

An introduction to systems of DEs

Lanchester's equations for battle

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A goal of military analysis is a means of reliably predicting the outcome of military encounters, given some basic information about the forces' status. The case of two combatants in an "aimed fire" battle was solved during World War I by Frederick William Lanchester, a British engineer in the Royal Air Force, who discovered a way to model battle-field casualties using systems of differential equations. He assumed that if two armies fight, with $x(t)$ troops on one side and $y(t)$ on the other, the rate at which soldiers in one army are put out of action is proportional to the troop strength of their enemy. This give rise to the system of differential equations

$$\begin{cases} x'(t) = -Ay(t), & x(0) = x_0, \\ y'(t) = -Bx(t), & y(0) = y_0, \end{cases}$$

where $A > 0$ and $B > 0$ are constants (called their *fighting effectiveness coefficients*) and x_0 and y_0 are the intial troop strengths. For some historical examples of actual battles modeled using Lanchester's equations, please see references in the paper by McKay [M].

We show here how to solve these using Laplace transforms.

Example: A battle is modeled by

$$\begin{cases} x' = -4y, & x(0) = 150, \\ y' = -x, & y(0) = 90. \end{cases}$$

(a) Write the solutions in parameteric form. (b) Who wins? When? State the losses for each side.

soln: Take Laplace transforms of both sides:

$$s\mathcal{L}[x(t)](s) - x(0) = -4\mathcal{L}[y(t)](s),$$

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$$s\mathcal{L}[y(t)](s) - x(0) = -\mathcal{L}[x(t)](s).$$

For convenience, let $X(s) = \mathcal{L}[x(t)](s)$ and $Y(s) = \mathcal{L}[y(t)](s)$. Solving these equations gives

$$X(s) = \frac{sx(0) - 4y(0)}{s^2 - 4} = \frac{150s - 360}{s^2 - 4},$$

$$Y(s) = -\frac{-sy(0) + x(0)}{s^2 - 4} = \frac{90s - 150}{s^2 - 4}.$$

Laplace transform tables give the solutions to the system:

$$x(t) = -15e^{2t} + 165e^{-2t}$$

$$y(t) = 90 \cosh(2t) - 75 \sinh(2t)$$

These functions, superimposed on the same graph, look like

The “ y -army” wins. Solving for $x(t) = 0$ gives $t_{win} = \log(11)/4 = .5994738182\dots$, so the number of survivors is $y(t_{win}) = 49.7493718$, so 49 survive.

Remark 1 *These is a more naive solution to this system which is possible. Simply differentiate the first DE, $x'' = -Ay'$ and then plug the 2nd DE into it, $x'' = -A(-Bx) = ABx$. This leads to*

$$x'' - ABx = 0, \quad x(0) = x_0, \quad x'(0) = -Ay(0) = -Ay_0.$$

This can be solved using the method of characteristic polynomials. Once you know $x = x(t)$, you can solve for y using $y(t) = -\frac{1}{A}x'(t)$.

Lanchester’s square law: Suppose that if you are more interested in y as a function of x , instead of x and y as functions of t . One can use the chain rule from calculus to derive from the system $x'(t) = -Ay(t)$, $y'(t) = -Bx(t)$ the single equation

$$\frac{dy}{dx} = \frac{Bx}{Ay}.$$

This differential equation can be solved by the method of separation of variables: $Aydy = Bxdx$, so

