Solving ODEs:

using the power series method, I

Prof. Joyner[1]

In this part, we recall some basic facts about power series and Taylor series. We will turn to solving DEs in part II.

Roughly speaking, power series are simply infinite degree polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... = \sum_{k=0}^{\infty} a_k x^k, \tag{1}$$

for some real or complex numbers $a_0, a_1, ...$ The number a_k is called the **coefficient** of x^k, for $k = 0, 1, ...$ Let us ignore for the moment the precise meaning of this infinite sum (How do you associate a value to an infinite sum? Does the sum converge for some values of x? If so, for which values? ...) We will return to that later.

First, some motivation. Why study these? This type of function is convenient for several reasons

- it is easy to differentiate (term-by-term):

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + ... = \sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k,$$

- it is easy to integrate (term-by-term):

$$\int f(x)\, dx = a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + ... = \sum_{k=0}^{\infty} \frac{1}{k+1} a_k x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} x^k,$$

[1]These notes licensed under Attribution-ShareAlike Creative Commons license, `http://creativecommons.org/about/licenses/meet-the-licenses`. The graphs were created using SAGE and and GIMP `http://www.gimp.org/` by the author. Originally written 9-25-2007. Some of the latex code is taken from the excellent (public domain!) text by Sean Mauch [M].

- if (as is often the case) the a_k's tend to zero very quickly, then the sum of the first few terms of the series are often a good numerical approximation for the function itself,
- power series enable one to reduce the solution of certain differential equations down to (often the much easier problem of) solving certain recurrance relations.
- Power series expansions arise naturally in *Taylor's Theorem of the Mean*: If $f(x)$ is $n+1$ times continuously differentiable in (a, x) then there exists a point $\xi \in (a, x)$ such that

$$\begin{aligned} f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \\ + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \end{aligned} \tag{2}$$

  The sum

$$T_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a),$$

  is called the **n-th degree Taylor polynomial of f centered at a**. For the case $n = 0$, the formula is

$$f(x) = f(a) + (x-a)f'(\xi),$$

  which is just a rearrangement of the terms in the *theorem of the mean*,

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}.$$

Some examples:

- Geometric series:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots \\ &= \sum_{n=0}^{\infty} x^n \end{aligned} \tag{3}$$

To see this, assume $|x| < 1$ and let $n \to \infty$ in the polynomial identity

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^{n+1}}{1 - x}.$$

For $x \geq 1$, the series does not converge.

- The exponential function:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned} \tag{4}$$

To see this, take $f(x) = e^x$ and $a = 0$ in Taylor's theorem (2), using the fact that $\frac{d}{dx} e^x = e^x$ and $e^0 = 1$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{\xi^{n+1}}{(n+1)!},$$

for some ξ between 0 and x. Perhaps it is not clear to everyone that as n becomes larger and larger (x fixed), the last ("remainder") term in this sum goes to 0. However, Stirling's formula tells us how large the factorial function grows,

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + O(\frac{1}{n})),$$

so we may indeed take the limit as $n \to \infty$ to get (4).

Wikipedia's entry on "Power series" [P1] has a nice animation showing how more and more terms in the Taylor polynomials approximate e^x better and better.

- The cosine function:

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad (5)\end{aligned}$$

  This too follows from Taylor's theorem (take $f(x) = \cos x$ and $a = 0$). However, there is another trick: Replace x in (4) by ix and use the fact ("Euler's formula") that $e^{ix} = \cos(x) + i\sin(x)$. Taking real parts gives (5). Taking imaginary parts gives (6), below.

- The sine function:

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \\ &= 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad (6)\end{aligned}$$

  Indeed, you can formally check (using formal term-by-term differentiation) that

$$-\frac{d}{dx}\cos(x) = \sin(x).$$

  (Alternatively, you can use this fact to deduce (6) from (5).)

- The logarithm function:

$$\begin{aligned}\log(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ &= -\sum_{n=0}^{\infty} \frac{1}{n} x^n \qquad (7)\end{aligned}$$

  This follows from (3) since (using formal term-by-term integration)

$$\int_0^x \frac{1}{1-t} = -\log(1-x).$$

SAGE

```
sage: taylor(sin(x), x, 0, 5)
 x - x^3/6 + x^5/120
sage: P1 = plot(sin(x),0,pi)
sage: P2 = plot(x,0,pi,linestyle="--")
sage: P3 = plot(x-x^3/6,0,pi,linestyle="-.")
sage: P4 = plot(x-x^3/6+x^5/120,0,pi,linestyle=":")
sage: T1 = text("x",(3,2.5))
sage: T2 = text("x-x^3/3!",(3.5,-1))
sage: T3 = text("x-x^3/3!+x^5/5!",(3.7,0.8))
sage: T4 = text("sin(x)",(3.4,0.1))
sage: show(P1+P2+P3+P4+T1+T2+T3+T4)
```

