On stable characters of tamely ramified supercuspidal representations of small unitary groups

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Abstract

The motivation for this note lies in the following simple question: given a discrete series representation of a p-adic reductive group, is there a purely local algorithm which will produce its L-packet? As a means of producing an L-packet, the behaviour of the "stabilized character" of a discrete series representation π of G is examined in the special case of the (unramified) unitary groups $G = U(1,1)(F)$, $U(2)(F)$, and $U(2,1)(F)$, F a p-adic field, $p \neq 2, 3$.

Contents

1 Introduction

It is hoped that the reasoning in the modest special case investigated in this article [1] will shed a little more light on the relationship between the Hecke algebra approach of Howe-Moy [M] and the Langlands philosophy [Rog] to the construction and classification of irreducible admissible representations of p-adic groups. The Theorem below, and the discussion following it, suggest a way to use the Hecke algebra approach of Howe-Moy, as developed in [Ja], to construct Langlands L-packets, at least in some simple cases.

For example, for the groups $U(1,1)$ and $U(2)$ it turns out that the Theorem below is a simple consequence of Jabon's classification of the supercuspidals of these groups given in [Ja]. In the case of $U(2,1)$ [Ja] (see also [M]) is used to verify Theorem 1 for the supercuspidal representations. Using different ideas the Theorem is verified for the special representations. Unfortunately, to do this it was necessary to rely on a character identity whose proof requires the global stable trace formula for $U(2,1)$. Thus for the special representations the results are philosophically less satisfactory.

Let F denote a p-adic field (p a prime) with uniformizer π_F, ring of integers $\mathcal{O}_F$, and maximal ideal $\mathcal{P}_F$. (We shall also use the letter π for representations but hopefully the reader will be able to distinguish the uniformizers from the representatioons by the context.) Let G denote the F-rational points of a connected reductive group $\underline{G}$ defined over F, let T denote an elliptic Cartan of $\underline{G}$, and let $\mathfrak{D}_G(T/F)$ be as in [L]. Define the **stabilized character** (in

[1] Written in 1991-92 while the second author held a post-doctoral fellowship at the University of Maryland, College Park. Revised in 1998.

distinction with what we shall call the "character of the L-packet") by

$$\chi_\pi^{st}(\gamma) := \sum_{w \in \mathfrak{D}_G(T/F)} \chi_\pi(\gamma^w),$$

for regular $\gamma \in T(F)$. We must define $\chi_\pi^{st}(\gamma)$ when γ is regular but not elliptic. Suppose γ is M-elliptic for some Levi M of G. By a formula of Casselman [Cas2], $\chi_\pi(\gamma) = \chi_{\pi_N}(\gamma)$ where N is the nilpotent radical associated to a suitable parabolic P attached to M and π_N denotes a representation of M associted to π via the Jacquet functor. We define $\chi_{\pi_N}^{st}(\gamma)$ as before and set $\chi_\pi^{st}(\gamma) := \chi_{\pi_N}^{st}(\gamma)$ in this case. Our main result is the following

Theorem 1 *Let G denote $U(2)(F)$, $U(1,1)(F)$, or $U(2,1)(F)$, associated to the unramified quadratic extension of F, and assume $p \neq 2, 3$. For any irreducible discrete series representation π of G and any elliptic Cartan T, we have*

$$\chi_\pi^{st}(\gamma) = \sum_i c_i \chi_{\pi_i}(\gamma), \quad \gamma \in T(F) \text{ regular}, \tag{1}$$

where $\{c_i\} = \{c_{i,T,\pi}\} \subset \mathbb{C}$ is a finite set of constants (independent of γ but not necessarily T) and $\{\pi_i\}$ is a finite set of admissible representations of G.

This is verified in §§2-4 below by explicit calculations involving the Frobenius formula and the constructions in [Ja]. For $G = U(1,1)(F)$, we also show that if π is an irreducible tempered representation which is not in the discrete series then χ_π^{st} vanishes on the elliptic set.

It should be remarked that the obvious analog of the character relation in the above theorem seems to be easily provable in the cases $SL(n)$ and $PGL(n)$. It is conjectured to hold for arbitrary connected reductive groups over a p-adic field.

Given π, the set $\{\pi_i\}$ satisfying the equation in the theorem is not unique. Although we have not done so, as a by-product of the method it seems that with a little more work one could give explicit formulas for at least some of the stabilized supercuspidal characters.

We now offer some speculations on how this theorem relates to the construction of Langlands L-packets of the discrete series. Let Π_π be the finite set (independent of both γ and T) of irreducible discrete series representations which

- containing π,

- if the $\{\pi_i\}$ satisfying (1) is not unique then Π_π is to contain all possible choices,

- if π' is an irreducible admissible representation of G which satisfies $\chi^{st}_\pi(\gamma) = \chi^{st}_{\pi'}(\gamma)$ for all elliptic regular γ then Π_π is to contain π', and

- Π_π is minimal with respect to the the above three properties.

It seems reasonable to expect (based on the examples we considered) that a suitable linear combination of the $\chi^{st}_{\pi'}$'s (for $\pi' \in \Pi_\pi$) should be the (stable) character of the L-packet Π of π. More precisely, it is expected that for each Cartan T (elliptic or not) there is a suitable collection of constants $\{a_{\pi',T} \mid \pi' \in \Pi_\pi\}$,

$$\sum_{\pi' \in \Pi_\pi} a_{\pi',T} \chi^{st}_{\pi'}(\gamma) = \sum_{\pi' \in \Pi_\pi} n(\pi') \chi_{\pi'}(\gamma) = \chi_\Pi(\gamma), \quad \gamma \in T(F) \text{ regular}, \tag{2}$$

for any $\pi \in \hat{G}_d$, where $\{n(\pi')\} \subset \mathbb{C}$ is a finite set of constants (independent of γ and T), and χ_Π is the character of the L-packet Π of π [Rog] and $\Pi_\pi \subset \hat{G}$ is the set above. In the case of $U(2)$ and $U(1,1)$ (and presumably, though we have not done so, $U(2,1)$ as well), one may use [Ja] to calculate the $a_{\pi',T}$ in terms of the $c_{i,\pi,T}$.

