}}\sqrt{y_1y_2y_1'y_2'}\cdot Q(r_{11}, r_{22}) + o(1)$$

and

$$\sum_{a\in\mathcal{O}_K} k\big(z, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} T_\tau z'\big) = \frac{1}{2\sqrt{D}}\sqrt{y_1y_2y_1'y_2'}\cdot Q(r_{12}, r_{21}) + o(1)\,.$$

Since we need approximate formulas for

$$\sum_{\gamma\in\Gamma_i} k(z, \gamma z'), \quad \text{and} \sum_{\gamma\in\Gamma_i} k(z, \gamma T_\tau z')\,,$$

this lemma isn't sufficient. However, it implies both

$$\sum_{\gamma\in\Gamma_\infty} k(z, \gamma z') = \frac{1}{2\sqrt{D}}\sqrt{y_1y_2y_1'y_2'}\cdot \sum_{k=-\infty}^{\infty} Q(r_{11}(k), r_{22}(k)) + o(1)\,,$$

and

$$\sum_{\gamma\in\Gamma_\infty} k(z, \gamma T_\tau z') = \frac{1}{2\sqrt{D}}\sqrt{y_1y_2y_1'y_2'}\cdot \sum_{k=-\infty}^{\infty} Q(r_{12}(k), r_{21}(k)) + o(1)\,,$$

as $z, z' \to \infty$. Here

$$\begin{aligned} r_{11}(k) &:= \tfrac{(\epsilon^{2k}y_1-y_1')^2}{\epsilon^{2k}y_1y_1'} = e^{u_{11}} + e^{-u_{11}} - 2 \quad , \quad u_{11} := \log\left(\tfrac{\epsilon^{2k}y_1}{y_1'}\right) \\ r_{22} &:= \tfrac{(\epsilon^{-2k}y_2-y_2')^2}{\epsilon^{-2k}y_2y_2'} = e^{u_{22}} + e^{-u_{22}} - 2 \quad , \quad u_{22} := \log\left(\tfrac{\epsilon^{-2k}y_2}{y_2'}\right) \\ r_{12} &:= \tfrac{(\epsilon^{2k}y_1-y_2')^2}{\epsilon^{2k}y_1y_2'} = e^{u_{12}} + e^{-u_{12}} - 2 \quad , \quad u_{12} := \log\left(\tfrac{\epsilon^{2k}y_1}{y_2'}\right) \\ r_{21} &:= \tfrac{(\epsilon^{-2k}y_2-y_1')^2}{\epsilon^{-2k}y_2y_1'} = e^{u_{21}} + e^{-u_{21}} - 2 \quad , \quad u_{21} := \log\left(\tfrac{\epsilon^{-2k}y_2}{y_1'}\right). \end{aligned}$$

From these we (finally) obtain the desired approximate formula describing the cuspidal contribution:

Proposition 3.10. *As $z, z' \to \infty$,*

$$\sum_{i=1}^{h} \sum_{\gamma \in \Gamma_i} k(z, \gamma z') = \frac{1}{2\sqrt{D}} \sum_{i=1}^{h} G_i(z, z') + o(1)\,,$$

and

$$\sum_{i=1}^{h} \sum_{\gamma \in \Gamma_i} k(z, \gamma T_\tau z') = \frac{1}{2\sqrt{D}} \sum_{i=1}^{h} G_i(z, T_\tau z') + o(1)\,,$$

where $G_i(z, z')$ equals

$$\sqrt{y_1(\sigma_i^{-1} z) y_2(\sigma_i^{-1} z) y_1'(\sigma_i^{-1} z') y_2'(\sigma_i^{-1} z')} \sum_{k=-\infty}^{\infty} Q\big(r_{11}(k,i), r_{22}(k,i)\big),$$

and where

$$r_{11}(k,i) := \tfrac{(\epsilon^{2k} y_1(\sigma_i^{-1} z) - y_1'(\sigma_i^{-1} z))^2}{\epsilon^{2k} y_1(\sigma_i^{-1} z) y_1'(\sigma_i^{-1} z)} = e^{u_{11}} + e^{-u_{11}} - 2, \quad u_{11} := \log\big(\tfrac{\epsilon^{2k} y_1(\sigma_i^{-1} z)}{y_1'(\sigma_i^{-1} z)}\big)$$
$$r_{22}(k,i) := \tfrac{(\epsilon^{-2k} y_2(\sigma_i^{-1} z) - y_2'(\sigma_i^{-1} z))^2}{\epsilon^{-2k} y_2(\sigma_i^{-1} z) y_2'(\sigma_i^{-1} z)} = e^{u_{22}} + e^{-u_{22}} - 2, \quad u_{22} := \log\big(\tfrac{\epsilon^{-2k} y_2(\sigma_i^{-1} z)}{y_2'(\sigma_i^{-1} z)}\big)$$

3.2. Eisenstein series and the continuous spectrum

Thoughout $K/\mathbb{Q}$ denotes a real quadratic extension $\mathbb{Q}(\sqrt{D})$, $D \equiv 2, 3 \pmod 4$ square–free, and $\epsilon > 1$ denotes a fixed fundamental unit. In this section, the class number $h = h(D)$ of K is arbitrary.

Again, the results of this subsection are known (see [Ef], [Kub], or [Zo]). A **Poincaré series** is a sum of the form

$$E_{\psi,n,j}(z) := \sum_{\gamma \in \Gamma_j \backslash \Gamma} \psi(y_1(\sigma_j^{-1}\gamma z) y_2(\sigma_j^{-1}\gamma z)) \Big[\frac{y_1(\sigma_j^{-1}\gamma z)}{y_2(\sigma_j^{-1}\gamma z)}\Big]^{\frac{i\pi n}{2 log \epsilon}}, \tag{3.23}$$

for $1 \leq j \leq h$, $z \in H^2$, $n \in \mathbb{Z}$, and $\psi \in C_c^\infty(\mathbb{R})$. These are also known as incomplete theta series.

It is easy to see that $E_{\psi,n,j} \in L^2(F, dz)$. In Kubota's notation, let Θ denote the L^2–closure of the subspace of all mock–Eisenstein series and denote by $\mathcal{H}_0$ its orthogonal complement. Θ turns out to be the continuous plus residual part of L^2 (see subsection 4.3 below). It is not hard to see that these spaces Θ and $\mathcal{H}_0$ are preserved by the action of T_τ.

Lemma 3.11. *Let $f \in C^\infty(\Gamma \setminus H^2)$. Then $f \in \mathcal{H}_0$ if and only if*

$$\int_{\mathcal{D}} f(\sigma_i z) dx = 0 \ , \quad dx := dx_1 dx_2 \, ,$$

for all $1 \leq i \leq h$, where

$$\mathbb{D} = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid -\frac{1}{2} \leq \frac{x_1 + x_2}{2} \leq \frac{1}{2}, \ -\frac{1}{2} \leq \frac{x_1 - x_2}{2\sqrt{D}} \leq \frac{1}{2}\}$$

is the "x–part" of F_∞.

Thanks to this lemma, it can be shown that

$$\mathcal{H}_0 = L^2_{\text{cusp}}(F, dz) \tag{3.24}$$

is the completion of the space spanned by all cusp forms. From the symmetry of $\mathbb{D}$ under T_τ, one can verify that the projections

$$L^2(F, dz) \to L^2_{\text{cusp}}(F, dz) \, ,$$

and

$$L^2(F, dz) \to \Theta \, ,$$

commute with K_Γ, T_τ, and hence also K_Γ^τ. To investigate Θ further we must introduce Eisenstein series.

Let

$$E_j(z, s, k) := \sum_{\gamma \in \Gamma_j \backslash \Gamma} (y_1(\sigma_j^{-1}\gamma z) y_2(\sigma_j^{-1}\gamma z))^s \left[\frac{y_1(\sigma_j^{-1}\gamma z)}{y_2(\sigma_j^{-1}\gamma z)}\right]^{\frac{i\pi k}{2 log \epsilon}} , \tag{3.25}$$

for $s \in \mathbb{C}, \ z \in H^2, \ 1 \leq j \leq h$. Since D_j commutes with

$$y_i(z) \longmapsto y_i(\sigma_j^{-1}\gamma z) \, ,$$

it is easily seen that each $E_j(z,s,k)$ is a common eigenfunction of $D_1,\ D_2$ with eigenvalues

$$\begin{aligned} \lambda_1 = \lambda_1(s,k) &:= -\left(s + \tfrac{i\pi k}{2\log\epsilon}\right)\left(s + \tfrac{i\pi k}{2\log\epsilon} - 1\right), \\ \lambda_2 = \lambda_2(s,k) &:= -\left(s - \tfrac{i\pi k}{2\log\epsilon}\right)\left(s - \tfrac{i\pi k}{2\log\epsilon} - 1\right). \end{aligned} \tag{3.26}$$

Observe that $T_\tau E_j(z,s,k) = E_j(T_\tau z,s,k) = E_j(z,s,-k)$.

Let me derive a few basic properties of these Eisenstein series needed to investigate Θ and hence $K_\Gamma,\ K_\Gamma^\tau$. For this we need Fourier series. Let

$$e_\nu(x) = e_\nu(x_1,x_2) = e^{i\pi[(x_1+x_2)m + (x_1-x_2)n/\sqrt{D}]} = e^{i\pi(\nu x_1 + \overline{\nu} x_2)}, \tag{3.27}$$

where

$$\nu = m + n/\sqrt{D}.$$

A function $f(x_1,x_2)$ is called $\mathcal{O}_\mathbf{K}$**–periodic** if

$$f(x_1+a, x_2+\overline{a}) = f(x_1,x_2),$$

for all $a \in \mathcal{O}_K$. In this case, it is defined on the domain $\mathcal{D}$ (in (3.11)) and, if it is sufficiently well–behaved, it has a **Fourier expansion**

$$f(x_1,x_2) = \sum_{\nu \in \hat{\mathcal{O}_K}} a_\nu e_\nu(x_1,x_2), \tag{3.28}$$

where

$$a_\nu := \frac{1}{2\sqrt{D}} \int_{\mathcal{D}} f(x) e_{-\nu}(x)\, dx.$$

Of course, each $E_j(z,s,k)$ has a Fourier expansion at each of the h cusps. Let us first obtain the Fourier expansion of $E_1(z,s,k)$ at the cusp $\zeta_1 = \infty \in \hat{K} \hookrightarrow \hat{\mathbb{R}}^2$. We have

$$a_\nu(y,s,k) := \frac{1}{2\sqrt{D}} \int_{\mathcal{D}} E_1(z,s,k) e_{-\nu}(x)\, dx. \tag{3.29}$$

We have a **Bruhat decomposition**:

Lemma 3.12. • $\Gamma_\infty \setminus \Gamma = \Gamma_\infty \cup \left(\bigcup_{c,d} \Gamma_\infty \gamma \Gamma_0\right)$, *where the union runs over representatives* γ *of*

$$((c), d(\mathrm{mod}(c))) \;\; s.t. \;\; \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \;\; (i.e., (c,d) = 1).$$

- *If $\gamma \notin \Gamma_\infty$ and $\sigma_1 \neq \sigma_2$, with $\sigma_i \in \Gamma_0$ then*

$$\Gamma_\infty \cdot \gamma_1\sigma_1 \neq \Gamma_\infty \cdot \gamma_2\sigma_2 \,.$$

Since $E_j(z,s,k)$ is Γ_0–invariant, it is $\mathcal{O}_K$–periodic in x. This implies

$$\begin{aligned} a_\nu(y,s,k) &= \delta_{0m}\delta_{0n} y_1^{s+\frac{i\pi k}{2\log\epsilon}} y_2^{s-\frac{i\pi k}{2\log\epsilon}} \\ &+\tfrac{1}{2\sqrt{D}} \textstyle\int_{\mathbb{R}^2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (y_1(\gamma z)y_2(\gamma z))^s \left[\frac{y_1(\gamma z)}{y_2(\gamma z)}\right]^{\frac{i\pi k}{2log\,\epsilon}} e_{-\nu}(x)\,dx\,, \end{aligned} \tag{3.30}$$

where γ runs over representatives

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma\,,$$

with $c \neq 0$, and ν is as in (3.27). If $\nu = 0$ then $a_0(y,s,k)$ equals

$$y_1^{s+\frac{i\pi k}{2\log\epsilon}} y_2^{s-\frac{i\pi k}{2\log\epsilon}} + \phi_0(s,k) y_1^{1-s-\frac{i\pi k}{2\log\epsilon}} y_2^{1-s+\frac{i\pi k}{2\log\epsilon}} \tag{3.31}$$

where

$$\phi_0(s,k) = \frac{\pi}{2\sqrt{D}} \Big(\sum_c \xi_{-k}(c)\phi(c)N(c)^{-2s}\Big) \frac{\Gamma(-\frac{1}{2}+s+\frac{i\pi k}{2\log\epsilon})\Gamma(-\frac{1}{2}+s-\frac{i\pi k}{2log\epsilon})}{\Gamma(s+\frac{i\pi k}{2\log\epsilon})\Gamma(s-\frac{i\pi k}{2\log\epsilon})}\,,$$

and

$$\phi(c) := |\{d \bmod (c) \mid (c,d)=1\}|\,.$$

This expression was obtained by Zograf [Zo], p. 1642. An Euler product argument allows one to conclude that

$$\sum_c \xi_{-k}(c)\phi(c)N(c)^{-2s} = L(2s-1,\xi_{-k})L(2s,\xi_{-k})^{-1}\,, \tag{3.32}$$

where $L(s,\xi)$ denotes the Hecke L–series for ξ.

