SM 212 Final Examination
Fall 1991

1. Find the explicit general solution to

   (a) $y' - 4y = e^{4x}$,

   (b) $x^2 y' = e^y$,

   (c) $4x^2 y'' + 16xy' + 9y = 0$.

2. (a) Use graphical analysis to sketch solutions to the differential equation $y' = y(y-2)$ corresponding to the four initial conditions $y(0) = -1, 0.5, 1.5, 3$. Show correct slope and concavity for each solution that you sketch. Also state what the equilibrium solutions are. Classify each equilibrium solution as stable or unstable.

   (b) Use two steps of Euler's method to find an approximation to $y(0.2)$ for the initial value problem $y' + y^2 = x$, $y(0) = 1$.

3. (a) Find the general solution to

      (i) $y'' - 4y' - 5y = 0$,

      (ii) $y'' + 4y' + 13y = 0$,

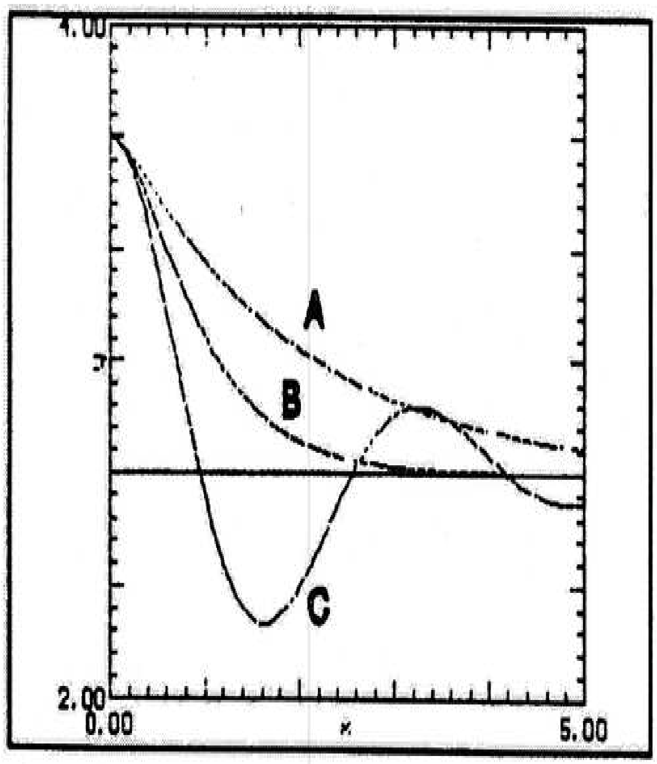
      (iii) $y'' - 6y' + 9y = 0$.

   (b) A damped mass-spring system hanging from a ceiling is governed by the differential equation $my'' + by' + ky = 0$.

      (i) From what point is the distance y measured?

      (ii) State conditions on the parameters m, b, and k which will make the motion of the mass critically damped.

(iii) The figure below shows an example of the motion of a mass-spring system under each of the three types of damping. Which is which?



4. Use the method of undetermined coefficients to solve the initial value problem

$$y'' - y' = 2 + 6x, \quad y(0) = 10, \quad y'(0) = 0.$$

5. (a) Find

(i) $\mathcal{L}\{2t^3 + 3\sin(2t) + 4e^{-5t}\}$,

(ii) $\mathcal{L}\{e^{-2t}\cos(3t)\}$,

(iii) $\mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} 1, & 0 \leq t < 3, \\ t, & 3 \leq t. \end{cases}$$

(b) Use Laplace transoforms to solve

$$y'' + y = e^{-t}, \quad y(0) = -1, y'(0) = 2.$$

6. (a) Use the convolution theorem to find $\mathcal{L}^{-1}\{1/(s^2(s+2))$.

   (b) Find $f(t) = \mathcal{L}^{-1}\{1/((1-e^{-s})(s+4))\}$. Write your answer in summation notation. What is $f(2.5)$?

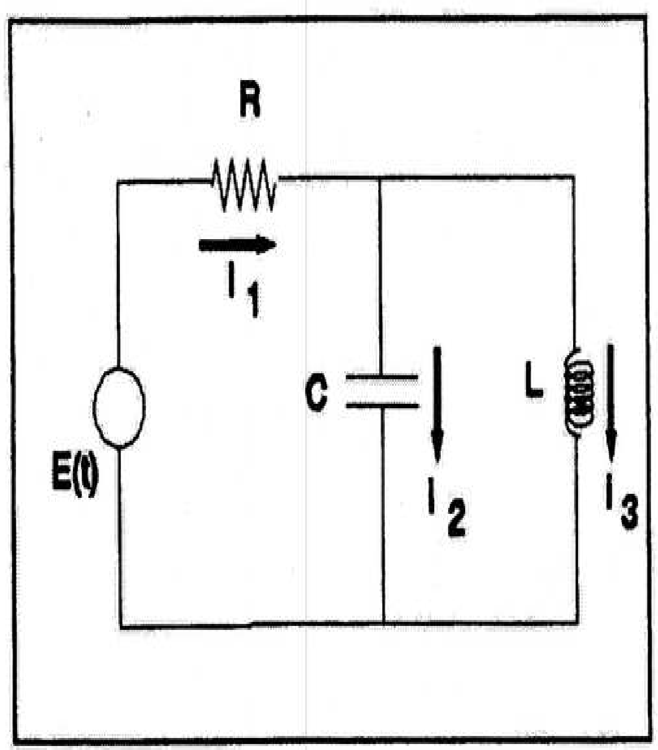
7. (a) (i) A mass-spring system subject to an external force is governed by the differential equation

$$y'' + y' + y = \sin(2t).$$

   Find the steady state motion of the spring.

   (ii) Write $y(t) = -4\cos(t) + 3\sin(t)$ in the form $y(t) = A\cos(t-\phi)$. What is the first positive value of t for which y is zero?

   (b) In the circuit shown below R, C, L and $E(t)$ stand for the numerical values of the resistance, capacitance, inductance and electromotive force, respectively. Find a 2×2 system of differential equations for the unknown current i_3 and the unknown charge q_2 on the capacitor. Do not solve this system.



8. Solve the initial value problem

$$dx/dt = -4x - 2y, \quad x(0) = -3,$$

$$dy/dt = 3x + y, \quad y(0) = 4$$

9. (a) For the differential equation

$$y'' - 4xy' + 6y = 0$$

   (i) Find the recursive relation for the coefficients c_n of a power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$

   (ii) Use the recursive relation in (a.i) to find the first three non-zero terms of each of two linearly independent solutions to the differential equation.

   (b) What is the smallest radius of convergence that a power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ to the differential equation $(1-2x^2)y'' - 3y = 0$ could have?

   (c) (i) Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0, \\ 2, & 0 < x \leq \pi. \end{cases}$$

   (ii) State the values to which the Fourier series in (c.i) converges at each of the points $x = 0, \pi/2, \pi$.

10. The temperature distribution $u(x,t)$ on a thin bar satisfies the following conditions

$$u_{xx} = u_t, \quad u_x(0,t) = u_x(\pi,t) = 0, \quad u(x,0) = x.$$

   (a) Interpret the boundary conditions physically.

   (b) Find $u(x,t)$. Show all steps of the separation of variables process clearly. Write your answer in summation notation.

   (c) Use the first three terms of your answer to (b) to find the termperature at the middle of the bar after two seconds.