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2 The Frobenius formula

Next we recall the following general version of the Frobenius formula.

Proposition 2 *Let H denote the group of F-rational points of a connected reductive group defined over F and let K denote a compact open subgroup of G. If π is an irreducible admissible representation of H induced from a finite dimensional representation σ of K then*

$$\chi_\pi(x) = \sum_{\substack{z \text{in} \\ K\backslash H/K}} [\sum_{\substack{g \text{in} \\ K\backslash KzK}} \chi^\circ_\sigma(gxg^{-1})]$$

converges for each regular $x \in H$. (The convergence is not uniform.) Here

$$\chi_\sigma^\circ(g) = \begin{cases} \chi_\sigma(g), & g \in K, \\ 0, & g \notin K. \end{cases}$$

The proof of an analogous version of this may be found in [Kut] or in [J].

3 Stabilized Characters of $U(1,1)$

Now let E/F denote an unramified extension of p-adic fields, $p \neq 2$, $E = F(\sqrt{\epsilon})$ with ϵ a unit, let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and let

$$G = U(1,1)(F) = \{g \in GL_2(E) \mid g \cdot J \cdot {}^t\overline{g} = J\},$$

where the conjugation (denoted by an overline) denotes the Galois action of E/F. Let

$$K = U(1,1)(\mathcal{O}_F) = G \cap GL_3(\mathcal{O}_E),$$

let

$$L = \{g \in G \mid g \in \begin{pmatrix} \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E \end{pmatrix},$$

and let

$$I = \{g \in G \mid g \in \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}$$

denote the Iwahori subgroup. Let $K_0 = K$ and

$$K_i = \{k \in K \mid k \equiv 1 \ (\mathrm{mod} \mathcal{P}_E^i)\}, \qquad i \geq 1.$$

Let $L_0 = L$ and

$$L_i = \{k \in K \mid k \in 1 + \pi^i \begin{pmatrix} \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}\}, \qquad i \geq 1.$$

Let $t = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$, let $I_0 = I$, and let

$$L_j = \{k \in K \mid k \in 1 + t^j \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}\}, \qquad j \geq 1.$$

Clearly if $g \in G$ then $N_{E/F}(det\ g) = 1$, where $N_{E/F}$ denotes the norm. Note that the torus

$$T(F) = \{g = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix} \mid \alpha \in E^\times\} \cong E^\times$$

is a subgroup of G and, by Hilbert's Theorem 90, the map $det : G \to T'(F)$ is surjective. Here $T'(F) := E^{\times 1}$ denotes the kernel of the norm map. Thus we have an exact sequence

$$1 \to SU(1,1)(F) \to G \to T'(F) \to 1,$$

where the third arrow is the det, and

$$\begin{aligned} SU(1,1)(F) &= G \cap SL_2(E) && (3)\\ &= \{\begin{pmatrix} a & \frac{b}{\epsilon}\sqrt{\epsilon} \\ c\sqrt{\epsilon} & d \end{pmatrix} \mid a,b,c,d \in F^\times,\ ad - bc = 1\} && (4)\\ &= \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}^{-1} SL_2(F) \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}. && (5) \end{aligned}$$

Since

$$G/SU(1,1)(F) \cong E^{\times 1},$$

$SU(1,1)(F)E^{\times 1}$ is a subgroup of G of finite index; in fact,

$$G/SU(1,1)(F)E^{\times 1} \cong E^{\times 1}/(E^{\times 1})^2.$$

The only discrete series representations of G are supercuspidal representations and twists of the Steinberg representation by a character (if there were others they would show up upon restricting to $SU(1,1) \cong SL(2)$). First we describe the L-parameters associated to the supercuspidals.

Let $\sigma \in Gal(E/F)$ denote the non-trivial element and let $Gal(E/F)$ act on $GL_2(\mathbb{C})$ by $\sigma(zg) := z^{-1}J^*gJ^*$, for all $z \in \mathbb{C}^\times$ and $g \in SL_2(\mathbb{C})$, where $J^* := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (a similar case is treated in [Kud] and [F1] for example). We write

$${}^LU(1,1) = GL_2(\mathbb{C}) \rtimes Gal(E/F), (\mathrm{semi-direct\ product})$$

$${}^LSU(1,1) = PGL_2(\mathbb{C}) \times Gal(E/F)(\mathrm{direct\ product}).$$

Let W_F denote the Weil group. Using Clifford-Mackey theory, it is not hard to verify that the L-parameters

$$\phi : W_F \to {}^L U(1,1)$$

associated to the supercuspidal packets of G are precisely the "lifts" of the L-parameters

$$\overline{\phi} : W_F \to {}^L SU(1,1)$$

associated to the supercuspidal packets of G. From the Labesse-Langlands theory for $SL(2,F)$ [LL], the latter L-packets are well-known.

We use [Ja] and the Frobenius formula below to examine the characters of the supercuspidal representations. In the notation of [Ja], we have two maximal parahorics, K and $L = \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix}^{-1} K \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix}$, and the Iwahori $I = K \cap L$. Here π_F denotes a uniformizer for F. Regarding $\begin{pmatrix} \sqrt{\pi_F} & 0 \\ 0 & \sqrt{\pi_F}^{-1} \end{pmatrix} = \pi_F^{-1/2} \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix}$ as an element of $SU(1,1)(F(\sqrt{\pi_F})) \subset U(1,1)(\overline{F})$, we will therefore say that K and L are **stably conjugate** parahorics. Consider a supercuspidal $\pi = Ind_K^G \ \rho$ or $\pi = Ind_L^G \ \rho$, where $\rho \in \hat{K}$ or $\rho \in \hat{L}$ is induced from a non-degenerate representation (of level one, unramified type, or ramified type [Ja]).