The case $\nu \neq 0$ is not substantially different. If $\nu = m + n/\sqrt{D}$, then

$$G_\nu(c) := \sum_{\substack{(d) \pmod{(c)} \\ (c,d)=1}} e_{-\nu}\Big(\frac{d}{c}, \frac{\overline{d}}{\overline{c}}\Big)$$

is the Gauss sum attached to our real quadratic field. It can be calculated that

$$\begin{aligned} a_\nu(y,s,k) \quad &= \tfrac{4\pi^{\alpha+\beta}}{2^{\alpha+\beta}\sqrt{D}}\sqrt{y_1y_2}(\nu\overline{\nu})^{s-\frac{1}{2}}\xi_k(\nu)\times \\ &\times\phi_\nu(s,k)\tfrac{K_{\alpha-\frac{1}{2}}(\pi y_1|\nu|)K_{\beta-\frac{1}{2}}(\pi y_2|\overline{\nu}|)}{\Gamma(\alpha)\Gamma(\beta)}\,, \end{aligned} \tag{3.33}$$

where

$$\nu := m + n/\sqrt{D} \quad \overline{\nu} := m - n/\sqrt{D}\,,$$

and

$$\phi_\nu(s,k) := \sum_c \xi_{-k}(c)G_\nu(c)N(c)^{-2s}\,.$$

(The expression (3.33) differs from Zograf's by a factor of 2.)

The Fourier expansion of the Eisenstein series is given by

$$E_i(\sigma_j z,s,k) := \sum_\nu a_\nu(y,s,k,i,j)e_\nu(x)\,, \tag{3.34}$$

where

$$a_\nu(y,s,k,i,j) := \frac{1}{2\sqrt{D}}\int_{\mathcal{D}} E_i(\sigma_j z,s,k)e_{-\nu}(x)\,dx\,.$$

The computation is almost the same as when $i=j=1$, except that the Bruhat decomposition must be replaced by

$$\Gamma_\infty \setminus \sigma_i^{-1}\Gamma\sigma_j = \begin{cases} \Gamma_\infty \cup \left(\bigcup_{c,d}\Gamma_\infty\gamma\Gamma_0\right), & \text{for } i=j \\ \bigcup_{c,d}\Gamma_\infty\gamma\Gamma_0, & \text{for } i \neq j\,, \end{cases} \tag{3.35}$$

where the union runs over representatives of

$$((c)\,,\,(d) \bmod (c)) \quad s.t. \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j\,.$$

For this, I refer to [Kub], pp. 14-16.

Lemma 3.13. *We have*

$$a_0(y,s,k,i,j) = \delta_{ij}y_1^\alpha y_2^\beta + \phi(s,k,i,j)y_1^{1-\alpha}y_2^{1-\beta}\,,$$

where

$$\begin{array}{rl} \phi(s,k,i,j) & := \sqrt{\pi}\,\Gamma(s)^{-1}\phi_0(s,k,i,j)\,, \\ \phi_\nu(s,k,i,j) & := \sum_c \xi_{-k}(c) G_\nu(c,i,j) N(c)^{-2s}\,, \\ G_\nu(c,i,j) & := \sum_{\substack{(d)\ mod(c) \\ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in\Gamma}} e_{-\nu}(d/c, \overline{d}/\overline{c})\,, \end{array}$$

and

$$a_\nu(y,s,k,i,j) = \frac{4\pi^{\alpha+\beta}\sqrt{y_1 y_2}}{2^{\alpha+\beta}\sqrt{D}}(\nu\overline{\nu})^{s-\frac{1}{2}}\xi_k(\nu)\phi_\nu(s,k,i,j)\frac{K_{\alpha-\frac{1}{2}}(\pi y_1|\nu|)K_{\beta-\frac{1}{2}}(\pi y_2|\overline{\nu}|)}{\Gamma(\alpha)\Gamma(\beta)}\,,$$

and where α and β are as above.

The functions $\phi_\nu(s,k,i,j)$ are all Dirichlet series and the $h\times h$ matrix

$\Phi_k(s) := \Big(beginarrayccc\phi_0(s,k,1,1)...\phi_0(s,k,1,h) \vdots \ddots \vdots \phi_0(s,k,h,1)...\phi_0(s,k,h,h)$

*iscalledthe***constanttermmatrix***.Itisasymmetricmatrixwhichbecomesunitaryonthe"criticalline"*
s = 1_2 (see [Kub], [Ef]). The diagonal entries of the constant term matrix are important for the following reason:

Lemma 3.14. *$E_i(z,s,k)$ is holomorphic in $\{Re\, s > 0\}$ except possibly for a finite number of simple poles which must all lie in the interval $(\frac{1}{2},1]$ and must also coincide with the (possible) simple poles of $\phi_0(s,k,i,i)$. Furthermore, $E_i(z,s,k)$ has a unique finite limit as $s\to\frac{1}{2}+it$ in $\{Re\, s > \frac{1}{2}\}$.*

Remark 2. *By Zograf [Zo] p. 1643, $E_i(z,s,k)$ has no poles on $(\frac{1}{2},1]$ if $k\neq 0$; see also [Ef]. For the proof of (3.28), see [Kub] Theorems 4.3-4.5.*

We've finally compiled enough information to return to the study of the space $\Theta\subset L^2(F,dx)$.

Let $\Theta_0\subset\Theta$ be the L^2–closure of the subspace of Θ spanned by the simultaneous eigenfunctions of D_1, D_2 in Θ. It is known that Θ_0 is spanned by the residues of $E_j(z,s,0)$ at the poles of $\phi_0(s,0,i,i)$ ([Kub], pp. 52-53). Let Θ_0' denote the orthgonal complement of Θ_0 in Θ. Since the $E_j(z,s,0)$ are T_τ–invariant, the decomposition $\Theta=\Theta_0\oplus\Theta_0'$ commutes with T_τ, K_Γ, and K_Γ^τ. It can be shown that

$$\Theta_0 = L^2_{\mathrm{res}}(F,dz) = \mathbb{C}^n\,, \tag{3.36}$$

where

$$n := \{\rho \in (\frac{1}{2}, 1] \mid \rho \text{ is a pole of } \phi_0(s,0,i,i), \text{ for some } 1 \le i \le h\}\,.$$

We also have $\Theta_0' = L^2_{\rm res}(F, dz)$. Furthermore, it is known that $n = 1$ and the only contribution to $L^2_{\rm res}(F, dz)$ comes from the simple pole of $\phi_0(s,0) = \phi_0(s,0,1,1)$ at $s = 1$ [Zo] p. 1644, [Ef] p. 64.

Theorem 3.15. *We have the following K_Γ^τ–invariant and K_Γ–invariant decomposition:*

$$\begin{aligned} L^2(F, dz) &= \mathcal{H}_0 \oplus \Theta_0 \oplus \Theta_0' \\ &= L^2_{\rm cusp}(F, dz) \oplus L^2_{\rm res}(F, dz) \oplus L^2_{\rm cont}(F, dz)\,. \end{aligned}$$

3.3. The spectral decomposition of the kernel

The notation of the previous subsection is preserved; in particular, we assume that $D \equiv 2, 3\ mod 4$ is square–free.

We breifly recall some results of Zograf [Zo] and obtain their twisted analogs. In Zograf's notation, set

$$\begin{aligned} T_\Gamma(z, z') \;:=\; & \tfrac{1}{16\pi\sqrt{D}\log\epsilon} \textstyle\sum_{j=1}^{h} \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(r + \frac{\pi k}{2\log\epsilon}, r - \frac{\pi k}{2\log\epsilon}) \times \\ & \times E_j(z, \tfrac{1}{2} + ir, k) E_j(z', \tfrac{1}{2} - ir, -k)\, dr\,, \end{aligned} \tag{3.37}$$

and in the twisted case set

$$T_\Gamma^\tau(z, z') := T_\Gamma(z, T_\tau z') \tag{3.38}$$

(i.e., replace $E_j(z', \frac{1}{2} - ir, -k)$ by $E_j(z', \frac{1}{2} - ir, k)$ in the integrand). Set

$$S_\Gamma(z, z') := k_\Gamma(z, z') - T_\Gamma(z, z') \tag{3.39}$$

and

$$S_\Gamma^\tau(z, z') := S_\Gamma(z, T_\tau z')\,. \tag{3.40}$$

The following fact is essentially well-known.

Lemma 3.16. *$S_\Gamma(z, z')$ and $S_\Gamma^\tau(z, z')$ are bounded on $F \times F$, and the integral operators*

$$\begin{aligned} S_\Gamma f(z) &:= \textstyle\int_F S_\Gamma(z, z') f(z')\, dz'\,, \\ S_\Gamma^\tau f(z) &:= \textstyle\int_F S_\Gamma^\tau(z, z') f(z')\, dz' = S_\Gamma T_\tau f(z)\,, \end{aligned}$$

for $f \in C_c^\infty(\Gamma\backslash H^2)$, are normal, compact, Hilbert–Schmidt operators on $L^2(F, dz)$.

The following corollary will be used in the spectral decomposition below.

Lemma 3.17. *(a) The restriction of S_Γ, S_Γ^τ to $L^2_{\text{cont}}(F,dz)$ is trivial,i.e.,*

$$\begin{aligned} K_\Gamma|_{L^2_{\text{cont}}} &= T_\Gamma|_{L^2_{\text{cont}}} , \\ K_\Gamma^\tau|_{L^2_{\text{cont}}} &= T_\Gamma^\tau|_{L^2_{\text{cont}}} . \end{aligned}$$

(b) The restriction of T_Γ, T_Γ^τ to $L^2_{\text{cusp}}(F,dz) \oplus L^2_{\text{res}}(F,dz)$ is trivial, i.e.,

$$T_\Gamma f = T_\Gamma^\tau f = 0 ,$$

for all $f \in L^2_{\text{cusp}}(F,dz) \oplus L^2_{\text{res}}(F,dz)$.

proof: See Kubota [Kub] p. 59, for example. □

In $L^2_{\text{cusp}}(F,dz) \oplus L^2_{\text{res}}(F,dz)$ we can select an orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of common eigenfunctions of D_1, D_2. We denote the corresponding eigenvalues by $\lambda_j^{(i)}$:

$$D_i f_j = \lambda_j^{(i)} f_j, \quad j \in \mathbb{N},\ i = 1,2 . \tag{3.41}$$

Using the separation of variables technique from differential equations, it is easy enough to obtain some information about the Fourier coefficients of the $f_j \in L^2_{\text{cusp}}(F,dz)$. Following Zagier [Z1] p. 307, write

$$D_i f_j = (1/4 + r_{ij}^2) f_j,\ f_j \in L^2_{\text{cusp}}(F,dz) . \tag{3.42}$$

Since D_i is positive definite, one has $r_{ij}^2 \geq -1/4$. The Fourier expansion may be written

$$f_j(z) = \sum_\nu A_\nu(y_1,y_2,j) e_\nu(x) . \tag{3.43}$$

Since, in the notation of Lemma 2.1, (2.1),

$$D_j e_\nu(x) = \tau_j(\nu)^2 = \begin{cases} y_1^2\pi^2\nu^2 , & j=1, \\ y_2^2\pi^2\overline{\nu}^2, & j=2 , \end{cases}$$

$$\text{and} \qquad D_i A_\nu(y_1,y_2,j) = -y_i^2 \tfrac{\partial^2}{\partial y_i^2} A_\nu(y_1,y_2,j) ,$$

it follows that $A_\nu(y)$ satisfies the Bessel equations, for i = 1,2,

$$-y_i^2 \frac{\partial^2}{\partial y_i^2} A_\nu(y_1,y_2,j) + \pi^2 y_i^2 \tau_i(\nu)^2 A_\nu(y_1,y_2,j) - (1/4 + r_{ij}^2) A_\nu(y_1,y_2,j) = 0 .$$

The bounded solutions to this are of the form

$$A_\nu(y_1, y_2, j) = \rho_j(\nu)\sqrt{y_1 y_2} K_{ir_{1j}}(\pi y_1|\nu|) K_{ir_{2j}}(\pi y_2|\overline{\nu}|)\,, \tag{3.44}$$

where $\nu := m + n/\sqrt{D}$, $\overline{\nu} := m - n/\sqrt{D}$. This gives the **Fourier coefficients** $\rho_j(\nu)$ of the Hilbert modular cusp forms f_j. This expansion (3.43) can also be derived as a consequence of Shalika's multipicity one theorem for GL(n) (see, e.g., Bump [B]). A priori, the r_{ij} are either real or pure imaginary of absolute value $|r_{ij}| \le 1/2$. Whether or not the latter possibility occurs depends on whether or not complementary series contribute to the (residual) discrete spectrum.

