SM 212 Final Examination
11 December 1996

1. (a) Find the general solution to
   (i) $x^2 dy/dx = 4 - 3xy$,
   (ii) $(1 - x\cos(xy))dy/dx = 1 + y\cos(xy)$,

   (b) Solve the initial value problem $dy/dx = 2x\cos^2(y), \quad y(0) = \pi/4$.

2. (a) A tank contains 100 gallons of water with 100 lbs. of salt dissolved in it. Saltwater with a concentration of 4 lb/gal is pumped into the tank at a rate of 1 gal/min. The well-stirred mixture is pumped out at the rate of 2 gal/min. At what time does the tank contain the largest amount of salt? How much salt is in the tank at that time?

   (b) Use the improved Euler method with a step size of $h = 0.1$ to approximate $y(0.1)$ for the initial value problem.

$$dy/dx = x - y^2, \quad y(0) = -1.$$

3. Find the general solution to:
   $y'' + 2y' - 8y = 0$,
   (b) $y'' + 8y' + 25y = 0$,
   (c) $y^{(4)} + 8y'' + 16y = 0$.

4. Use undetermined coefficients (or annihilators) to solve the initial value problem

$$y'' - 3y' + 2y = 6e^x, \quad y(0) = 1, \quad y'(0) = -2$$

5. (a) An 8 lb weight stretches a spring 2 ft upon coming to rest at equilibrium. From equilibrium the weight is raised 1 ft and released from rest. The motion of the weight is resisted by a damping force that in numerically equal to 2 times the weight's instantaneous velocity.
   (i) Find the position of the weight as a function of time.
   (ii) What type of damping does this mass-spring system possess?

   (b) The position of a weight in a mass-spring system subject to an external force is given by $x(t) = e^{-t}\cos(3t) + e^{-t}\sin(3t) + 6\cos(2t) + 4\sin(2t)$.

(i) What are the amplitude and period of the steady-state part of the solution?
(ii) Write the transient part of the solution in the form $Ae^{-t}\sin(3t+\phi)$.
(iii) Find the time past which the magnitude of the transient part of the solution is less than one-percent of that of the steady-state part of the solution.

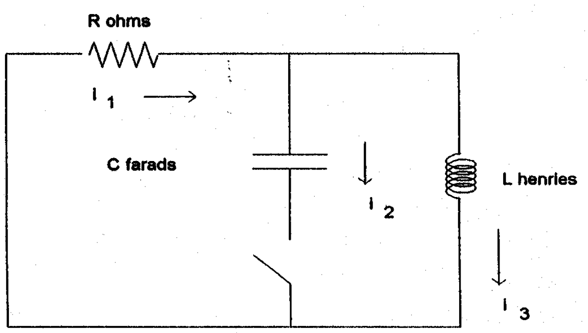
6. (a) Find:
(i) $\mathcal{L}\{4t^3 - 4e^{-t}\sin(2t)\}$,
(ii) $\mathcal{L}\{t^2 U(t-1)\}$,
(iii) $\mathcal{L}\{t\cos t\}$.

(b) Find $\mathcal{L}\{f(t)\}$ for the periodic function given over one period by

$$f(t) = \begin{cases} 1, & 0 < t \le 2, \\ 0, & 2 < t < 3. \end{cases}$$

7. (a) Find
(i) $\mathcal{L}^{-1}\{1/(s^2+9) - e^{-3s}/(s-4)^2\}$,
(ii) $\mathcal{L}^{-1}\{(s+4)/(s^2+6s+25)\}$.

(b) Use the convolution theorem to find $\mathcal{L}^{-1}\{1/((s-1)^2(s-3))\}$.



8. (a) For the circuit in the circuit above show that the charge q on the capacitor and the current i_3 in the right branch satisfy the system of differential equations

$$q' + (1/RC)q + i_3 = 0,$$
$$i_3' + (1/LC)q = 0.$$

(b) When the switch in the circuit is closed at time $t = 0$, the current i_3 is 0 amps and the charge on the capacitor is 5 coulombs. With $R = 2$, $L = 3$, $C = 1/6$ use Laplace transforms to find the charge $q(t)$ on the capacitor.