Let $\overline{x}$ denote the E/F-Galois conjugation of a matrix x with entries in E. If $g \in G$ satisfies $g \cdot J \cdot {}^t\overline{g} = J$ and if $\delta = \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix}$ then one can check that $g^\delta = \delta^{-1} g \delta$ satisfies $g^\delta \cdot J \cdot {}^t\overline{g}^\delta = J$. Thus δ defines an automorphism of

G which is not inner. Note

$$\begin{aligned}
\chi_\pi(\delta^{-1}x\delta) &= \sum_{\substack{z \text{in} \\ Q\backslash G/Q}} \left[\sum_{g\in Q\backslash QzQ} \chi_\rho^o(g^{-1}\delta^{-1}x\delta g) \right] \\
&= \sum_{\substack{z \text{in} \\ Q\backslash G/Q}} \left[\sum_{g\in Q\backslash QzQ} \chi_\rho^o(\delta^{-1}(g^\delta)^{-1}xg^\delta\delta) \right] \\
&= \sum_{\substack{z \text{in} \\ Q\backslash G/Q}} \left[\sum_{g\in (Q\backslash QzQ)^\delta} \chi_\rho^o(\delta^{-1}g^{-1}xg\delta) \right] \\
&= \sum_{\substack{z \text{in} \\ Q^\delta\backslash G/Q^\delta}} \left[\sum_{g\in Q^\delta\backslash Q^\delta zQ^\delta} \chi_\rho^o(\delta^{-1}g^{-1}xg\delta) \right]. \qquad (6)
\end{aligned}$$

In particular, if $\pi = Ind_K^G\ \rho$ and $\pi' = Ind_L^G\ (\rho^\delta)$, with δ as above, then the Frobenius formula implies $\chi_{\pi'}(x) = \chi_\pi^\delta(x)$, for all $x \in G$. On the other hand, if $\delta = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, with u a unit in F, $\pi = Ind_K^G\ \rho$ and $\pi' = Ind_K^G\ (\rho^\delta)$, or $\pi = Ind_L^G\ \rho$ and $\pi' = Ind_L^G\ (\rho^\delta)$, then likewise the Frobenius formula implies $\chi_{\pi'}(x) = \chi_\pi^\delta(x)$, for all $x \in G$.

To prove Theorem 1 in the case of $U(1,1)$, we enumerate the elliptic Cartans of G. Each elliptic Cartan of $U(1,1)$ is associated to a quadratic extension F'/F, $F' \neq E$. There are four conjugacy classes of anisotropic tori of G, each stable conjugacy class of an elliptic Cartan contains two conjugacy classes of elliptic Cartans. Let T_i, $i = 1,2,3,4$, denote the four conjugacy classes of elliptic Cartans. We may label these in such a way that T_1, T_2 are stably conjugate, T_3, T_4 are stably conjugate, and $T_1(F) \hookrightarrow K$, $T_2(F) \hookrightarrow L$, $T_3(F) \hookrightarrow I$, and $T_4(F) \hookrightarrow I$. If T is an elliptic Cartan of $U(1,1)$ associated to F'/F then there is a natural bijection $\mathfrak{D}_G(T/F) \cong F'^\times/N_{F'/F}(F'^\times)$.

Let $x \in T(F)$ be regular and let

$$\chi_\pi^{st}(x) = \sum_{\delta\in F'^\times/N_{F'/F}(F'^\times)} \chi_\pi(x^\delta),$$

where $x^\delta := \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}^{-1} x \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}$. If $F^\times/N_{F'/F}(F'^\times)$ can be represented by 1 and a non-unit (e.g., π_F) then

$$\chi_{\pi}^{st}(x) = \chi_{\pi}(x) + \chi_{\pi'}(x), \quad x \in T(F) \text{ regular},$$

where $\pi = Ind_K^G\ \rho$, $\pi' = Ind_L^G\ (\rho^\delta)$, and $\delta = \pi_F$. On the other hand, if $F^\times / N_{F'/F}(F'^\times)$ can be represented by 1 and a unit u then for $x \in T(F)$ regular, we have

$$\chi_{\pi}^{st}(x) = \chi_{\pi}(x) + \chi_{\pi'}(x),$$

where $\delta = u$, $\pi = Ind_K^G\ \rho$ (and $\pi' = Ind_K^G\ (\rho^\delta)$), or $\pi = Ind_L^G\ \rho$ (and $\pi' = Ind_L^G\ (\rho^\delta)$). In any case, we've verified Theorem 1 for the supercuspidals of $U(1,1)(F)$.

Since the Steinberg representation on $SU(1,1)(F) \cong SL(2,F)$ has a singleton L-packet, the restriction of the Steinberg representation St_G of $G(F) = U(1,1)(F)$ to $SU(1,1)(F)$ must remain irreducible (by Clifford-Mackey theory). By a well-known result of Borel-Serre-Casselman [Cas1] the character of the Steinberg representation is actually given as a certain alternating sum of characters each of which is invariant under conjugation by elements of the form $\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}$. Therefore the Theorem is (trivially) true in this case. (Alternatively, the value of the Steinberg character is known on the elliptic set: see [Sil], §4.7.) This proves Theorem 1 for every element of the discrete series of $U(1,1)(F)$.