Finally, we can write down the spectral decomposition.

The Selberg/Harish–Chandra transform satisfies

$$K_\Gamma f_j = h(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j \tag{3.45}$$

and

$$K_\Gamma^\tau f_j = h^\tau(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j = h(\lambda_j^{(2)}, \lambda_j^{(1)}) f_j\,. \tag{3.46}$$

Denote the operator norm of K_Γ to be given by

$$\| K_\Gamma \| := \Big\{ \int_{F\times F} |k_\Gamma(z,z')|^2 \; dz dz' \Big\}^{1/2},$$

so that

$$\| K_\Gamma^\tau \| = \| K_\Gamma \|\,.$$

Theorem 3.18. *(a) (Zograf) Considered as operators in the usual way, the limits*

$$\lim_{J\to\infty} \sum_{j=1}^{J} h^\tau(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j(z)\overline{f_j(z')}\,,$$

and

$$\begin{aligned}\lim\nolimits_{T,T'\to\infty} \tfrac{1}{16\pi\sqrt{D}\log\epsilon} \quad & \textstyle\sum_{j=1}^{h} \sum_{-T'<k<T'} \int_{-T}^{T} h(r + \frac{\pi k}{2\log\epsilon}, r - \frac{\pi k}{2\log\epsilon}) \times \\ & \times E_j(z, \tfrac{1}{2} + ir, k) E_j(z', \tfrac{1}{2} - ir, -k)\, dr\,,\end{aligned}$$

exist in the operator norm and these limits satisfy

$$\begin{aligned} k_\Gamma(z,z') = \quad & \textstyle\sum_{j=1}^{\infty} h^\tau(\lambda_j^{(1)}, \lambda_j^{(2)}) f_j(z)\overline{f_j(z')} \\ & + \tfrac{1}{16\pi\sqrt{D}\log\epsilon} \textstyle\sum_{j=1}^{h} \sum_{-\infty<k<\infty} \int_{-\infty}^{\infty} h(r + \frac{\pi k}{2\log\epsilon}, r - \frac{\pi k}{2\log\epsilon}) \times \\ & \times E_j(z, e\tfrac{1}{2} + ir, k) E_j(z', \tfrac{1}{2} - ir, -k)\, dr\,.\end{aligned}$$

(b) In the twisted case, we have

$$k_\Gamma^\tau(z,z') = \begin{array}{c} \sum_{j=1}^\infty h^\tau(\lambda_j^{(2)},\lambda_j^{(1)}) f_j(z)\overline{f_j(z')} \\ +\frac{1}{16\pi\sqrt{D}\log\epsilon}\sum_{j=1}^h \sum_{-\infty<k<\infty}\int_{-\infty}^\infty h(r-\frac{\pi k}{2\log\epsilon}, r+\frac{\pi k}{2\log\epsilon})\times \\ \times E_j(z,\frac{1}{2}+ir,k)E_j(z',\frac{1}{2}-ir,k)\,dr\,. \end{array}$$

proof: This follows from the results above; for more details, see Kubota [Kub] or Efrat [Ef]. The fact that the "twisting" T_τ exchanges $E(z',\frac{1}{2}-ir,-k)$ with $E(z',\frac{1}{2}-ir,k)$ is a trivial consequence of the definition of the Eisenstein series. □

4. Zagier's method

4.1. Introducing the Kloostermann sums

As a simplifying assumption, let us assume that K has class number one. This implies that $\mathcal{O}_K$ is a principle ideal domain and that $\Gamma\setminus H^2$ has only one cusp. As $D\equiv 2,3 \pmod 4$, we also have $\mathcal{O}_K=\mathbb{Z}+\sqrt{D}\mathbb{Z}$ and its dual lattice is given by

$$Diff^{-1}={\mathcal{O}_K}^\wedge=\mathbb{Z}+\frac{1}{\sqrt{D}}\mathbb{Z}.$$

In Zagier's method one of the steps is to transform

$$\frac{1}{4D}\int_{\mathcal{D}}\int_{\mathcal{D}} k_\Gamma(x+iy,x'+iy')\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx', \tag{4.1}$$

or, in the twisted case,

$$\frac{1}{4D}\int_{\mathcal{D}}\int_{\mathcal{D}} k_\Gamma^\tau(x+iy,x'+iy')\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx', \tag{4.2}$$

into an "abelian sum" (i.e., one over ideals in $\mathcal{O}_K$, not over elements of Γ) involving Kloostermann sums. This subsection will be devoted to carrying out this first step.

First, write k_Γ, k_Γ^τ in (4.1, 4.2) as a sum over (c) and $d \pmod c$, as follows.

Lemma 4.1. *Suppose $f(\gamma z,\gamma z')=f(z,z')$, for all $\gamma\in\Gamma$. We have*

$$\int_{\mathcal{D}}\int_{\mathcal{D}}\sum_{\gamma\in\Gamma} f(z,\gamma z')\ dxdx' = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sum_{c,d} f(z,\gamma_{c,d}z')\ dxdx',$$

where the sum runs over (c), with $c = 0$ and d an arbitrary unit, and representatives $(c) \neq 0,\ (d) \pmod{c}$, with $(c,d) = 1$, and

$$\gamma_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$$

(we will show that the right–hand integral is well–defined, independent of the choice of $\gamma_{c,d}$).

proof Since

$$\Gamma_\infty = \bigcup_{k=-\infty}^{\infty} \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} | a \in \mathcal{O}_K \right\} \cdot \begin{pmatrix} \epsilon^k & 0 \\ 0 & \epsilon^{-k} \end{pmatrix}, \tag{4.3}$$

the coset Γ/Γ_∞ can be expressed as the union of all possible pairs

$$(c,d) \in (A \times A)/{\mathcal{O}_K}^\times,\ (c,d) \neq (0,0), \tag{4.4}$$

with (c),(d) having no common factors, A being any ideal of (the p.i.d.) $\mathcal{O}_K$. Here $\mathcal{O}_K^\times$–equivalence means

$$(c,d) \sim (c',d') \iff c' = uc,\ d' = ud, \text{ for some } u \in \mathcal{O}_K^\times.$$

Phrased in another way: $\gamma_{c,d} \sim \gamma_{c',d'}$ (i.e., they belong to the same coset modulo Γ_∞) if and only if

$$\begin{pmatrix} * & * \\ c' & d' \end{pmatrix} = \begin{pmatrix} \epsilon^k & 0 \\ 0 & \epsilon^{-k} \end{pmatrix} \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

for some $k \in \mathbb{Z}$. If $ad - bc = 1$ and $a'd - b'c = 1$, so

$$\gamma_1 := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ ,\ \gamma_2 := \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

are two possible $\gamma_{c,d}$, we have to show $f(z, \gamma_{c,d} z')$ makes sense. We have $b = b' + nd,\ a = a' + nc$, for some $n \in \mathcal{O}_K$, if and only if $\gamma_1,\ \gamma_2$ both represent $\gamma_{c,d}$. A simple calculation shows that

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z' = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z'.$$

On the other hand, our hypothesis on f implies that

$$\begin{aligned}\int_{\mathbb{R}^2} f\big(z, \begin{pmatrix} a & b \\ c & d \end{pmatrix} z'\big)\ dx &= \int_{\mathbb{R}^2} f\big(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} z, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z'\big)\ dx \\ &= \int_{\mathbb{R}^2} f\big(z, \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} z'\big)\ dx,\end{aligned}$$

The lemma follows. $\square$

From (4.1) it follows that

$$\begin{aligned}&\int_{\mathcal{D}}\int_{\mathcal{D}} k_\Gamma(z,z')\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx' \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big(\textstyle\sum_{c,d} k(z,\gamma_{c,d}z')\big)\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx',\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\int_{\mathcal{D}}\int_{\mathcal{D}} k_\Gamma^\tau(z,z')\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx' \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big(\textstyle\sum_{c,d} k(z,\gamma_{c,d}T_\tau z')\big)\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx'.\end{aligned} \tag{4.6}$$

As in section 1, we want to treat the "c=0 term" in this expression separately. For $c \neq 0$, in the right–hand side of (4.5), replace x_1 by $x_1 + a/c$, x_2 by $x_2 + \overline{a}/\overline{c}$, x_1' by $x_1' - d/c$, x_2' by $x_2' - \overline{d}/\overline{c}$, i.e., replace $\gamma_{c,d}$ by

$$\begin{pmatrix} 1 & -a/c \\ 0 & 1 \end{pmatrix} \gamma_{c,d} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix},$$

where $\gamma_{c,d}$ is represented by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Neglecting the "c = 0 term", this yields for (4.17) the expression

$$\begin{aligned}&\textstyle\sum_{(c)\neq 0}\Big(\sum_{\substack{(d) \pmod{(c)} \\ ad\equiv 1 \pmod{(c)} \\ (c,d)\in(A\times A)/\mathcal{O}_K{}^\times}} \overline{e_\nu(a/c,\overline{a}/\overline{c})e_{\nu'}(d/c,\overline{d}/\overline{c})}\Big)\times \\ &\times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big(\textstyle\sum_{c,d} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z')\big)\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx'.\end{aligned} \tag{4.7}$$

The "c = 0 term" of $k(z,\gamma_{c,d}z')$ in (4.5) is

$$\begin{aligned}&\textstyle\sum_{\gamma\in\Gamma_\infty} k(z,\gamma z') \\ &= \textstyle\sum_{b\in\mathcal{O}_K} k_0\big(\frac{(x_1-x_1')^2+(y_1-y_1')^2}{2y_1y_1'}, \frac{(x_2-x_2')^2+(y_2-y_2')^2}{2y_2y_2'}\big) \\ &= \textstyle\sum_{\mu\in\mathcal{O}_K{}^\wedge} c_\mu(y,y')e_\mu(x-x'),\end{aligned}$$

where

$$c_\mu(y,y') := \int_{\mathbb{R}^2} e_\mu(u) k_0\Big(\frac{u_1^2+(y_1-y_1')^2}{2y_1y_1'}, \frac{u_2^2+(y_2-y_2')^2}{2y_2y_2'}\Big) du.$$

Thus the "c = 0 term" of (4.5) is

$$\delta_{\nu,\nu'} c_\nu(y,y'). \tag{4.8}$$

For $c \neq 0$, in (4.6), replace x_1 by $x_1 + a/c$, x_2 by $x_2 + \overline{a}/\overline{c}$, x_1' by $x_1' - \overline{d}/\overline{c}$, x_2' by $x_2' - d/c$. Neglecting the $c = 0$ term, this yields the analogous expression

$$\begin{array}{c} \sum_{(c)\neq 0} \left(\sum_{\substack{(d) \pmod{(c)} \\ ad\equiv 1 \pmod{(c)} \\ (c,d)\in (A\times A)/\mathcal{O}_K{}^\times}} \overline{e_\nu(a/c,\overline{a}/\overline{c})} \overline{e_{\nu'}(\overline{d}/\overline{c}, d/c)} \right) \times \\ \times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \big(\sum_{c,d} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z') \big) \overline{e_\nu(x)} e_{\nu'}(x') \; dx dx'. \end{array} \tag{4.9}$$

The "c = 0 term" of $k^\tau(z, \gamma_{c,d} z')$ in (4.6) is

$$\begin{array}{c} \sum_{\gamma \in \Gamma_\infty} k^\tau(z, \gamma z') \\ = \sum_{b \in \mathcal{O}_K} k_0\big(\frac{(x_1-x_2')^2+(y_1-y_2')^2}{2y_1y_2'}, \frac{(x_2-x_1')^2+(y_2-y_1')^2}{2y_2y_1'} \big) \\ = \sum_{\mu \in \mathcal{O}_K{}^\wedge} c_\mu^\tau(y,y') e_\mu(x - T_\tau x'), \end{array}$$

where

$$c_\mu^\tau(y,y') := \int_{\mathbb{R}^2} e_\mu(u) k_0\Big(\frac{u_1^2+(y_1-y_2')^2}{2y_1y_2'}, \frac{u_2^2+(y_2-y_1')^2}{2y_2y_1'}\Big) du.$$

Thus the "c = 0 term" of (4.6) is

$$\delta_{\nu,\overline{\nu'}} c_\nu^\tau(y,y'). \tag{4.10}$$

Here

$$S(\nu,\nu';c) := \sum_{\substack{(d) \pmod{(c)} \\ ad\equiv 1 \pmod{(c)} \\ (c,d)\in (A\times A)/\mathcal{O}_K{}^\times}} \overline{e_\nu(a/c,\overline{a}/\overline{c})} \overline{e_{\nu'}(d/c, \overline{d}/\overline{c})}, \tag{4.11}$$

and its twisted analog in (4.9), which we denote by S^τ, can be shown to be ("real quadratic") Kloostermann sums, so Weil's estimate is applicable as in the classical case. Indeed, if $(c) = (p) \subset \mathcal{O}_K$ is a prime ideal of K then $d \pmod{(p)}$ runs through a field of $N(p)$ elements and, moreover, we have an additive character of the field $\mathcal{O}_K/(p)$,

$$\psi_{\nu,\nu'}(a) := e_\nu(a/p, \overline{a}/\overline{p})e_{\nu'}(d/p, \overline{d}/\overline{p}) = \exp\left(\pi i tr\left(\frac{a\nu + d\nu'}{p}\right)\right), \quad (ad \equiv 1 \pmod{(p)}),$$

which yields the usual expression for a Kloostermann sum over a finite field. The twisted sum S^τ is treated similarly, except that ν' gets replaced by $\overline{\nu'}$.