$$Ay^2 = Bx^2 + C,$$

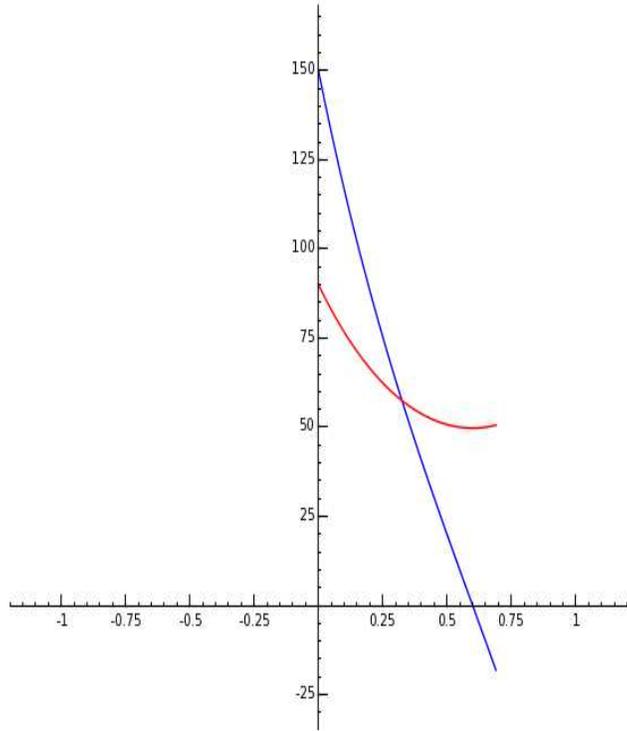


Figure 1: Lanchester’s model for the x vs. y battle.

where C is an unknown constant. (To solve for C you must be given some initial conditions.) The quantity Bx^2 is called the **fighting strength of the X-men** and the quantity Ay^2 is called the **fighting strength of the Y-men** (“fighting strength” is not to be confused with “troop strength”). The quantity A is called the **fighting effectiveness of the Y-men** and quantity B is called the **fighting effectiveness of the X-men**. This relationship between the troop strengths is sometimes called **Lanchester’s square law** and is sometimes expressed as saying the relative fighting strength is a constant:

$$Ay^2 - Bx^2 = \text{constant}.$$

Suppose your total number of troops is some number T , where $x(0)$ are initially placed in a fighting capacity and $T - x(0)$ are in a support role. It is often assumed that the fighting effectiveness is proportional to some fixed power of $(T - x(0))/x(0)$ (for example, $B = B_0 \cdot \frac{T-x(0)}{x(0)}$, for a fixed $B_0 > 0$).

If your troops outnumber the enemy then you want to choose the number of support units to be the smallest number such that the fighting effectiveness is not decreasing (therefore is roughly constant). The remainder should be engaged with the enemy in battle [M].

A battle between three forces gives rise to the differential equations

$$\begin{cases} x'(t) = -A_1y(t) - A_2z(t), & x(0) = x_0, \\ y'(t) = -B_1x(t) - B_2z(t), & y(0) = y_0, \\ z'(t) = -C_1x(t) - C_2y(t), & z(0) = z_0, \end{cases}$$

where $A_i > 0$, $B_i > 0$, and $C_i > 0$ are constants and x_0 , y_0 and z_0 are the initial troop strengths.

Example: Consider the battle modeled by

$$\begin{cases} x'(t) = -y(t) - z(t), & x(0) = 100, \\ y'(t) = -2x(t) - 3z(t), & y(0) = 100, \\ z'(t) = -2x(t) - 3y(t), & z(0) = 100. \end{cases}$$

The Y-men and Z-men are better fighters than the X-men, in the sense that the coefficient of z in 2nd DE (describing their battle with y) is higher than that coefficient of x , and the coefficient of y in 3rd DE is also higher than that coefficient of x . However, as we will see, the worst fighter wins! (Can you give an intuitive reason why this is true in this case?)

Taking Laplace transforms, we obtain the system

$$\begin{cases} sX(s) + Y(s) + Z(s) = 100 \\ 2X(s) + sY(s) + 3Z(s) = 100, \\ 2X(s) + 3Y(s) + sZ(s) = 100, \end{cases}$$

which we solve by row reduction using the augmented matrix

$$\begin{pmatrix} s & 1 & 1 & 100 \\ 2 & s & 3 & 100 \\ 2 & 3 & s & 100 \end{pmatrix}$$

This has row-reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & (100s + 100)/(s^2 + 3s - 4) \\ 0 & 1 & 0 & (100s - 200)/(s^2 + 3s - 4) \\ 0 & 0 & 1 & (100s - 200)/(s^2 + 3s - 4) \end{pmatrix}$$

This means $X(s) = \frac{100s+100}{s^2+3s-4}$ and $Y(s) = Z(s) = \frac{100s-200}{s^2+3s-4}$. Taking inverse LTs, we get the solution: $x(t) = 40e^t + 60e^{-4t}$ and $y(t) = z(t) = -20e^t + 120e^{-4t}$. In other words, the worst fighter wins!