Our final goal of this section is to examine the stablized character of the tempered non-discrete series representations of $U(1,1)$. In fact, we prove the following

Lemma 3 *Let π denote a reducible unitary principal series representation of $U(1,1)(F)$ and let $T(F)$ denote an elliptic Cartan subgroup of $U(1,1)(F)$ associated to a quadratic extension F'/F. Define the stablilized character on $T(F)$ by*

$$\chi_{\pi}^{st}(x) = \chi_{\pi}(\gamma) + \chi_{\pi}(\overline{\gamma}),$$

where $\gamma \in T(F)$ is regular and where conjugation denotes the Galois automorphism of F'/F on $T(F)$ (which is separate from the action of $Gal(E/F)$ on $T(F)$). If

$$1 \to \pi_1 \to \pi \to \pi_2 \to 1,$$

for irreducible π_i, then

$$\chi_{\pi_1}^{st}(\gamma) = \chi_{\pi_2}^{st}(\gamma) = \chi_{\pi}(\gamma) = 0,$$

for all regular $\gamma \in T(F)$.

proof: Let τ denote a reducible unitary principal series representation of $SL_2(F)$ and let $T'(F)$ denote an elliptic Cartan subgroup of $SL_2(F)$ associated to a quadratic extension F'/F. It is known that the composition series of τ is of length two:

$$1 \to \tau_1 \to \tau \to \tau_2 \to 1,$$

for irreducible τ_i. By the theory of Labesse-Langlands [LL], it is well-known that

$$\chi_{\tau_1}(\gamma) = \chi_{\tau_2}(\overline{\gamma}),$$

for all $\gamma \in T'(F)$, where conjugation denotes the Galois automorphism of F'/F on $T(F)$. Since $SU(1,1)(F) \cong SL_2(F)$, this identity transfers over without change to $SU(1,1)(F)$, provided one takes care to keep the Galois automorphism of F'/F on $T'(F)$ separate from the action of $Gal(E/F)$ on $T'(F)$.

Now let π be as in the statement of the Lemma. Since $U(1,1)(F)$ is semisimple-rank one, π must also have a composition series of length two. Since $E^{\times 1}$ belongs to the center of $U(1,1)(F)$ and since both π_i must have the same central character, the above character identity between τ_1 and τ_2 must extend to a character identity on $SU(1,1)(F)E^{\times 1}$:

$$\chi_{\pi_1}(\gamma) = \chi_{\pi_2}(\overline{\gamma}),$$

for all regular $\gamma \in T(F) \cap (SU(1,1)(F)E^{\times 1})$. Since

$$U(1,1)(F)/SU(1,1)(F)E^{\times 1} \cong E^{\times 1}/(E^{\times 1})^2,$$

the extension of the latter identity to all of $U(1,1)(F)$ is an immediate consequence of Clifford-Mackey theory. $\square$

4 Stablized Characters of $U(2)$

Again, E/F denotes an unramified extension of p-adic fields, $p \neq 2$, $E = F(\sqrt{\epsilon})$, let $H = \begin{pmatrix} 1 & 0 \\ 0 & \pi_F \end{pmatrix}$, and let

$$G = U(2)(F) = \{g \in GL_2(E) \mid g \cdot H \cdot {}^t\overline{g} = H\},$$

where the conjugation denotes the Galois action of E/F. Clearly if $g \in G$ then $N_{E/F}(det\ g) = 1$. Note that the torus

$$T(F) = \{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in E^{\times 1}\},$$

is a subgroup of G and, by Hilbert's Theorem 90, the map $det : G \to T'(F)$ is surjective. Here $T'(F) = E^{\times 1}$ denotes the kernel of the norm map. One can also show that

$$\begin{aligned} SU(2)(F) &= G \cap SL_2(E) \\ &= \{ \begin{pmatrix} \alpha & \beta \\ -\pi\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in E^{\times},\ \alpha\overline{\alpha} + \pi\beta\overline{\beta} = 1 \}. \end{aligned} \tag{7}$$

Thus we have an exact sequence

$$1 \to SU(2)(F) \to G \to T'(F) \to 1$$

where the third arrow is the det map.

Now we describe the L-group of G. Let $\sigma \in Gal(E/F)$ denote the non-trivial element and let $Gal(E/F)$ act on $GL_2(\mathbb{C})$ by $\sigma(zg) := z^{-1}H^* g H^*$, for all $z \in \mathbb{C}^{\times}$ and $g \in SL_2(\mathbb{C})$, where $H^* := \begin{pmatrix} 0 & 1 \\ -\pi & 0 \end{pmatrix}$ (see [F1] for a similar example). We write

$${}^{L}U(2) = GL_2(\mathbb{C}) \rtimes Gal(E/F), (\text{semi} - \text{direct product})$$

$${}^{L}SU(2) = PGL_2(\mathbb{C}) \rtimes Gal(E/F) (\text{semi} - \text{direct product}).$$

To describe the stable "packets" Π_π of supercuspidal representations, we proceed as in the previous section. There is only one conjugacy class of parahoric subgroups of G. Moreover, if $g \in G$ then $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} g \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \in G$ if and only if $x \in E^{\times 1}$.

To verify Theorem 1 in the case of $U(2)$, we enumerate the elliptic Cartans of $U(2)$. Every Cartan of $U(2)$ is of the form

$$T''(F) = Res_{EF'/F}((EF')^{\times 1}),$$

where F'/F is a quadratic extension of F. If $F' = E$ then $T''(F) = E^{\times 1} \times E^{\times 1}$. If $F' \neq E$ then the action of $Gal(F'/F)$ on $T''(F)$ can be affected by conjugation with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (see [Ja], p. 61, (3-8)).

Now define the stabilized character on $T''(F)$ by

$$\chi_\pi^{st}(x) = \chi_\pi(\gamma) + \chi_\pi(\overline{\gamma}),$$

where $\gamma \in T''(F)$ is regular and where conjugation denotes the Galois automorphism of F'/F on $T''(F)$. Since the matrix $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defines an automorphism of $U(2,F)$ which is not inner, but does preserve the filtrations K_i of the parahoric, the reasoning of the $U(1,1)$ case can be carried out here (and is in fact a bit easier). We obtain

$$\chi_\pi^{st}(x) = \chi_\pi(x) + \chi_{\pi'}(x),$$

where $\pi = Ind_{K_i}^G \rho$ and $\pi' = Ind_{K_i}^G(\rho^w)$. From this, Theorem 1 follows easily in this case.