4.2. A Change of Variables

Next, we need a clearer understanding of the expression

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\sum_{c\neq 0,d} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z')\Big)\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx' \tag{4.12}$$

and its twisted analog

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\sum_{c\neq 0,d} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z')\Big)\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx'. \tag{4.13}$$

This involves a rather lengthy sequence of change–of–variables, analogous to one carried out in the previous section, but nothing more. Once this is accomplished, we can turn to the second step of Zagier's method.

For brevity, let

$$\begin{array}{lll} \nu & := m + n/\sqrt{D} \quad , & \overline{\nu} := m - n/\sqrt{D}, \\ \nu' & := m' + n'/\sqrt{D} \quad , & \overline{\nu}' := m' - n'/\sqrt{D}. \end{array} \tag{4.14}$$

As in the previous section, we assume that $\nu \neq 0$, $\nu' \neq 0$. In the notation of (4.14), we have

$$\overline{e_\nu(x)}e_{\nu'}(x') = \exp[-\pi i(x_1\nu - x_1'\nu' + x_2\overline{\nu} - x_2'\overline{\nu}')]. \tag{4.15}$$

To reduce (4.12) to something which can be more easily understood, it is useful to change variables in such a way that x_i and x_i' then have the same coefficients in the exponent of (4.15). This amounts to replacing

$$\begin{array}{l} x_1 \longmapsto \frac{1}{c}\sqrt{\nu'/\nu}\cdot x_1,\ \ x_2 \longmapsto \frac{1}{\overline{c}}\sqrt{\overline{\nu}'/\overline{\nu}}\cdot x_2 \\ x_1' \longmapsto \frac{1}{c}\sqrt{\nu/\nu'}\cdot x_1', x_2' \longmapsto \frac{1}{\overline{c}}\sqrt{\overline{\nu}/\overline{\nu}'}\cdot x_2', \end{array} \tag{4.16}$$

provided $\nu'/\nu \in K^\times$ is totally positive (we take the positive value of the square root). Then $\overline{e_\nu} e_{\nu'}$ is replaced by

$$E_{\nu,\nu',c}(x,x') := \exp\big[-\frac{i\pi}{c}\sqrt{\nu\nu'}\cdot(x_1-x_1')-\frac{i\pi}{c}\sqrt{\overline{\nu}\overline{\nu}'}\cdot(x_2-x_2')\big], \tag{4.17}$$

and $k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z')$ is replaced by

$$k(\frac{\sqrt{\nu'/\nu}}{c}\cdot x_1+iy_1, \frac{\sqrt{\overline{\nu}'/\overline{\nu}}}{\overline{c}}\cdot x_2+iy_2, \frac{-1/c^2}{\frac{\sqrt{\nu/\nu'}}{c}\cdot x_1'+iy_1'}, \frac{-1/\overline{c}^{-2}}{\frac{\sqrt{\overline{\nu}/\overline{\nu}'}}{\overline{c}}\cdot x_2'+iy_2'}). \tag{4.18}$$

Now set $v_1 := \nu y_1$, $v_2 := \overline{\nu} y_2$, $v_1' := \nu' y_1'$, $v_2' := \overline{\nu}' y_2'$, and use the fact that $k(\lambda z, \lambda z') = k(z,z')$ to see that (4.18) is equal to

$$k\big(x_1+\frac{ic}{\sqrt{\nu\nu'}}v_1, x_2+\frac{i\overline{c}}{\sqrt{\overline{\nu}\overline{\nu}'}}v_2, -(x_1'+\frac{ic}{\sqrt{\nu\nu'}}v_1')^{-1}, -(x_2'+\frac{i\overline{c}}{\sqrt{\overline{\nu}\overline{\nu}'}}v_2')^{-1}\big). \tag{4.19}$$

Combining (4.16-4.19), we arrive at the following

Lemma 4.2. *If $\nu\nu' \in K^\times$ is totally positive then*

$$\begin{aligned}\int_{\nu y_1=\nu' y_1'>0}\int_{\overline{\nu} y_2=\overline{\nu}' y_2'>0}\tfrac{1}{y_1 y_1' y_2 y_2'}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z')\overline{e_\nu(x)}e_{\nu'}(x')\, dx dx' dv_1 dv_2 \\ = N(c)^{-2}\int_{v_1=v_1'>0}\int_{v_2=v_2'>0}\tfrac{1}{\sqrt{y_1 y_1' y_2 y_2'}}\times \\ \times\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} E_{\nu,\nu',c}(x,x') k\left(w_1, w_2, \tfrac{-1}{w_1'}, \tfrac{-1}{w_2'}\right) dx dx' dv_1 dv_2,\end{aligned}$$

where $E_{\nu,\nu',c}$ is as in (4.17) and

$$\begin{aligned} w_1 &:= x_1+\tfrac{ic}{\sqrt{\nu\nu'}}v_1,\ w_1' := x_1'+\tfrac{ic}{\sqrt{\nu\nu'}}v_1', \\ w_2 &:= x_2+\tfrac{i\overline{c}}{\sqrt{\overline{\nu}\overline{\nu}'}}v_2,\ w_2' := x_2'+\tfrac{i\overline{c}}{\sqrt{\overline{\nu}\overline{\nu}'}}v_2'. \end{aligned}$$

Lemma 4.3. *If $\nu\nu' \in K^\times$ is totally positive then*

$$\begin{aligned}\delta_{\nu,\nu'}\int_{\nu y_1=\nu' y_1'>0}\int_{\overline{\nu} y_2=\overline{\nu}' y_2'>0}\tfrac{1}{\sqrt{y_1 y_1' y_2 y_2'}} c_\nu(y,y') dv_1 dv_2 \\ = \int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2} e_\nu(u_1 y_1, u_2 y_2) k_0(u_1^2/2, u_2^2/2) du\ dy \\ = \delta_{\nu,\nu'} 64 k_0(0,0).\end{aligned}$$

Remark 3. *(a) The integrals over $\nu y_1 = \nu' y_1'$, etc., are with respect to $dv_1 dv_2$, not $dy_1 dy_2$, so there is a suppressed constant factor which will appear explicitly later.*
(b) Our assumptions on $k(z,z')$ in §1 easily imply absolute convergence of the integrals.

We need the twisted analog of Lemma 4.2. In this case, we want to do the same as in (4.16), except that the roles of x_1' and x_2' are exchanged. This amounts to replacing

$$\begin{array}{c} x_1 \longmapsto \frac{1}{c}\sqrt{\overline{\nu}'/\nu}\cdot x_1,\ x_2 \longmapsto \frac{1}{\overline{c}}\sqrt{\nu'/\overline{\nu}}\cdot x_2 \\ x_1' \longmapsto \frac{1}{\overline{c}}\sqrt{\overline{\nu}/\nu'}\cdot x_1', x_2' \longmapsto \frac{1}{c}\sqrt{\nu/\overline{\nu}'}\cdot x_2', \end{array} \tag{4.20}$$

provided $\nu'\overline{\nu} \in K^\times$ is totally positive. Then $\overline{e_\nu} e_{\nu'}$ is replaced by

$$E^\tau_{\nu,\nu',c}(x,x') := \exp\big[-\frac{i\pi}{c}\sqrt{\nu\overline{\nu}'}\cdot(x_1 - x_2') - -\frac{i\pi}{c}\sqrt{\overline{\nu}\nu'}\cdot(x_2 - x_1')\big], \tag{4.21}$$

and $k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z')$ is replaced by

$$k(\frac{\sqrt{\overline{\nu}'/\nu}}{c}\cdot x_1 + iy_1, \frac{\sqrt{\nu'/\overline{\nu}}}{\overline{c}}\cdot x_2 + iy_2, \frac{-1/c^2}{\frac{\sqrt{\nu/\overline{\nu}'}}{c}\cdot x_2' + iy_2'}, \frac{-1/\overline{c}^2}{\frac{\sqrt{\overline{\nu}/\nu'}}{\overline{c}}\cdot x_1' + iy_1'}). \tag{4.22}$$

Now set $v_1 := \nu y_1,\ v_2 := \overline{\nu} y_2,\ v_1' := \nu' y_1',\ v_2' := \overline{\nu}' y_2'$, as before, and use the fact that $k(\lambda z, \lambda T_\tau z') = k(z, T_\tau z')$ to see that (4.22) is equal to

$$k\big(x_1 + \frac{ic}{\sqrt{\nu\overline{\nu}'}}v_1, x_2 + \frac{i\overline{c}}{\sqrt{\nu\overline{\nu}'}}v_2, \frac{-1}{x_2' + \frac{ic}{\sqrt{\nu\overline{\nu}'}}v_2'}, \frac{-1}{x_1' + \frac{i\overline{c}}{\sqrt{\overline{\nu}\nu'}}v_1'}\big). \tag{4.23}$$

Whereas in the untwisted case, Zagier's method dictated that we integrate over $v_1 = v_1',\ v_2 = v_2'$, in the twisted case we must integrate over $v_1 = v_2',\ v_2 = v_1'$, i.e., $\nu y_1 = \overline{\nu}' y_2',\ \overline{\nu} y_2 = \nu' y_1'$. Summarizing (4.20-4.23), we obtain

Lemma 4.4. *If $\nu'\overline{\nu} \in K^\times$ is totally positive then*

$$\begin{aligned} \int_{\nu y_1 = \overline{\nu}' y_2' > 0} \int_{\overline{\nu} y_2 = \overline{\nu}' y_1' > 0} \frac{1}{\sqrt{y_1 y_1' y_2 y_2'}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} T_\tau z')\overline{e_\nu(x)} e_{\nu'}(x')\ dx dx' dv_1 dv_2 \\ = N(c)^{-2} \int_{v_1 = v_2' > 0}\int_{v_2 = v_1' > 0} \frac{1}{\sqrt{y_1 y_1' y_2 y_2'}} \times \\ \times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} E^\tau_{\nu,\nu',c}(x,x') k(\zeta_1, \zeta_2, \tfrac{-1}{\zeta_1'}, \tfrac{-1}{\zeta_2'})\ dx dx' dv_1 dv_2, \end{aligned}$$

where $E^{\tau}_{\nu,\nu',c}$ *is as in (4.21) and*

$$\begin{array}{ll} \zeta_1 & := x_1 + \frac{ic}{\sqrt{\nu\overline{\nu}'}} v_1, \quad \zeta_1' := x_2' + \frac{ic}{\sqrt{\nu\overline{\nu}'}} v_2', \\ \zeta_2 & := x_2 + \frac{i\overline{c}}{\sqrt{\overline{\nu}\nu'}} v_2, \quad \zeta_2' := x_1' + \frac{i\overline{c}}{\sqrt{\overline{\nu}\nu'}} v_1'. \end{array}$$

Lemma 4.5. *If* $\nu'\overline{\nu} \in K^\times$ *is totally positive then*

$$\begin{array}{c} \delta_{\nu,\overline{\nu}'} \int_{\nu y_1 = \overline{\nu}' y_2' > 0} \int_{\overline{\nu} y_2 = \overline{\nu}' y_1' > 0} \frac{1}{\sqrt{y_1 y_1' y_2 y_2'}} c^\tau_\nu(y, y') dv_1 dv_2 \\ = \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2} e_\nu(u_1 y_1, u_2 y_2) k_0(u_1^2/2, u_2^2/2) du \ dy \\ = \delta_{\nu,\overline{\nu}'} 64 k_0(0,0). \end{array}$$

Now only one more transformation remains to be performed. Define the **Zagier transform** by

$$(Zk)(t_1, t_2) := \int_{H^2} k(z_1 + t_1, z_2 + t_2, -1/z_1, -1/z_2) y_1 y_2 \ dz_1 dz_2, \tag{4.24}$$

and let

$$(Zk)^\wedge(r_1, r_2) := \int_{\mathbb{R}^2} e^{i(r_1 t_1 + r_2 t_2)/2} (Zk)(t_1, t_2) \ dt_1 dt_2. \tag{4.25}$$

This Fourier transform differs slightly from the Fourier transform in (3.22). From Lemma 4.2, we obtain

$$\begin{array}{c} \sqrt{\nu\nu'\overline{\nu\nu'}} (Zk)^\wedge(2\pi \frac{\sqrt{\nu\nu'}}{c}, 2\pi \frac{\sqrt{\overline{\nu\nu'}}}{\overline{c}}) \\ = \quad \int_{\nu y_1 = \nu' y_1' > 0} \int_{\overline{\nu} y_2 = \overline{\nu}' y_2' > 0} \frac{1}{\sqrt{y_1 y_1' y_2 y_2'}} \times \\ \times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z') \overline{e_\nu(x)} e_{\nu'}(x') \ dx dx' dy_1 dy_2, \end{array} \tag{4.26}$$

if $\nu\nu' \in K^\times$ is totally positive.