In fact, the battle is over at $t = \log(6)/5 = 0.35\dots$ and at this time, $x(t) = 71.54\dots$. Therefore, the worst fighters, the X-men, not only won but have lost less than 30% of their men!

Since $y(t) = z(t)$, we have the following parametric plots of these functions.

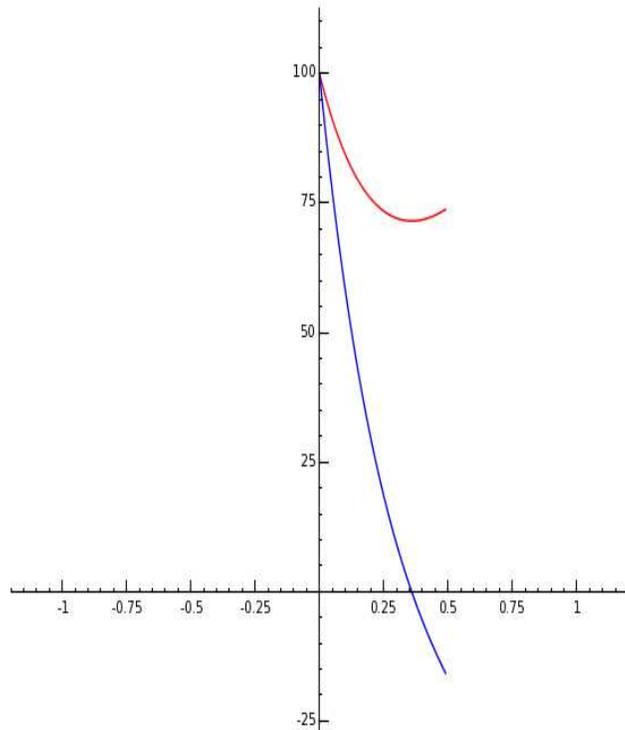


Figure 2: Lanchester's model for the x vs. y vs z battle.

SAGE

```
sage: t = var('t')
sage: s = var('s')
sage: SR = SymbolicExpressionRing()
sage: MS = MatrixSpace(SR, 3, 4)
```

```

sage: A = MS([[s,1,1,100],[2,s,3,100],[2,3,s,100]])
sage: B = A.echelon_form()
sage: X = B[0][3]
sage: X.inverse_laplace(s,t)
40*e^t + 60*e^(-(4*t))
sage: Y = B[1][3]
sage: Y.inverse_laplace(s,t)
120*e^(-(4*t)) - 20*e^t
sage: Z = B[2][3]
sage: Z.inverse_laplace(s,t)
120*e^(-(4*t)) - 20*e^t
sage: a = log(6)/5
sage: a
0.358351893846
sage: 40*exp(a)+60*exp(-4*a)
71.5484540553

```

Exercise: A battle is modeled by

$$\begin{cases} x' = -4y, & x(0) = 150, \\ y' = -x, & y(0) = 40. \end{cases}$$

(a) Write the solutions in parameteric form. (b) Who wins? When? State the losses for each side.

Use **SAGE** to solve this.

References

[KL] Everything2 entry for Lanchester's equations:

<http://www.everything2.com/index.pl?node=Lanchester%20Systems%20and%20the%20Lanchester%20Laws%20of%20Combat>

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http://en.wikipedia.org/wiki/Frederick_William_Lanchester

[K] Wikipedia entry for predator-prey equations:

http://en.wikipedia.org/wiki/Volterra-Lotka_equations

[N] Lanchester automobile information:

http://www.amwmag.com/L/Lanchester_World/lanchester_world.html

- [M] N. J. MacKay, “Lanchester combat models”,
<http://arxiv.org/abs/math.H0/0606300>
- [La] Frederick William Lanchester, **Aviation in Warfare: The Dawn of the Fourth Arm**, Constable and Co., London, 1916.