5 Stablized Characters of $U(2,1)$

The purpose of this section is to verify Theorem 1 in the special case of the (unramified) unitary group of three variables $U(2,1)$ attached to the unramified quadratic extension of a p-adic field F, $p \neq 2,\ 3$. Since $p \neq 2,\ 3$, every quadratic or cubic extension is tamely ramified and may be embedded in a finite, tamely ramified, Galois extension.

Let E/F denote the unramified quadratic extension and let

$$G = U(2,1)(F) = \{g \in GL_3(E) \mid gJ\,{}^t\overline{g} = J\}.$$

5.1 Basic facts about parahorics

Let us begin by briefly recalling some facts about the parahorics of $GL(3,F)$ which we shall need later. In the notation of [T], §3.10, the special parahorics $P_0,\ P_1$, and P_2 are all conjugate. They are the stabilizers of the three vertices of a certain fixed facet C in the standard apartment A of the building $B_{GL(3,F)}$. The Iwahori $B = P_o \cap P_1 \cap P_2$ fixes the facet itself. All vertices of $B_{GL(3,F)}$ are hyperspecial, hence P_0, P_1, and P_2 are all hyperspecial, maximal bounded, compact subgroups of $GL(3,F)$.

Now let us recall some facts about the parahorics of G which we shall need later. We use the notation of [Ja]. There are three conjugacy classes of parahorics: the hyperspecial maximal bounded, compact subgroup

$$K = U(2,1)(\mathcal{O}_F) = G \cap GL_3(\mathcal{O}_E),$$

the special (but not hyperspecial) maximal bounded, compact subgroup

$$L = \{x \in G \mid x \in \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}\},$$

and the Iwarori

$$I = K \cap L == \{x \in K \mid x \in \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}\}.$$

Their "Levi components" are

$$\begin{aligned} &K/K_1 \cong U(2,1)(k_F), \\ &L/L_1 \cong U(1,1)(k_F) \times U(1,k_F), \\ &I/I_1 \cong A(k_F), \end{aligned} \tag{8}$$

where $k_F = \mathcal{O}_F/P_F$ denotes the residue field and where K_1, L_1, I_1 are defined below. Extending scalars from k_F to k_E, we find that

$$\begin{aligned} &(K/K_1) \otimes k_E \cong GL(3,k_F), \\ &(L/L_1) \otimes k_E \cong GL(2,k_F) \times GL(1,k_F), \\ &(I/I_1) \otimes k_E \cong GL(1,k_F)^3. \end{aligned} \tag{9}$$

Thus K "corresponds to" P_0, L "corresponds to" $P_0 \cap P_1$, and I "corresponds to" B, under "base-change" E/F.

The filtrations of these subgroups which we shall need are as follows. Let $K_0 = K$ and

$$K_j = \{k \in K \mid k \equiv 1 \pmod{\mathcal{P}_E^j}\}, \qquad j \geq 1.$$

Let $L_0 = L$ and let

$$\ell_0 = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}, \qquad \ell_1 = \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

and let $\ell_{2j+k} = \pi_F^j \ell_k$, for $k = 0, 1$ and $j \geq 0$. Let $L_m = \{x \in L \mid x \in 1 + \ell_m\}$, for $m > 0$. Let $I_0 = I$ and let

$$i_0 = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}, \quad i_1 = \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix}, \quad i_2 = \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

and let $i_{3j+k} = \pi_F^j i_k$, for $k = 0, 1, 2$ and $j \geq 0$. Let $I_m = \{x \in L \mid x \in 1+i_m\}$, for $m > 0$. Let $I_0^\flat = I$ and let

$$i_0^\flat = i_0, \qquad i_1^\flat = i_1, \qquad i_2^\flat = \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix}, \qquad i_3^\flat = \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

and let $i_{3j+k}^\flat = \pi_F^j i_k^\flat$, for $k = 0, 1, 2, 3$ and $j \geq 0$. Let $I_m^\flat = \{x \in L \mid x \in 1 + i_m^\flat\}$, for $m > 0$.

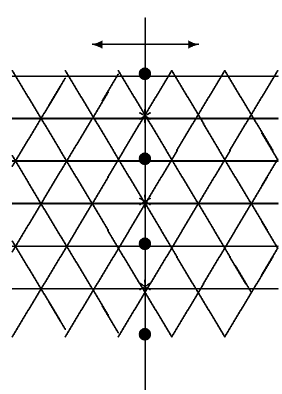
For the reader's convenience, we next recall from [T], §1.15, some relevant facts about the group $SU(2,1)(F)$. Let

$$e_1(t) := \begin{pmatrix} t^{-1} & & \\ & 1 & \\ & & t \end{pmatrix}, \qquad e_{-1}(t) := e_1(t)^{-1},$$

and let a_1, a_{-1} be defined by $a_1(e_1(t)) := t$, $a_{-1}(e_1(t)) := t^{-1}$. The set of roots relative to the maximal split torus $S := \{e_1(t) \mid t \in F^\times\}$ is $\Phi = \{a_1, 2a_1, a_{-1}, 2a_{-1}\}$. The local Dynkin diagram is given on page 42 of [T]. The standard apartment for $SU(2,1)(F)$ is the affine space

$$A := \{ve_1 \mid v \in \mathbb{R}\}.$$

A point $a \in A$ is special if and only if (i) $v \in \mathbb{Z}$, or (ii) $v \in \mathbb{Z} + 1/2$. A point $a \in A$ is hyperspecial if and only if $v \in \mathbb{Z}$ (and E/F is unramified). Thus the embedding of A into the standard apartment for $SL(3,E)$ (with the Galois action which fixes A) may be pictured as :



The building for $SU(2,1)(F)$ is a tree, each vertex being either of order q^3+1 (special vertices) or $q+1$ (hyperspecial vertices). Here $\cdot$ labels hyperspecial vertices and $*$ labels special (but not hyperspecial) vertices.