In the twisted situation, define the **twisted Zagier transform** by

$$(Z^\tau k)(t_1, t_2) := \int_{H^2} k(z_1 + t_1, z_2 + t_2, -1/z_2, -1/z_1) y_1 y_2 \ dz_1 dz_2, \tag{4.27}$$

and let

$$(Z^\tau k)^\wedge(r_1, r_2) := \int_{\mathbb{R}^2} e^{i(r_1 t_1 + r_2 t_2)/2} (Z^\tau k)(t_1, t_2) \ dt_1 dt_2. \tag{4.28}$$

From Lemma 4.2, we obtain

$$\begin{array}{rl} = & \sqrt{\nu\nu'\overline{\nu}\overline{\nu}'}(Z^{\tau}k)^{\wedge}(2\pi\frac{\sqrt{\nu\overline{\nu}'}}{c}, 2\pi\frac{\sqrt{\overline{\nu}\nu'}}{\overline{c}}) \\ & \int_{\nu y_1=\overline{\nu}' y_2'>0}\int_{\overline{\nu} y_2=\nu' y_1'>0}\frac{1}{\sqrt{y_1y_1'y_2y_2'}}\times \\ \times & \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z')\overline{e_{\nu}(x)}e_{\nu'}(x')\ dxdx'dy_1dy_2, \end{array} \tag{4.29}$$

if $\nu'\overline{\nu} \in K^{\times}$ is totally positive.

This concludes the first step.

4.3. Contribution of the Eisenstein series and the f_j

Here we examine the terms

$$\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda_1(r,k),\lambda_2(r,k))E(z,\frac{1}{2}+ir,k)E(z',\frac{1}{2}-ir,-k)\ dr \tag{4.30}$$

and

$$\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda_1(r,k),\lambda_2(r,k))E(z,\frac{1}{2}+ir,k)E(z',\frac{1}{2}-ir,k)\ dr \tag{4.31}$$

from the spectral decomposition of the kernel function, where

$$\begin{array}{l} \lambda_1 = \lambda_1(r,k) := -(s+\frac{i\pi k}{2\log\epsilon})(s+\frac{i\pi k}{2\log\epsilon}-1), \\ \lambda_2 = \lambda_2(r,k) := -(s-\frac{i\pi k}{2\log\epsilon})(s-\frac{i\pi k}{2\log\epsilon}-1). \end{array}$$

However, we are only interested in the term arising from the ν^{th} and ν'^{th} Fourier coefficients of the Eisenstein series. In other words, we consider

$$\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda_1,\lambda_2)a_{\nu}(z,\frac{1}{2}+ir,k)a_{\nu'}(z',\frac{1}{2}-ir,-k)\ dr \tag{4.32}$$

and, in the twisted case,

$$\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda_1,\lambda_2)a_{\nu}(z,\frac{1}{2}+ir,k)a_{\nu'}(z',\frac{1}{2}-ir,k)\ dr. \tag{4.33}$$

Since these Fourier coefficients involve K–Bessel functions, which converge very rapidly, there are no convergence problems here.

Let us begin with (4.32). For brevity, denote

$$\begin{array}{ll} \alpha & := \frac{1}{2} + ir + \frac{i\pi k}{2\log\epsilon} \\ \beta & := \overline{\alpha} = \frac{1}{2} - ir - \frac{i\pi k}{2\log\epsilon}, \end{array}$$

and recall that, for some constant $C = C(k, r, \nu, \nu') \in \mathbb{C}^\times$ not depending on y, we have, by (4.19),

$$\begin{aligned} & a_\nu(y, \tfrac{1}{2} + ir, k) a_{\nu'}(y', \tfrac{1}{2} - ir, -k) \\ &= C\sqrt{y_1 y_2 y_1' y_2'} \cdot K_{\alpha-1/2}(\pi y_1 |\nu|) K_{\alpha-1/2}(\pi y_1' |\nu'|) K_{\beta-1/2}(\pi y_2 |\overline{\nu}|) K_{\beta-1/2}(\pi y_2' |\overline{\nu}'|) \\ &= 4C\sqrt{\tfrac{\mu_1 \mu_2 \mu_1' \mu_2'}{\nu\nu'\overline{\nu}\overline{\nu}'}} \cdot K_{\alpha-1/2}(2\pi\mu_1) K_{\alpha-1/2}(2\pi\mu_1') K_{\beta-1/2}(2\pi\mu_2) K_{\beta-1/2}(2\pi\mu_2'), \end{aligned} \tag{4.34}$$

where

$$\mu_1 := |\nu| y_1/2,\ \mu_2 := |\overline{\nu}| y_2/2,\ \mu_1' := |\nu'| y_1',\ \mu_2' := |\overline{\nu}'| y_2'.$$

By the formula following (2.6), it is easy to see that

$$\int_0^\infty \int_0^\infty K_{\alpha-1/2}(2\pi x)^2 K_{\beta-1/2}(2\pi y)^2 \, dx dy = \frac{(\pi/8)^2}{\cosh[\pi(r + \frac{\pi k}{2\log\epsilon})]^2}.$$

This fact will be used later.

In the twisted case, one has, for some constant $C' = C'(k, r, \nu, \nu') \in \mathbb{C}^\times$ independent of y,

$$\begin{aligned} & a_\nu(y, \tfrac{1}{2} + ir, k) a_{\nu'}(y', \tfrac{1}{2} - ir, k) \\ &= C\sqrt{y_1 y_2 y_1' y_2'} \cdot K_{\alpha-1/2}(\pi y_1 |\nu|) K_{\alpha-1/2}(\pi y_1' |\nu'|) K_{\beta-1/2}(\pi y_2 |\overline{\nu}|) K_{\beta-1/2}(\pi y_2' |\overline{\nu}'|) \\ &= 4C'\sqrt{\tfrac{\mu_1 \mu_2 \mu_1' \mu_2'}{\nu\nu'\overline{\nu}\overline{\nu}'}} \cdot K_{\alpha-1/2}(2\pi\mu_1) K_{\alpha-1/2}(2\pi\mu_1') K_{\beta-1/2}(2\pi\mu_2) K_{\beta-1/2}(2\pi\mu_2'), \end{aligned} \tag{4.35}$$

where, as above,

$$\mu_1 := |\nu| y_1/2,\ \mu_2 := |\overline{\nu}| y_2/2,\ \mu_1' := |\nu'| y_1',\ \mu_2' := |\overline{\nu}'| y_2'.$$

Now we turn to the f_j. Those $f_j \in L^2_{res}(F, dz)$ may be ignored, as they must be constant and we have assumed $\nu \neq 0,\ \nu' \neq 0$. For those $f_j \in L^2_{cusp}(F, dz)$, we have

$$f_j(z) = \sum_\nu A_\nu(y, j) e_\nu(x), \tag{4.36}$$

where A_ν is given by (3.44). By the results of section 3, we have

$$\begin{array}{c} A_\nu(y,j)\overline{A_{\nu'}(y',j)} \\ = \rho_j(\nu)\overline{\rho_j(\nu')}\sqrt{y_1y_2y_1'y_2'}\times \\ \times K_{ir_{1j}}(\pi y_1|\nu|)K_{ir_{1j}}(\pi y_1'|\nu'|)K_{ir_{2j}}(\pi y_2|\overline{\nu}|)K_{ir_{2j}}(\pi y_2'|\overline{\nu}'|) \\ = 4\rho_j(\nu)\overline{\rho_j(\nu')}\sqrt{\frac{\mu_1\mu_2\mu_1'\mu_2'}{\nu\nu'\overline{\nu}\overline{\nu}'}}\times \\ \times K_{ir_{1j}}(2\pi\mu_1)K_{ir_{1j}}(2\pi\mu_1')K_{ir_{2j}}(2\pi\mu_2)K_{ir_{2j}}(2\pi\mu_2'), \end{array} \tag{4.37}$$

in the notation above. Combining the spectral decomposition with (4.3, 4.37), we find that the $f_j \in L^2_{cusp}(F, dz)$ contribute (in the untwisted case)

$$\sum_j h(\lambda_j^{(1)}, \lambda_j^{(2)})\rho_j(\nu)\overline{\rho_j(\nu')}K_{ir_{1j}}(2\pi\mu_1)K_{ir_{1j}}(2\pi\mu_1')K_{ir_{2j}}(2\pi\mu_2)K_{ir_{2j}}(2\pi\mu_2'), \tag{4.38}$$

and the Eisenstein–Maass series contribute

$$\begin{array}{c} (4\pi\sqrt{D}\log\epsilon)^{-1}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} C_{\nu,\nu'}(k,r)h(\lambda_1(r,k),\lambda_2(r,k))\times \\ \times K_{ir}(2\pi\mu_1)K_{ir}(2\pi\mu_1')K_{ir}(2\pi\mu_2)K_{ir}(2\pi\mu_2')\ dr, \end{array} \tag{4.39}$$

where

$$\begin{array}{c} \alpha := \frac{1}{2} + ir + \frac{i\pi k}{2\log\epsilon} \\ \beta := \overline{\alpha} = \frac{1}{2} - ir - \frac{i\pi k}{2\log\epsilon}, \\ \lambda_1 := -(s + \frac{i\pi k}{2\log\epsilon})(s + \frac{i\pi k}{2\log\epsilon} - 1), \\ \lambda_2 := -(s - \frac{i\pi k}{2\log\epsilon})(s - \frac{i\pi k}{2\log\epsilon} - 1). \end{array}$$

Finally, (4.7) contributes to the untwisted case

$$\begin{array}{c} \frac{1}{16D}\sqrt{\frac{\nu\nu'\overline{\nu}'\overline{\nu}}{\mu_1\mu_2\mu_1'\mu_2'}}\sum_{(c)} S(\nu,\nu';c)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k(z, \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} z')\overline{e_\nu(x)}e_{\nu'}(x')\ dxdx' \\ +\delta_{\nu,\nu'}(y_1y_2y_1'y_2')^{-1}c_\nu(y,y') \end{array} \tag{4.40}$$

Here we have cancelled $4|\nu\nu'\overline{\nu}'\overline{\nu}|^{-1/2}(\mu_1\mu_2\mu_1'\mu_2')^{-1/2}$ from both sides of the spectral decomposition and the usual assumptions on ν, ν' are in force.

In the twisted case the terms are

$$\sum_j h(\lambda_j^{(2)}, \lambda_j^{(1)})\rho_j(\nu)\overline{\rho_j(\nu')}K_{ir_{1j}}(2\pi\mu_1)K_{ir_{1j}}(2\pi\mu_1')K_{ir_{2j}}(2\pi\mu_2)K_{ir_{2j}}(2\pi\mu_2'); \tag{4.41}$$

$$\frac{1}{4\pi\sqrt{D}\log\epsilon}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} C'_{\nu,\nu'}(k,r)h^{\tau}(\lambda_1(r,k),\lambda_2(r,k))\times \\ \times K_{ir}(2\pi\mu_1)K_{ir}(2\pi\mu'_1)K_{ir}(2\pi\mu_2)K_{ir}(2\pi\mu'_2)\ dr; \tag{4.42}$$

and,

$$\frac{1}{16D}\sqrt{\frac{\nu\nu'\overline{\nu\nu'}}{\mu_1\mu_2\mu'_1\mu'_2}}\sum_{(c)} S^{\tau}(\nu,\nu';c)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k(z,\begin{pmatrix} 0 & -1/c \\ c & 0\end{pmatrix} T_{\tau}z')\overline{e_{\nu}(x)}e_{\nu'}(x')\ dxdx' \\ +\delta_{\nu,\overline{\nu'}}(y_1y_2y'_1y'_2)^{-1}c^{\tau}_{\nu}(y,y'). \tag{4.43}$$

Here again, the usual assumptions on $\nu,\ \nu'$ remain in force.