This is displayed below:

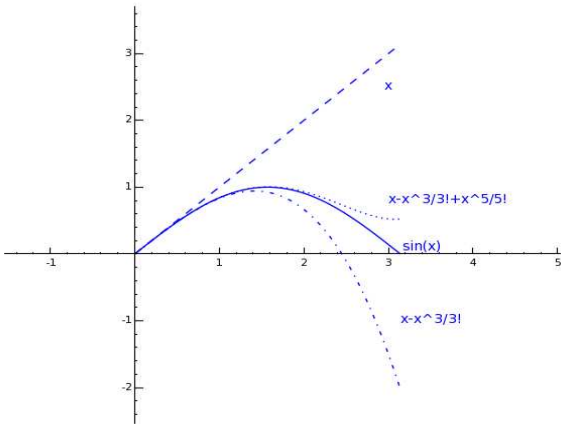

Figure 1: Taylor polynomial approximations for $\sin(x)$.

Exercise: Use SAGE to plot successive Taylor polynomial approximations for $\cos(x)$.

Finally, we turn to the meaning of these sums. How do you associate a value to an infinite sum? Does the sum converge for some values of x? If so, for which values? . We will (for the most part) answer all of these.

First, consider our infinite power series $f(x)$ in (1), where the a_k are all given and x is fixed for the momemnt. The **partial sums** of this series are

$$f_0(x) = a_0, \quad f_1(x) = a_0 + a_1 x, \quad f_2(x) = a_0 + a_1 x + a_2 x^2, \cdots .$$

We say that the series in (1) **converges at** x if the limit of partial sums

$$\lim_{n\to\infty} f_n(x)$$

exists. There are several tests for determining whether or not a series converges. One of the most commonly used tests is the

Root test: Assume

$$L = \lim_{k\to\infty} |a_k x^k|^{1/k} = |x| \lim_{k\to\infty} |a_k|^{1/k}$$

exists. If $L < 1$ then the infinite power series $f(x)$ in (1) converges at x. In general, (1) converges for all x satisfying

$$-\lim_{k\to\infty} |a_k|^{-1/k} < x < \lim_{k\to\infty} |a_k|^{-1/k}.$$

The number $\lim_{k\to\infty} |a_k|^{-1/k}$ (if it exists, though it can be ∞) is called the **radius of convergence**.

Example: The radius of convergence of e^x (and $\cos(x)$ and $\sin(x)$) is ∞:

$$\begin{aligned}
&\lim_{k\to\infty} |a_k|^{-1/k} = \lim_{k\to\infty} |1/k!|^{-1/k} \\
&= \lim_{k\to\infty} k!^{1/k} = \lim_{k\to\infty} [\sqrt{2\pi k}\left(\tfrac{k}{e}\right)^k (1 + O(\tfrac{1}{k}))]^{1/k} \\
&= \lim_{k\to\infty} (\sqrt{2\pi k})^{1/k} \left(\tfrac{k}{e}\right) = \infty.
\end{aligned}$$

The radius of convergence of $1/(1-x)$ (and $\log(1+x)$) is 1.

Example: The radius of convergence of

$$f(x) = \sum_{k=0}^{\infty} \frac{k^7 + k + 1}{2^k + k^2} x^k$$

can be determined with the help of SAGE. We want to compute

$$\lim_{k\to\infty} |\frac{k^7+k+1}{2^k+k^2}|^{-1/k}.$$

SAGE

```
sage: k = var('k')
sage: limit(((k^7+k+1)/(2^k+k^2))^(-1/k),k=infinity)
2
```

In other words, the series converges for all x satisfying $-2 < x < 2$.

Exercise: Use SAGE to find the radius of convergence of

$$f(x) = \sum_{k=0}^{\infty} \frac{k^3+1}{3^k+1} x^{2k}$$

References

[P1] Wikipedia entry for Power series: `http://en.wikipedia.org/wiki/Power_series`

[P2] Wikipedia entry for the power series method: `http://en.wikipedia.org/wiki/Power_series_method`

[M] Sean Mauch, *Introduction to methods of Applied Mathematics*, `http://www.its.caltech.edu/~sean/book/unabridged.html`