5.2 Basic facts about elliptic Cartans

To verify Theorem 1 in the case of $U(2,1)$, we enumerate the different types of Cartans T which can arise in the construction of supercuspidals in this case. We can have

- $T = Res_{EM/F}(ker\ N_{EM/M})$, M/F the unramified cubic extension,
- $Res_{EM/F}(ker_{EM/M})$, where M/F is a ramified cubic extension of F,
- $T = E^{\times 1} \times Res_{EM/F}(ker_{EM/M})$, where M/F is the unramified quadratic extension,
- $T = E^{\times 1} \times Res_{EM/F}(ker_{EM/M})$, where M/F is a ramified quadratic extension,
- $T = E^{\times 1} \times E^{\times 1} \times E^{\times 1}$.

With these preliminaries out of the way, let T_i, $i = 1,2,3,4$, denote a set of conjugacy classes of stably conjugate elliptic Cartans of G which split over the unramified extension E/F, so $T_i(F) \cong E^{\times 1} \times E^{\times 1} \times E^{\times 1}$. Suppose $T_i(F) \hookrightarrow K$, $i = 1,2$, and $T_j(F) \hookrightarrow L$, $j = 3,4$. From [T], §2.5, it follows that $T_i(F)$ fixes some hyperspecial vertex $x = x_T$ in the building B_G for G. Note that the building for $U(2,1)(E)$ is isomorphic to the building for $GL(3,E)$ and, by [T], §2.6.1, or [R] ("Galois descent") the $Gal(E/F)$-fixed points of the building for $U(2,1)(E)$ is the building for G. Let $g \in U(2,1)(E) \cong GL(3,E)$ be an element such that $T_j(F) = T_1(F)^g$, $j = 3,4$. Clearly

$$T_j(F) \hookrightarrow (K^g \cap K^{\overline{g}})^{Gal(E/F)} = (K^g)^{Gal(E/F)}.$$

Here $\overline{g}$ denotes the action of $Gal(E/F)$ on $g \in U(2,1)(E)$ fixing G, $(K^g)^{Gal(E/F)}$ stabilizes the vertex in the building for G "corresponding to" (via "Galois descent") the edge in the building for $U(2,1)(E)$ which is fixed by $K^g \cap K^{\overline{g}}$.

This proves the following

Lemma 4 *Take $p \neq 2$ and let $U(2,1)/F$ denote the unramified unitary group in three variables associated to the unramified quadratic extension E/F. Suppose T_i and T_j are stably conjugate elliptic Cartans which split over E and suppose (without loss of generality) that $T_i(F) \hookrightarrow K$. Then there is a $g \in U(2,1)(E) \cong GL(3,E)$ for which*

$$T_j(F) \hookrightarrow (K^g)^{Gal(E/F)}, \quad T_j(F) = T_1(F)^g.$$

In fact, essentially the same argument yields the following result:

Lemma 5 *Let $U(2,1)/F$ denote the unramified unitary group as above. Suppose T and T' are stably conjugate elliptic Cartans which split over a tamely ramified Galois extension of F and suppose that $T(F) \hookrightarrow K$. Then there is a $g \in U(2,1)(E) \cong GL(3,E)$ for which*

$$T'(F) \hookrightarrow (K^g)^{Gal(E/F)}, \quad T'(F) = T(F)^g.$$

Moreover, one can replace K by L or even I in the above argument, with only minor modifications. The result is the following

Lemma 6 *Let $U(2,1)/F$ denote the unramified unitary group as above. Suppose T and T' are stably conjugate elliptic Cartans which split over a tamely ramified Galois extension of F and suppose that $T(F) \hookrightarrow L$. Then there is a $g \in U(2,1)(E) \cong GL(3,E)$ for which*

$$T'(F) \hookrightarrow (L^g)^{Gal(E/F)}, \quad T'(F) = T(F)^g.$$

A similar statement holds if we replace L by I.

If we assume that $p \neq 2,3$ then every quadratic or cubic extension of F must be tamely ramified. Let F'/F denote such an extension. It is easy to see that there always exists a tamely ramified Galois extension F''/F such that $F \subset F' \subset F''$.

Now, to verify Theorem 1 for the supercuspidals of G, we discuss the various possibilities in detail using [Ja].

First, consider the case where π is of non-degenerate type (K_i, Ω_α), i odd, $\varpi^{i+1}\alpha$ elliptic. In this case, either $G' = Cent(\alpha, G) = G_\alpha$ is isomorphic to $Res_{EM/F}(ker\, N_{EM/M})$, M/F the unramified cubic extension, or to $E^{\times 1} \times E^{\times 1} \times E^{\times 1}$. We fix an embedding of G' into K. If $G' \cong Res_{EM/F}(ker\, N_{EM/M})$ then G' is stable and there is an embedding $G' \hookrightarrow K$. In this case, if $\gamma \in G'$

then clearly $\chi_\pi^{st}(\gamma)$ satisfies the Theorem. As mentioned above, if $G' \cong E^{\times 1} \times E^{\times 1} \times E^{\times 1}$ then there are two non-conjugate embeddings of G' into K and two into L. In either case, we have (in the notation of [Ja], p. 37)

$$\pi = Ind_{G' K_{\frac{i+1}{2}}}^{G} \rho_\theta.$$

We use the Frobenius formula to write the character of π in terms of the values of ρ_θ.

5.2.1 Case $Res_{EM/F}(ker\, N_{EM/M})$, M/F the unramified cubic extension

Suppose first that $G' \cong Res_{EM/F}(ker\, N_{EM/M})$. We have already considered $\chi_\pi^{st}(\gamma)$ with $\gamma \in G'$, so we may without loss assume that γ belongs to some other elliptic Cartan $T(F)$. Since

$$\rho_\theta(g'k) = \theta(g')\Omega_\alpha(k),$$

$k \in K_{\frac{i+1}{2}}$, $g' \in G'$, we claim that by using the Frobenius formula it can be shown that $\chi_\pi(\gamma) = 0$ unless $\gamma \in T(F) \cap K_{\frac{i+1}{2}}$. We claim that $\Omega_\alpha(\sigma\gamma) = \Omega_\alpha(\gamma)$ for $\sigma \in Aut_F\, T(F)$ (this follows from the definition of Ω_α and the fact that $T(F) \neq G'$). We claim that this implies $\chi_\pi^{st}(\gamma)$ satisfies the Theorem. This finishes the case $G' \cong Res_{EM/F}(ker\, N_{EM/M})$.