4.4. The real quadratic Kuznetsov-Bruggeman formula

In the untwisted case, one must simply integrate

$$(4.38)+(4.39)=(4.40)+(4.8)$$

over the quarter–plane $\mu_1=\mu'_1>0,\ \mu_2=\mu'_2>0$, assuming that the interchange order of integration is justified for our fixed $\nu,\ \nu'\in\mathcal{O}_K{}^{\wedge}$, and using the two–dimensional analog of the result in the appendix to section 1 to express the formula in its final form. The interchange of orders of integration can be verified just as in the $SL(2,\mathbb{Z})$ case (see the appendix to section 1). For (4.38), we obtain

$$\left(\frac{\pi}{8}\right)^2\sum_j h(\lambda_j^{(1)},\lambda_j^{(2)})\frac{\rho_j(\nu)\overline{\rho_j(\nu')}}{\cosh(\pi r_{1j})\cosh(\pi r_{2j})}, \tag{4.44}$$

whereas (4.39) yields

$$\left(\frac{\pi}{8}\right)^2(4\pi\sqrt{D}\log\epsilon)^{-1}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} C(k,r)h(\lambda_1(r,k),\lambda_2(r,k))\frac{dr}{\cosh[\pi(r+\frac{\pi k}{2\log\epsilon})]^2}, \tag{4.45}$$

and finally, if $\nu\nu'$ is totally positive then (4.40) formally yields (by (4.26))

$$\frac{1}{8D}\sqrt{\nu\nu'\overline{\nu\nu'}}\cdot\sum_{(c)}\frac{S(\nu,\nu';c)}{N(c)^2}(Zk)^{\wedge}(\frac{2\pi\sqrt{\nu\nu'}}{c},\frac{2\pi\sqrt{\overline{\nu'}\overline{\nu}}}{\overline{c}}).$$

Let $\phi_J(r_1, r_2)$ and $\phi_H(x_1, x_2)$ be the obvious two-variable generalizations of the transforms defined in (2.9). Since, for $h(r_1, r_2) = \phi_J(r_1, r_2)$, the appendix to section 2 yields the identity

$$\frac{4x_1x_2}{\pi^4}(Zk)^\wedge(x_1, x_2) = \phi_H(x_1, x_2),$$

(see (4.25) for the definitions),the contribution of (4.40) may be rewritten

$$\frac{\pi^4}{32D} \cdot \sum_{(c)} \frac{S(\nu, \nu'; c)}{N(c)} \phi_H\Big(\frac{2\pi\sqrt{\nu\nu'}}{c}, \frac{2\pi\sqrt{\overline{\nu}'\overline{\nu}}}{c}\Big). \tag{4.46}$$

The **real quadratic Kutznetsov formula** is the identity

$$(4.44) + (4.45) = (4.46) + \delta_{\nu,\nu'} k_0(0,0),$$

assuming $\nu\nu'$ is totally positive.

In the twisted case, we integrate over the quarter–plane $\mu_1 = \mu_2' > 0,\ \mu_2 = \mu_1' > 0$, formally interchanging orders of integration. Again, the validity of this can be verified by the method in the appendix to section 2. The contribution of (4.41) is

$$\sum_j h(\lambda_j^{(2)}, \lambda_j^{(1)}) \rho_j(\nu) \overline{\rho_j(\nu')} R_j^2, \tag{4.47}$$

where

$$R_j := \int_0^\infty K_{ir_{1j}}(2\pi x) K_{ir_{2j}}(2\pi x)\ dx.$$

If $|Re\, r_{1j}| + |Re\, r_{2j}| < 1$, this has an expression in terms of $\Gamma-$functions and Gauss' hypergeometric function: from [GR] p. 693, we have

$$\begin{aligned} &\int_0^\infty K_\mu(x) K_\nu(x)\ dx \\ &= \tfrac{1}{4\sqrt{\pi}} \Gamma\big(\tfrac{1+\mu+\nu}{2}\big)\Gamma\big(\tfrac{1-\mu+\nu}{2}\big)\Gamma\big(\tfrac{1+\mu-\nu}{2}\big)\Gamma\big(\tfrac{1-\mu-\nu}{2}\big) F\big(\tfrac{1+\mu+\nu}{2}, \tfrac{1-\mu+\nu}{2}; 1; 1\big), \end{aligned} \tag{4.48}$$

for $|Re\, \mu| + |Re\, \nu| < 1$. The asymptotic behavior of this can be completely determined (in fact, in [EMOT] p. 371, one can find an expression for this as a product of Γ-functions alone). Since K_{ir} is even in r, corresponding to (4.42) we have the contribution

$$\Big(\frac{\pi}{8}\Big)^2 (4\pi\sqrt{D}\log\epsilon)^{-1} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} C'(k, r) h^\tau(\lambda_1(r, k), \lambda_2(r, k)) \frac{dr}{\cosh[\pi(r + \frac{\pi k}{2\log\epsilon})]^2}, \tag{4.49}$$

whereas (4.43) yields, if $N\overline{\nu}'$ is totally positive, the contribution

$$\frac{\pi^4}{32D} \cdot \sum_{(c)} \frac{S^\tau(\nu, \nu'; c)}{N(c)^2} (Zk^\tau)^\wedge \left(\frac{2\pi\sqrt{\nu\nu'}}{c}, \frac{2\pi\sqrt{\overline{\nu}'\overline{\nu}}}{\overline{c}}\right). \tag{4.50}$$

Unfortunately, we have not been able to apply the result of the appendix to section 2 to rewrite (4.50) in an explicit form useful for applications. The **twisted real quadratic Kuznetsov formula** is

$$(4.47) + (4.49) = (4.50) + \delta_{\nu,\overline{\nu}'} k_0(0,0),$$

assuming that $\nu\overline{\nu}'$ is totally positive.

5. The method of Poincaré series and applications

5.1. Introduction

This section briefly sketches the real quadratic analog of some of the parts of [Kuz].

The purpose of this section is to discuss applications to estimating eigenvalues of Maass wave forms for $SL(2, \mathcal{O}_K)$ and sums of Kloosterman sums. In 5.1, we compute the Fourier coefficients of the Poincaré series; in 5.2, we state the Petersson formula for the inner product of two Poincaré series, involving "real quadratic" Kloosterman sums; in section 5.3, we use the Selberg–Parseval equality (derived from the Selberg spectral decomposition) to obtain another inner product formula, involving Fourier coefficients of Eisenstein series and an orthonormal basis of cusp forms of weight zero; in 5.4, we obtain the estimate

$$\begin{aligned} \sum_{r_{1j}\le T_1} \sum_{r_{2j}\le T_2} \frac{|\rho_j(\nu)|^2}{\cosh(\pi r_{1j})\cosh(\pi r_{2j})} &= cD^{-1/2}T_1^2T_2^2 + O_{\epsilon,D}((T_1T_2)^{1+\epsilon}) + \\ &\quad + O_{\epsilon,D}(T_1T_2N(\nu)^\epsilon + N(\nu)^{1/2+\epsilon}), \end{aligned} \tag{5.1}$$

for ν totally positive, where $c > 0$ is an absolute constant. This has the corollary

$$|\rho_j(\nu)| <<_{\epsilon,D} \exp[\pi(r_{1j} + r_{2j})/2] \cdot N(\nu)^{1/4+\epsilon}, \tag{5.2}$$

as $\nu \to \infty$ runs through totally positive elements. We state the real quadratic Kuznetsov-Bruggeman trace formula as Lemma 5.4, in the form presented in

[Kuz] pp. 327-328. This (untwisted) formula is, of course, exactly the same as that obtained in the previous section by Zagier's method. In 5.5, we apply our trace formula to prove the following estimate:

$$\sum_{\substack{0<c\leq T_1\\ 0<\overline{c}\leq T_2}} S(\nu,\nu',c)N(c)^{-1} <<_{\epsilon,D,\nu,\nu'} (T_1T_2)^{1/6+\epsilon}+ \sum_{\substack{0<\lambda_j^{(1)}<1/4,\\ 0<\lambda_j^{(2)}<1/4}} [(T_1T_2)^{2|r_{1j}|+\epsilon}+(T_1T_2)^{2|r_{2j}|+\epsilon}], \tag{5.3}$$

where $\nu,\ \nu'$ are fixed and totally positive, and $T_1 \approx T_2 \approx T \to \infty$ are both of the same order of magnitude.

5.2. Fourier expansion of Poincaré series

The notation of the previous section is preserved. Fix $\nu \in \hat{\mathcal{O}_K},\ z=(z_1,z_2)\in H^2$, $s=(s_1,s_2)\in \mathbb{C}^2, \mathrm{Im}(s_i)>1$. Let us denote the ν^{th} **non–holomorphic Poincaré series** by

$$U_\nu(z,s) := \sum_{\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\gamma z_1)^{s_1}(\mathrm{Im}\,\overline{\gamma}z_2)^{s_2}e_\nu(\gamma z_1,\overline{\gamma}z_2), \tag{5.4}$$

and let

$$f_\nu(z,s,c) := \sum_{\mu\in\mathcal{O}_K} |z_1+\mu|^{-2s_1}|z_2+\overline{\mu}|^{-2s_2}e_\nu\Big(\frac{-1/c^2}{z_1+\mu},\frac{-1/\overline{c}^2}{z_2+\overline{\mu}}\Big).$$

We have then that

$$\begin{aligned} U_\nu(z,s) &= y_1^{s_1}y_2^{s_2}e_\nu(z)+\\ +\sum_{\substack{c\neq 0,\\ c\in\mathcal{O}_K}} \Big(\tfrac{y_1}{|c|^2}\Big)^{s_1}\Big(\tfrac{y_2}{|\overline{c}|^2}\Big)^{s_2} &\sum_{\substack{c,\ d \pmod c,\\ (c,d)=1,\\ c\in\mathcal{O}_K}} e_\nu(a/c,\overline{a}/\overline{c})f_\nu(z_1+d/c,z_2+\overline{d}/\overline{c},c,s). \end{aligned} \tag{5.5}$$

Clearly $f_\nu(z)$ is $\mathcal{O}_K$–periodic in $x=\mathrm{Re}(z)$, so it has a Fourier expansion,

$$f_\nu(z,s,c) = \sum_{\alpha\in\hat{\mathcal{O}_K}} b_\nu(\alpha,y,s,c)e_\alpha(x),$$

where

$$\begin{aligned} b_\nu(\alpha,y,s,c) &:= \int\int_{\mathcal{D}} f_\nu(x+iy,s,c)\overline{e_\alpha(x)}\ dx\\ &= \int\int_{\mathbb{R}^2} |z_1|^{-2s_1}|z_2|^{-2s_2}e_\nu\big(\tfrac{-1/c^2}{z_1},\tfrac{-1/\overline{c}^2}{z_2}\big)\overline{e_\alpha(x)}\ dx. \end{aligned}$$

Let

$$J_{\nu,\alpha}(y_1,s_1,c) := y^{1-2s_1}\int_{\mathbb{R}} \exp i\pi[-\alpha\xi y_1 - \frac{\nu/c^2}{(i+\xi)y_1}]\frac{d\xi}{(1+\xi^2)^{s_1}},$$

and similarly for $J_{\overline{\nu},\overline{\alpha}}(y_2,s_2,\overline{c})$. Then we have, after a simple change of variables,

$$\begin{array}{ll} b_\nu(\alpha,y,s,c) & = J_{\nu,\alpha}(y_1,s_1,c)J_{\overline{\nu},\overline{\alpha}}(y_2,s_2,\overline{c}), \\ U_\nu(z,s) & = \sum_{\mu\in\hat{\mathcal{O}_K}} B_\nu(\nu,y,s)e_\mu(x), \\ B_\nu(\mu,y,s) & = y_1^{s_1}y_2^{s_2}+ \\ & +\sum_{\substack{c\neq 0\\ c\in\mathcal{O}_K}} S(\nu,\mu,c)|c|^{-2s_1}|\overline{c}|^{-2s_2}J_{\nu,\mu}(y_1,s_1,c)J_{\overline{\nu},\overline{\mu}}(y_2,s_2,\overline{c}). \end{array} \tag{5.6}$$

5.3. Petersson inner product formula

For ν, ν' totally positive, we will now compute the Petersson inner product of two non-holomorphic Poincaré series, using the "unfolding–of–the–integral" trick. Let $\mathcal{F}$ denote a fundamental domain for $\Gamma \setminus H^2$.

