Applications of DEs:
Spring problems, III

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If the frequency of the driving force of the spring matches the frequency of the homogeneous part $x_h(t)$, in other words if

$$x'' + \omega^2 x = F_0 \cos(\gamma t),$$

satisfies $\omega = \gamma$ then we say that the spring-mass system is in **(pure, mechanical) resonance**. For some time, it was believed that the collapse of the Tacoma Narrows Bridge [T] was explained by this phenomenon but this is false [R].
Fact: Consider

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \tag{1}$$

and

$$x'' + \omega^2 x = F_0 \sin(\gamma t). \tag{2}$$

Let $x_p = x_p(t)$ denote a particular solution to either (1) or (2). Then we have

- in case (1): if $\gamma \neq \omega$ then

$$x_p(t) = \frac{F_0}{\omega^2 - \gamma^2} \cos(\gamma t),$$

- in case (1): if $\gamma = \omega$ then

$$x_p(t) = \frac{F_0}{2\omega} t \sin(\gamma t),$$

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- in case (2): if $\gamma \neq \omega$ then

$$x_p(t) = \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t),$$

- in case (2): if $\gamma = \omega$ then

$$x_p(t) = -\frac{F_0}{2\omega} t \cos(\gamma t).$$

In particular, in case (1), if $\gamma \neq \omega$ then

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}(\cos(\gamma t) - \cos(\omega t))$$

is a solution and, in case (2), if $\gamma \neq \omega$ then

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}(\sin(\gamma t) - \sin(\omega t))$$

is a solution. In both of these, to derive the case $\gamma = \omega$, one can take limits $\gamma \to \omega$ in these expressions.

Example: Solve

$$x'' + \omega^2 x = \cos(\gamma t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $\omega = \gamma = 2$ (ie, mechanical resonance). We use SAGE for this:

SAGE

```
sage: t = var('t')
sage: x = function('x', t)
sage: (m,b,k,w,F0) = var("m,b,k,w,F0")
sage: de = lambda y: diff(y,t,t) + w^2*y - F0*cos(w*t)
sage: m = 1; b = 0; k = 4; F0 = 1; w = 2
sage: desolve(de(x(t)),[x,t])
 '(2*t*sin(2*t)+cos(2*t))/8+%k1*sin(2*t)+%k2*cos(2*t)'
sage: desolve_laplace(de(x(t)),["t","x"],[0,0,0])
 't*sin(2*t)/4'
sage: soln = lambda t : t*sin(2*t)/4
sage: P = plot(soln(t),0,10)
sage: show(P)
```

This is displayed below:

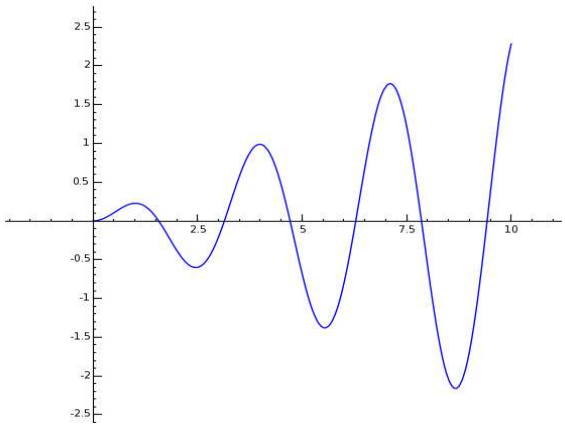

Figure 1: A forced undamped spring, with resonance.

Example: Solve

$$x'' + \omega^2 x = \cos(\gamma t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $\omega = 2$ and $\gamma = 3$ (ie, no mechanical resonance). We use SAGE for this:

SAGE

```
sage: t = var('t')
sage: x = function('x', t)
sage: (m,b,k,w,g,F0) = var("m,b,k,w,g,F0")
sage: de = lambda y: diff(y,t,t) + w^2*y - F0*cos(g*t)
sage: m = 1; b = 0; k = 4; F0 = 1; w = 2; g = 3
sage: desolve_laplace(de(x(t)),["t","x"],[0,0,0])
 'cos(2*t)/5-cos(3*t)/5'
sage: soln = lambda t : cos(2*t)/5-cos(3*t)/5
sage: P = plot(soln(t),0,10)
sage: show(P)
```

This is displayed below:

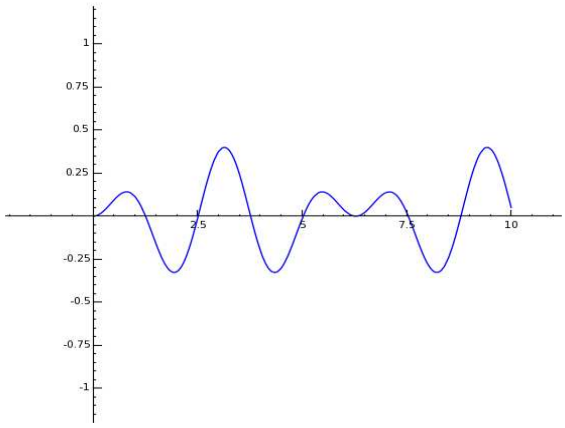

Figure 2: A forced undamped spring, no resonance.

Exercise: Using SAGE, solve

$$x'' + \omega^2 x = \cos(t), \quad x(0) = 0, \quad x'(0) = 0,$$

when $\omega = 1$ and when $\omega = 2$ (or any other value not equal to the resonance frequency).

References

[H] General wikipedia introduction to Hooke's Law `http://en.wikipedia.org/wiki/Hookes_law`

[R] Wikipedia entry for Resonance: `http://en.wikipedia.org/wiki/Resonance`

[S] Wikipedia entry for Simple harmonic motion: `http://en.wikipedia.org/wiki/Simple_harmonic_motion`

[T] Wikipedia entry for the Tacoma Narrows Bridge: `http://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge#Collapse`