5.2.2 Case $E^{\times 1} \times E^{\times 1} \times E^{\times 1}$

Now let us consider the case $G' \cong E^{\times 1} \times E^{\times 1} \times E^{\times 1}$. We fix an embedding of G' into K. Define

$$Aut_K(G') = N_K(G')/G'.$$

Observe that $K_{\frac{i+1}{2}}$ is normal in K and

$$N_K(G')K_{\frac{i+1}{2}} \subset N_K(G'K_{\frac{i+1}{2}}).$$

We may identify G' with

$$G' = \{ \begin{pmatrix} \alpha & 0 & \beta\sqrt{\epsilon} \\ 0 & \gamma & 0 \\ b\beta\sqrt{\epsilon} & 0 & \alpha \end{pmatrix} \in U(2,1)(\mathcal{O}_F)\},$$

where $b \in \mathcal{O}_F^{\times 2}$.

Lemma 7 *Let $T(F)$ denote an elliptic Cartan of G and let π be of non-degenerate type (K_i, Ω_α), i odd, $\varpi^{i+1}\alpha$ elliptic. If γ is a regular element of $T(F)$ then $\chi_\pi^{st}(\gamma)$ satisfies (1).*

proof: The elements of $\mathfrak{D}_G(T/F)$ may be represented by elements δ of G. Analogous to (6), we may express $\chi_\pi^{st}(\gamma)$ as a sum of elements $\chi_{\pi'}(\gamma)$, where each π' is also supercuspidal. $\square$

Note if $\gamma \in K_{\frac{i+1}{2}} \cap T(F)$ is regular then $\chi_\pi(\gamma) = \chi_\pi(1) = d_\pi$ is the formal degree of π, so the character is stable in this range. Let $T'(F)$ denote a Cartan of G which is stably conjugate to $T(F)$. If $\gamma \in T(F)$ and $\gamma' \in T'(F)$ have the same preimage belonging to G' but not to $K_{\frac{i+1}{2}}$ then the above corollary implies

$$\chi_\pi(\gamma) = \sum_{w \in Aut_F(T')} \chi^o_{\rho_\theta}(w^{-1}\gamma' w) = \sum_{w \in Aut_F(T)} \chi^o_{\rho_\theta}(w^{-1}\gamma w).$$

This implies that $\chi_\pi^{st}(\gamma)$ satisfies the Theorem in case $G' \cong E^{\times 1} \times E^{\times 1} \times E^{\times 1}$ and γ belongs to some elliptic Cartan stably conjugate to G'. If γ does not belong to some elliptic Cartan stably conjugate to G' then we claim that the same reasoning as in the previous case suggests that $\chi_\pi^{st}(\gamma)$ satisfies the Theorem in this case as well. This finishes the case where π is of type (K_i, Ω_α), i odd, α elliptic. The case where i is even is similar and omitted.

5.2.3 Case $E^{\times 1} \times Res_{EM/F}(ker_{EM/M})$, M/F is a ramified quadratic extension

Next, consider the case where π is of non-degenerate type $(L_{2i-1}, \Omega_\alpha)$, $\varpi^{i+1}\alpha$ elliptic. In this case the centralizer G' of α is isomorphic to $E^{\times 1} \times Res_{EM/F}(ker_{EM/M})$, where M/F is a ramified quadratic extension. There are two non-conjugate but stably conjugate embeddings of G' into G: one into L and another into the non-standard filtration on the Iwahori I. We fix an embedding of G' into L, let λ be an extension of $\Omega_\alpha|_{G' \cap L_{2i-1}}$ to G', and let $J = L_i$, where

$$G' = \{ \begin{bmatrix} \alpha & 0 & \pi^{-1}\beta\sqrt{\epsilon} \\ 0 & \gamma & 0 \\ \pi^2\beta\sqrt{\epsilon} & 0 & \alpha \end{bmatrix} \in L \}.$$

In this case we have

$$\pi = Ind_{G'L_i}^G \rho_\lambda,$$

where $\rho_\lambda(g'j) = \lambda(g')\Omega_\alpha(j)$ for $g' \in G'$ and $j \in L_i$. It is clear that the reasoning for the case (K_i, Ω_α) applies to this case as well, with minor modifications

Lemma 8 *Let $T(F)$ denote an elliptic Cartan of G and let π be of non-degenerate type $(L_{2i-1}, \Omega_\alpha)$, $\varpi^{i+1}\alpha$ elliptic. If γ is a regular element of $T(F)$ then $\chi_\pi^{st}(\gamma)$ satisfies (1).*

proof: Omitted. □

Next, consider the case where π is of unramified type (K, Ω), Ω cuspidal. Suppose γ and γ' are stably conjugate elliptic elements of K. Under that map $k \longmapsto k_* := k \bmod K_1$, the elements γ_* and γ'_* are conjugate in $U(2,1)(k_F)$ ($k_F = \mathcal{O}_F/P_F$ denotes the residue field). Thus the class function $\chi_\Omega^o(x)$ is stable. From the Frobenius formula it follows as in the case above that the character χ_π^{st} satisfies the Theorem.

Lemma 9 *Let $T(F)$ denote an elliptic Cartan of G and let π be of unramified type (K, Ω), Ω cuspidal. If γ is a regular element of $T(F)$ then $\chi_\pi^{st}(\gamma)$ satisfies (1).*

proof: Omitted. □

Next, consider the case where π is of ramified type (L, Ω), Ω cuspidal. Suppose γ and γ' are stably conjugate elliptic elements of L. Under that map $\ell \longmapsto \ell_* := \ell \bmod L_1$, the elements γ_* and γ'_* are conjugate in $U(1,1)(k_F) \times U(1, k_F)$. Thus the class function $\chi_\Omega^o(x)$ is stable. As in the case where π is of unramified type (K, Ω), it follows that the character χ_π^{st} satisfies the Theorem.