We have

$$\begin{array}{l} \int_{\Gamma\setminus H^2} U_\nu(z,s)\overline{U_{\nu'}(z,\overline{s'})}\ dz \\ = \sum_{\gamma\in\Gamma_\infty\setminus\Gamma}\int_{\mathcal{F}} U_\nu(z,s)(\operatorname{Im}\gamma z_1)^{s_1'}(\operatorname{Im}\overline{\gamma}z_2)^{s_2'}\overline{e_{\nu'}(\gamma z_1,\overline{\gamma}z_2)}\ dz \\ = \delta_{\nu,\nu'}\cdot\int_0^\infty\int_0^\infty y_1^{s_1+s_1'}y_2^{s_2+s_2'}\exp[-\pi(\nu'y_1+\overline{\nu'}y_2)]\frac{dy_1\,dy_2}{y_1^2y_2^2}+ \\ +\sum_{\substack{c\neq 0\\ c\in\mathcal{O}_K}} S(\nu',\nu,c)|c|^{-2s_1}|\overline{c}|^{-2s_2}\int_0^\infty\int_0^\infty y_1^{s_1'}J_{\nu,\nu'}(y_1,s_1,c)y_2^{s_2'}\times \\ \times J_{\overline{\nu},\overline{\nu'}}(y_2,s_2,\overline{c})\exp[-\pi(\nu'y_1+\overline{\nu'}y_2)]\frac{dy_1\,dy_2}{y_1^2y_2^2}. \end{array} \tag{5.7}$$

The first expression is of the form

$$\delta_{\nu,\nu'}\pi^{2-s_1-s_1'-s_2-s_2'}\nu'^{1-s_1-s_1'}\overline{\nu'}^{1-s_2-s_2'}\Gamma(s_1+s_1'-1)\Gamma(s_2+s_2'-1),$$

after a simple change of variables. The second expression is more complicated and a non–trivial change of variables is required. Following pp. 315-320 of Kuznetsov [Kuz], let us first consider an integral arising in the last expression in (5.7):

$$\begin{array}{l} \int_{\mathbb{R}}\int_0^\infty y_1^{s_1'-s_1-1}\exp[-i\pi(\xi y_1\nu'+\frac{\nu/c^2}{y_1(i+\xi)})]e^{-\pi\nu'y_1}\ dy_1\frac{d\xi}{(1+\xi^2)^{s_1}} \\ = \int_{\mathbb{R}}\int_0^\infty y_1^{s_1'-s_1-1}\exp[-\pi(i\xi y_1\nu'+y_1\nu'+\frac{\nu/c^2}{y_1(1-i\xi)})]\ dy_1\frac{d\xi}{(1+\xi^2)^{s_1}}. \end{array} \tag{5.8}$$

Note that $y_1\nu'(1+i\xi)+\frac{\nu}{c^2(1-i\xi)y_1}=\frac{\sqrt{\nu\nu'}}{|c|}\sqrt{\frac{1+i\xi}{1-i\xi}}(v_1+1/v_1)$, where we have changed variables: $v_1:=y_1\sqrt{\nu'/\nu}|c|\sqrt{1+\xi^2}$, $dy_1=\sqrt{\nu/\nu'}|c|^{-1}(1+\xi^2)^{-1/2}dv_1$.

With this change of variables, now use

$$K_s(z)=\frac{1}{2}\int_0^\infty \exp-\frac{z}{2}(u+1/u)\frac{du}{u^{s+1}},\quad \mathrm{Re}\, z>0, \tag{5.9}$$

to express (5.8) in terms of a K–Bessel function with a complex argument. (This can then be re–expressed in terms of J–Bessel functions of real arguments, as in [Kuz] pp. 319-320.) The integral (5.8) is equal to

$$2\nu^{(s_1'-s_1)/2}\nu'^{\,(s_2'-s_2)/2}|c|^{-s_1-s_1'}\int_{\mathbb{R}}(1+\xi^2)^{-(s_1+s_1')/2}K_{s_1-s_1'}(2\pi\frac{\sqrt{\nu\nu'}}{|c|}\sqrt{\frac{1+i\xi}{1-i\xi}})\ d\xi.$$

A similar expression holds for the term analogous to (5.8) obtained by replacing ν by $\overline{\nu}$, etc. In terms of the function Φ in [Kuz], (4.9) p. 315, we have

$$\begin{aligned}2|c|^{-s_1-s_1'}\left(\tfrac{\nu'}{\nu}\right)^{(s_1-s_1')/2}&\textstyle\int_{\mathbb{R}}(1+\xi^2)^{-(s_1+s_1')/2}K_{s_1-s_1'}(2\pi\frac{\sqrt{\nu\nu'}}{|c|}\sqrt{\frac{1+i\xi}{1-i\xi}})\ d\xi\\ &=\left(\tfrac{\nu'}{\nu}\right)^{(s_1-s_1')/2}\frac{2^{3-s_1-s_1'}}{\sin\pi(s_1-s_1')}|c|^{-s_1-s_1'}\Phi(s_1,s_1';2\pi\tfrac{\sqrt{\nu\nu'}}{|c|}).\end{aligned}$$

A similar expression holds for the analogous term involving $\overline{\nu}$, $\overline{\nu'}$, etc.. Collecting these computations together, we arrive at the following

Theorem 5.1. Petersson inner product formula

$$\begin{aligned}&(U_\nu(*,s),U_{\nu'}(*,\overline{s'}))\\ &=\delta_{\nu,\nu'}\pi^{2-s_1-s_1'-s_2-s_2'}\nu^{1-s_1-s_1'}\overline{\nu}^{1-s_2-s_2'}\Gamma(s_1+s_1'-1)\Gamma(s_2+s_2'-1)+\\ &\quad+\left(\tfrac{\nu'}{\nu}\right)^{(s_1-s_1')/2}\left(\tfrac{\overline{\nu'}}{\overline{\nu}}\right)^{(s_2-s_2')/2}\frac{2^{6-s_1-s_1'-s_2-s_2'}}{\sin\pi(s_1-s_1')\sin\pi(s_2-s_2')}\times\\ &\quad\times\sum_{\substack{c\neq 0\\ c\in\mathcal{O}_K}}\frac{S(\nu',\nu,c)}{|c|^{s_1+s_1'}|\overline{c}|^{s_2+s_2'}}\Phi(s_1,s_1';2\pi\tfrac{\sqrt{\nu\nu'}}{|c|})\Phi(s_2,s_2';2\pi\tfrac{\sqrt{\overline{\nu}\overline{\nu'}}}{|\overline{c}|}).\end{aligned}\tag{5.10}$$

5.4. An application of the Selberg–Parseval equality

For any $f,\ g\in L^2(\mathcal{F},dz)$, we have the **Selberg– Parseval equality**

$$(f,\,g)=\sum_{j=1}^{\infty}(f,\,\phi_j)\overline{(g,\,\phi_j)}+c(D)\cdot\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}(f,\,E(*,1/2+ir,k))\overline{(g,\,E(*,1/2+ir,k))}\,dr, \tag{5.11}$$

where $c(D):=1/(4\pi\sqrt{D}\log\epsilon)$. Applying this to $f(z):=U_\nu(z,s)$, $g(z):=U_{\nu'}(z,s')$, we must compute, after unfolding the integral, the inner product

$$\begin{aligned}(U_\nu(*,s),\phi_j) &= \int_0^\infty\int_0^\infty y_1^{s_1-2}y_2^{s_2-2}\int\int_{\mathcal{D}} e_\nu(z)\overline{\phi_j(z)}\;dx\;dy\\ &= \int_0^\infty\int_0^\infty y_1^{s_1-2}y_2^{s_2-2}e_\nu(iy_1,iy_2)\int\int_{\mathcal{D}} e_\nu(x)\overline{\phi_j(z)}\;dx\;dy\\ &= 2\sqrt{D}(\pi\nu)^{1/2-s_1}(\pi\overline{\nu})^{1/2-s_2}\overline{\rho_j(\nu)}\times\\ &\times\int_0^\infty\int_0^\infty e^{-y_1-y_2}\overline{K_{ir_{ij}}(y_1)K_{ir_{2j}}(y_2)}y_1^{s_1-3/2}y_2^{s_2-3/2}\;dy_1dy_2.\end{aligned} \tag{5.12}$$

Since ([GR], §4.62 pp. 712, $\mathrm{Re}(\alpha)>|\mathrm{Re}(\beta)|$)

$$\int_0^\infty x^{\alpha-1}e^{-x}K_\beta(x)\;dx=\frac{\sqrt{\pi}}{2^\alpha}\frac{\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha+1/2)},$$

we obtain

$$\begin{aligned}&(U_\nu(*,s),\phi_j)\\ &=(2\pi)^{2-s_1-s_2}\sqrt{D}\nu^{1/2-s_1}\overline{\nu}^{1/2-s_2}\times\\ &\times\tfrac{\Gamma(s_1-1/2+ir_{1j})\Gamma(s_1-1/2-ir_{1j})\Gamma(s_2-1/2+ir_{2j})\Gamma(s_2-1/2-ir_{2j})}{\Gamma(s_1)\Gamma(s_2)}\overline{\rho_j(\nu)}.\end{aligned} \tag{5.13}$$

Define the coefficients $a_\nu(r,y,k)$, $c_\nu(r,k)$ by

$$\begin{aligned}a_\nu(r,y,k)\quad &:=\tfrac{1}{2\sqrt{D}}\int\int_{\mathcal{D}} E(x+iy,1/2+ir,k)\overline{e_\nu(x)}\;dx\\ &=: c_\nu(r,k)\sqrt{y_1y_2}K_{\alpha-1/2}(\pi|\nu|y_1)K_{\beta-1/2}(\pi|\overline{\nu}|y_2),\end{aligned} \tag{5.14}$$

so

$$\begin{aligned}E(z,s,k) &= \text{constant term }+\\ &+\textstyle\sum_{\nu\in\hat{\mathcal{O}_K}} c_\nu(r,k)\sqrt{y_1y_2}K_{\alpha-1/2}(\pi|\nu|y_1)K_{\beta-1/2}(\pi|\overline{\nu}|y_2)e_\nu(x)\,.\end{aligned} \tag{5.15}$$

Similarly, we obtain

$$\begin{aligned}&(U_\nu(*,s),E(*,1/2+ir,k))\\ &=(2\pi)^{2-s_1-s_2}\sqrt{D}\nu^{1/2-s_1}\overline{\nu}^{1/2-s_2}\times\\ &\times\tfrac{\Gamma(s_1-1+\alpha)\Gamma(s_1-1-\alpha)\Gamma(s_2-1+\beta)\Gamma(s_2-1-\beta)}{\Gamma(s_1)\Gamma(s_2)}\overline{c_\nu(r,k)}\,.\end{aligned} \tag{5.16}$$

Putting these results together, we obtain the

Theorem 5.2. Selberg–Parseval inner product formula

$$\begin{aligned}&(U_\nu(*,s),U_{\nu'}(*,\overline{s'}))\\ &= 2\sqrt{D}\pi^{2-s_1-s_1'-s_2-s_2'}(\nu\nu')^{(1-s_1-s_1')/2}(\overline{\nu}\overline{\nu'})^{(1-s_2-s_2')/2}(\tfrac{\nu}{\nu'})^{(s_1'-s_1)/2}(\tfrac{\overline{\nu}}{\overline{\nu'}})^{(s_2'-s_2)/2}\times\\ &\times\Big\{\textstyle\sum_{j=1}^\infty \rho_j(\nu)\overline{\rho_j(\nu')}\Lambda(s,r_{1j})\overline{\Lambda(s',r_{2j})}+\\ &+\textstyle\sum_{k=-\infty}^\infty\int_{\mathbb{R}} c_\nu(r,k)\overline{c_{\nu'}(r,k)}\Lambda(s,(\alpha-1/2)/i)\overline{\Lambda(s',(\beta-1/2)/i)}\,dr\Big\}.\end{aligned} \tag{5.17}$$

Combining this inner product formula with the Petersson inner product formula we will obtain a special case of the Kuznetsov trace formula (see Lemma 5.4 below).

To prepare for this, first note that

$$\Lambda(1+it,1-it,r)=\frac{\pi\sinh(\pi t)}{t\cosh(\pi r)}H(r,t), \tag{5.18}$$

where

$$H(r,t):=\frac{\cosh(\pi r)}{\cosh\pi(r+t)\cosh\pi(r-t)}.$$

In case $s_1=1+it_1=\overline{s_1},\ s_2=1+it_2=\overline{s_2}$, the Petersson inner product formula reads

$$\begin{aligned}&(U_\nu(*,1+it_1,1-it_2),U_{\nu'}(*,1-it_1,1+it_2))\\ &=\delta_{\nu,\nu'}\pi^{-2}\nu^{-1}\overline{\nu}^{-1}\Gamma(1)^2+\left(\tfrac{\nu'}{\nu}\right)^{it_1}\left(\tfrac{\overline{\nu'}}{\overline{\nu}}\right)^{it_2}\tfrac{2^2}{\sin\pi(it_1)\sin\pi(it_2)}\times\\ &\times\sum_{\substack{c\neq 0\\ c\in\mathcal{O}_K}}\tfrac{S(\nu',\nu,c)}{N(c)^2}\Phi(1+it_1,1-it_1;2\pi\tfrac{\sqrt{\nu\nu'}}{|c|})\Phi(1+it_2,1-it_2;2\pi\tfrac{\sqrt{\overline{\nu}\overline{\nu'}}}{|\overline{c}|}),\end{aligned} \tag{5.19}$$

where

$$\Phi(1+it,1-it,x):=\pi\sin(i\pi t)\int_1^\infty[J_{2it}(xu)-J_{-2it}(xu)]\frac{du}{u}. \tag{5.20}$$

The inner product formula (5.17) combined with the inner product formula (5.19) give

Lemma 5.3. *For ν, ν' totally positive, $|Im\, t_i| < \frac{1}{4}$, we have*

$$2\sqrt{D}\frac{\sinh(\pi t_1)\sinh(\pi t_2)}{t_1 t_2}\Big\{\sum_{j=1}^{\infty}\rho_j(\nu)\overline{\rho_j(\nu')}\cosh(\pi r_{1j})^{-1}\cosh(\pi r_{2j})^{-1}H(r_{1j},t_1)H(r_{2j},t_2)+$$
$$+\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}c_\nu(r,k)\overline{c_{\nu'}(r,k)}H((\alpha-1/2)/i,t_1)H((\beta-1/2)/i,t_2)\;dr\Big\}$$
$$=\delta_{\nu,\nu'}\Gamma(1)^2++\pi^2N(\nu)^{1/2}N(\nu')^{1/2}\times$$
$$\times\sum_{\substack{c\neq 0\\ c\in\mathcal{O}_K}}\frac{S(\nu',\nu,c)}{N(c)^2}\Phi(1+it_1,1-it_1;2\pi\frac{\sqrt{\nu\nu'}}{|c|})\Phi(1+it_2,1-it_2;2\pi\frac{\sqrt{\overline{\nu}\overline{\nu'}}}{|\overline{c}|}).$$

This is the real quadratic analog of [Kuz], Lemma 4.5 p. 323.

5.5. Mean–square estimates and the trace formula

Multiply both sides of Lemma 5.3 by t_1t_2 and then integrate over $0\le t_i\le T_i$, for $\nu=\nu'$, or, equivalently, take $h(r)=\int_0^T H(r,t)\sinh(\pi t)\;dt$ in the real-quadratic formula of §5.4 above. In either case, we obtain

Lemma 5.4.

$$2\sqrt{D}\sum_{j=1}^{\infty}|\rho_j(\nu)|^2\cosh(\pi r_{1j})^{-1}\cosh(\pi r_{2j})^{-1}h_{T_1}(r_{1j})h_{T_2}(r_{2j})+$$
$$+\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}|c_\nu(r,k)|^2h_{T_1}(\alpha-1/2)/i)h_{T_2}(\beta-1/2)/i)\;dr$$
$$=\frac{1}{4}\Gamma(1)^2T_1^2T_2^2+$$
$$+\sum_{\substack{c\neq 0\\ c\in\mathcal{O}_K}}\frac{S(\nu',\nu,c)}{N(c)}\int_0^{T_1}\int_0^{T_2}\frac{t_1t_2}{\cosh(\pi t_1)\cosh(\pi t_2)}\Phi^*(2\pi\frac{\sqrt{\nu\nu'}}{|c|},t_1)\Phi^*(2\pi\frac{\sqrt{\overline{\nu}\overline{\nu'}}}{|\overline{c}|},t_2),$$

where

$$h_T(r)\quad:=\int_0^T H(r,t)\sinh(\pi t)\;dt,$$
$$\Phi^*(x,t)\quad:=x\Phi(1+it,1-it,x).$$

Following [Kuz] p. 324, we obtain

Lemma 5.5. *For $\nu\in\hat{\mathcal{O}}_K$ totally positive,*

$$\sum_{r_{1j}\le T_1}\sum_{r_{2j}\le T_2}\frac{|\rho_j(\nu)|^2}{\cosh(\pi r_{1j})\cosh(\pi r_{2j})}\qquad=\frac{\pi}{2\sqrt{D}}T_1^2T_2^2+O_{\epsilon,D}((T_1T_2)^{1+\epsilon})+$$
$$+O_{\epsilon,D}(T_1T_2N(\nu)^\epsilon+N(\nu)^{1/2+\epsilon}).$$

Corollary 5.6. *([Kuz] p. 327) As $\nu\to\infty$ runs through totally positive elements, we have*

$$|\rho_j(\nu)|<<_{\epsilon,D}e^{\pi(r_{1j}+r_{2j})/2}N(\nu)^{1/4+\epsilon}.$$

5.6. Estimating sums of Kloostermann sums

From section 3, we recall the real quadratic version of the Kuznetsov-Bruggeman formula:

Theorem 5.7. *Let $h(r_1, r_2) = h_1(r_1)h_2(r_2)$, with each h_i as in Theorem 2.2. For $\nu,\ \nu' \in \hat{\mathcal{O}}_K$ totally positive, we have*

$$\begin{aligned} &\textstyle\sum_{j=1}^{\infty} \rho_j(\nu)\overline{\rho_j(\nu')} \cosh(\pi r_{1j})^{-1} \cosh(\pi r_{2j})^{-1} h(r_{1j}, r_{2j}) + \\ &+c \textstyle\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} c_\nu(r,k)\overline{c_{\nu'}(r,k)} h(\alpha - 1/2)/i), (\beta - 1/2)/i)\ dr \\ &= \delta_{\nu,\nu'} c' \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tanh(\pi t_1) \tanh(\pi t_2) t_1 t_2 h_1(t_1) h_2(t_2)\ dt_1 dt_2 + \\ &+c'' \sum_{\substack{c \neq 0 \\ c \in \mathcal{O}_K}} \tfrac{S(\nu',\nu,c)}{N(c)} k^*(2\pi \tfrac{\sqrt{\nu\nu'}}{|c|}, 2\pi \tfrac{\sqrt{\overline{\nu}\overline{\nu'}}}{|\overline{c}|}), \end{aligned}$$

where

$$k^*(x_1, x_2) := \frac{2i^2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} J_{2it_1}(x_1) J_{2it_2}(x_2) \frac{t_1 t_2 h(t_1, t_2)}{\cosh(\pi t_1)\cosh(\pi t_2)}\ dt_1 dt_2,$$

and the c, c', c'' are certain constants, depending at most on D. The values of these constants are given in section 3 above. (Their values won't be needed for the application below.)

For estimates on Kloostermann sums, see Bruggemann and Miatello, "Sum formula for SL_2 over a number field and Selberg type estimates for exceptional eigenvalues," Geometric and Functional Analysis, 8(1998). The estimates given in the published version of the paper contain a gap and are valid only under the assumption that either both $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ are exceptional or both are not (i.e., "mixed" type eigenvalues don't exist) [2].

We apply this to prove the following estimate:

Lemma 5.8. Assume that either both $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ are exceptional or both are not (i.e., "mixed" type eigenvalues don't exist).

$$\sum_{\substack{0 < c \leq T_1 \\ 0 < \overline{c} \leq T_2}} S(\nu, \nu', c) N(c)^{-1} <<_{\epsilon, D, \nu, \nu'} (T_1 T_2)^{1/6+\epsilon} + \sum_{\substack{0 < \lambda_j^{(1)} < 1/4, \\ 0 < \lambda_j^{(2)} < 1/4}} [(T_1 T_2)^{2|r_{1j}|+\epsilon} + (T_1 T_2)^{2|r_{2j}|+\epsilon}],$$

where ν, ν' are fixed and totally positive, and $T_1 \approx T_2 \approx T \to \infty$ are both of the same order of magnitude.

[2] I am grateful to Professors Bruggemann and Miatello for pointing this out

proof Write $\phi = k^*$ and $\hat{\phi} = h$ in Theorem 5.7. Pick two bump functions $\phi_1,\ \phi_2$ on $\mathbb{R}_+$ obtained by taking, in the notation of [Kuz] p. 333, $a = a_1 := 2\pi\sqrt{\nu\nu'},\ T = T_1$ for ϕ_1 and taking $a = a_2 := 2\pi\sqrt{\overline{\nu}\overline{\nu'}},\ T = T_2$ for ϕ_2. Now consider the bump function on $\mathbb{R}_+^2$ defined by $\phi(x_1, x_2) := \phi_1(x_1)\phi_2(x_2)$. Fix, for now, a $T_0 << T^{2/3+\epsilon}$ and consider the "diamond"

$$D(T_1, T_2) := \{c \in \mathcal{O}_K \mid T_1 < c < T_1 + T_0,\ T_2 < \overline{c} < T_2 + T_0\}.$$

Using Weil's bound for $S(\nu, \nu', c)$, on easily finds that

$$\sum_{c \in D(T_1,T_2)} |S(\nu, \nu', c)| N(c)^{-1} <<_{\epsilon,D,\nu,\nu'} T_0^2 (T_1 T_2)^{-1/2+\epsilon}. \tag{5.21}$$

Set

$$Z_{\nu,\nu'}(\phi) := \sum_{\substack{c \neq 0 \\ c \in \mathcal{O}_K}} \frac{S(\nu, \nu', c)}{N(c)} \phi\Big(\frac{2\pi\sqrt{\nu\nu'}}{|c|}, \frac{2\pi\sqrt{\overline{\nu}\overline{\nu'}}}{|\overline{c}|}\Big),$$

and

$$S_{\nu,\nu'}(T_1, T_2) := \sum_{\substack{0 < c \leq T_1 \\ 0 < \overline{c} \leq T_2}} S(\nu, \nu', c) N(c)^{-1}.$$

From (5.21), we obtain

$$|Z_{\nu,\nu'}(\phi) - S_{\nu,\nu'}(T_1, T_2)| <<_{\epsilon,D,\nu,\nu'} T_0 (T_1 T_2)^{-1/2+\epsilon}. \tag{5.22}$$

By definition of ϕ and ϕ_H, Kuznetsov's argument [Kuz] p. 333, gives

$$|Z_{\nu,\nu'}(\phi) - Z_{\nu,\nu'}(\phi_H)| << \sqrt{\nu\nu'}. \tag{5.23}$$

(The key point is to keep in mind that the variables of ϕ are separated.) Fix, for the moment, any $M_1,\ M_2 > 1$. The definitions of ϕ and h along with the two–variable version of Kuznetsov's argument [Kuz] p. 334, gives

$$\sum_{\substack{r_{1j} \leq M_1 \\ r_{2j} \leq M_2}} \Big|\frac{\rho_j(\nu)\overline{\rho_j(\nu')}}{\cosh(\pi r_{1j})\cosh(\pi r_{2j})} h(r_{1j}, r_{2j})\Big| << \sqrt{M_1 M_2} + \sum_{\substack{0 < \lambda_j^{(1)} < 1/4, \\ 0 < \lambda_j^{(2)} < 1/4}} [(T_1 T_2)^{2|r_{1j}|+\epsilon} + (T_1 T_2)^{2|r_{2j}|+\epsilon}], \tag{5.24}$$

and

$$\sum_{\substack{r_{1j} \geq M_1 \\ r_{2j} \geq M_2}} \left| \frac{\rho_j(\nu)\overline{\rho_j(\nu')}}{\cosh(\pi r_{1j})^{-1}\cosh(\pi r_{2j})^{-1}} h(r_{1j}, r_{2j}) \right| << \frac{T_1 T_2}{T_0^2 \sqrt{M_1 M_2}}. \tag{5.25}$$

By putting together Theorem 5.7, (5.25), we obtain

$$S_{\nu,\nu'}(T_1, T_2) <<_{D,\epsilon,\nu,\nu'} T_0^2 (T_1 T_2)^{-1/2+\epsilon} +$$
$$\sqrt{M_1 M_2} + \frac{T_1 T_2}{T_0^2 \sqrt{M_1 M_2}} + \sum_{\substack{0<\lambda_j^{(1)}<1/4, \\ 0<\lambda_j^{(2)}<1/4}} [(T_1 T_2)^{2|r_{1j}|+\epsilon} + (T_1 T_2)^{2|r_{2j}|+\epsilon}].$$

Taking $M_1 = M_2 = T/T_0$, $T_0 = T^{2/3+\epsilon}$ gives the result stated in Lemma 5.8. $\square$

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