Lemma 10 *Let $T(F)$ denote an elliptic Cartan of G and let π be of ramified type (L, Ω), Ω cuspidal. If γ is a regular element of $T(F)$ then $\chi_\pi^{st}(\gamma)$ satisfies (1).*

proof: Omitted. □

5.2.4 Case $Res_{EM/F}(ker_{EM/M})$ M/F a ramified cubic extension of F

Next, consider the case where π is of non-degenerate type (I_j, Ω_α), $j = 3i-1$ or $j = 3i-2$, $\varpi^{i+1}\alpha$ elliptic. In this case the centralizer of α is isomorphic to $Res_{EM/F}(ker_{EM/M})$ where M/F is a ramified cubic extension of F. This Cartan is stable (i.e., contains one conjugacy class inside its stable conjugacy class) and has an embedding into I. We fix such an embedding. We may realize G' as

$$G' = \{ \begin{bmatrix} \alpha & \beta & \gamma \\ \pi\gamma & \alpha & \beta \\ \pi\beta & \pi\gamma & \alpha \end{bmatrix} \in I \}.$$

Let $d := [\frac{j+2}{2}]$ (the greatest integer). By [Ja], Prop. 3.35, or [M], Theorem 3.5, we have

$$\pi = Ind_{G' I_{j+1-d}}^{G} \rho,$$

where $\rho = \rho_{G'}$ is a character of G' which agrees with Ω_α upon restriction to G'_j.

Lemma 11 *Let $T(F)$ denote an elliptic Cartan of G and let π be of non-degenerate type (I_j, Ω_α), $j = 3i-1$ or $j = 3i-2$, $\varpi^{i+1}\alpha$ elliptic. If γ is a regular element of $T(F)$ then $\chi_\pi^{st}(\gamma)$ satisfies (1).*

proof: Omitted. □

5.2.5 Case $E^{\times 1} \times Res_{EM/F}(ker_{EM/M})$, M/F a ramified quadratic extension

Next, consider the case where π is of non-degenerate type $(I_j^\flat, \Omega_\alpha)$, $j = 4i-2$, $\varpi^{i+1}\alpha$ elliptic. In this case the centralizer G' of α is isomorphic to $E^{\times 1} \times Res_{EM/F}(ker_{EM/M})$, where M/F is a ramified quadratic extension. As mentioned above, there are two non-conjugate but stably conjugate embeddings of G' into G: one into L and another into the non-standard filtration on the Iwahori I. We fix an embedding of G' into the non-standard filtration on I. We have

$$G' = \{ \begin{pmatrix} \alpha & 0 & \beta\sqrt{\epsilon} \\ 0 & \gamma & 0 \\ \pi\beta\sqrt{\epsilon} & 0 & \alpha \end{pmatrix} \in I \}.$$

We have

$$\pi = Ind_{G' I_{2i}^\flat}^{G} \rho,$$

for some representation ρ of $G' I_{2i}^\flat$.

Lemma 12 *Let $T(F)$ denote an elliptic Cartan of G and let π be of non-degenerate type $(I_j^\flat, \Omega_\alpha)$, $j = 4i-2$, $\varpi^{i+1}\alpha$ elliptic. If γ is a regular element of $T(F)$ then $\chi_\pi^{st}(\gamma)$ satisfies (1).*

proof: Omitted. □

The remaining cases deal with various types of π with $\varpi^{i+1}\alpha$ non-elliptic. These are shown in [Ja] to reduce down to one of the cases we've considered already. Thus we have verified Theorem 1 for the supercuspidals in every case.

5.3 Special representations

To deal with the discrete series representations of G which are not supercuspidal (i.e., the "special representations"), different methods are required. From [JKM], for example, such a representation must be either a twist of the Steinberg representation St_G by a 1-dimensional character or a twist of one of countably many other square-integrable representations σ_1, $\sigma_2^{(h)}$, say, where $h \geq 1$ is the degree of the character λ_0 in [JKM], §3. It is known that St_G, σ_1, and $\sigma_2 = \sigma_2^{(h)}$ all embed into a (reducible) principal series representation whose composition series is of length 2. The other constituent in the composition series containing St_G (resp., σ_1, σ_2) is one-dimensional (resp., infinite dimensional and non-tempered). By [Rog], §12, or [F2], the Langlands L-packet of St_G is a singleton whereas the L-packet of σ_i consists of σ_i and a certain supercuspidal representation which we denote by π_i, $i = 1, 2$.

In the case of the Steinberg representation, and its twists, we may verify the Theorem using the Borel-Serre-Casselman character formula as in the case of $U(1,1)(F)$. (Alternatively, the value of the Steinberg character is known on the elliptic set.) In the cases of σ_i, we use the following character identity of J. Rogawski and Y. Flicker (whose proof uses the global stable trace formula on $U(2,1)$).

Lemma 13 *([Rog], Lemma 12.3.7) There is a supercuspidal representation π_i, an endoscopic group H of G, and a transfer factor τ such that*

$$\chi_{\sigma_i}(\delta) - \chi_{\pi_i}(\delta) = \tau(\delta)\chi_{St_H}(\gamma),$$

where $\delta \in G(F)$ and $\gamma \in H(F)$ are matching elliptic regular elements.

From the basic properties of the transfer factor [Rog] and the fact that St_H is stable and π_i satisfies the Theorem, it follows from this character relation that σ_i must also satisfy the Theorem, $i = 1, 2$.

This verifies the first part of Theorem 1 for all discrete series representations of G, $p \neq 2, 3$, as desired. Moreover, if π and π' are supercuspidal and $\pi' \notin \Pi_\pi$ then $\Pi_{\pi'} \cap \Pi_\pi = \phi$.

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