9. Use eigenvalues/eigenvectors to solve the initial value problem

$$dx/dt = 3y + 6, \quad x(0) = 1,$$

$$dy/dt = -3x, y(0) = 0.$$

10. (a) Find the Fourier sine series of the function

$$f(x) = \begin{cases} 0, & 0 \le x \le 2, \\ 50x, & 2 < x \le 4. \end{cases}$$

(b) The temperature $u(x,t)$ of a thin bar of length 4 satisfies the following conditions

$$\partial u/\partial t = 3\partial^2 u/\partial x^2,$$
$$u(0,t) = u(4,t) = 0,$$
$$u(x,0) = f(x)$$

where $f(x)$ is given in (10.a).
(i) Use your answer in (10.a) to find $u(x,t)$. Show all steps of the separation of variables process clearly. Write your answer in summation notation.
(ii) Use the first two non-zero terms of your answer in ((10)(b)1) to approximate $u(1,.5)$.

Answers for SM 212 Final Examination
11 December 1996

1. a. i. $x^2(dy/dx) = 4-3xy$, $dy/dx+(3/x)y = (4/x^2)$, $y = (\int(e^{\int(3/x)dx} * 4/x^2 + C)/e^{\int(3/x)dx}$, $y = (2x^2 + C)/x^3$.

1. a. ii. $(1 - x\cos(xy))y' = 1 + y\cos(xy)$, $(1 + y\cos(xy))dx - (1 - x\cos(xy))dy = 0$, $M_y = N_x$, $\sin(xy)+x \mid -y+\sin(yx)/x$, $\sin(xy)+x-y = C$

1.b. $dy/dx = 2x\cos^2(y)$, $y(0) = \pi/4$, $\int(dy/\cos^2 y) = \int(2x\ dx)$, $\tan y = x^2 + C$, $\tan y = x^2 + 1$, $y = arctan(x^2 + 1)$.

2. $A'(t) = (flowin)(Cin)-(A(t)flowout)/volume$, $A'(t)+2A(t)/(100-t) = 4$, $A(t) = (4*\int(e^{\int(2/(100-t))})+C)/(e^{\int(2/(100-t))})$, gives $A(t) = C*(100-t)^2 + 4*(100-t)$. The IC $A(0) = 100$, implies $A(t) = 100 + 2t - (3/100)t^2$.

2.b. $dy/dx = x - y^2$, $y(0) = -1$, $y_{new} = y + (h/2)(f(x, y) + f(x + h, y + hf(x, y)))$, $h = .1$, $y_{new} = -2211/2000$

3.a. $y'' + 2y' - 8y = 0$, $(r + 4)(r - 2) = 0$, $y = C_1e^{-4t} + C_2e^{2t}$

3.b. $y''+8y'+25 = 0$, $r^2+8r+25 = 0$¡ $r = -4\pm3i$, $y = c_1e^{-4x}\cos 3x + c_2e^{-4x}\sin 3x$,

3.c. $y^{(4)} + 8y'' + 16y = 0$, $(x^2 + 4)(x^2 + 4) = 0$, $r = \pm2i$ (double roots), $y = c_1\cos 2x + c_2\sin 2x + c_3x\cos 2x + c_4x\sin 2x$,

4. $y'' - 3y' + 2y = 6e^x$, $y(0) = 1$, $y'(0) = -2$ $(r - 2)(r - 1) = 0$, $y_h = c_1e^{2x} + c_2e^x$, $y_p = Axe^x$, $y'_p = Axe^x + Ae^x$, $y''_p = Axe^x + 2Ae^x$, $Axe^x + 2Ae^x - 3(Axe^x + Ae^x) + 2Axe^x = 6e^x$, $A = -6$, $y = y_h + y_p = c_1e^2x + c_2e^x - 6xe^x$, $y(0) = 1$, $c_1 + c_2 = 1$, $y'(0) = -2 = 2c_1 + c_2$, $c_1 = 3$, $c_2 = -2$, $y = 3e^2x - 2e^x - 6xe^x$

5.a. 1. $mx'' + bx' + kx = 0$, $k = 8/2 = 4$, $m = 8/32$ slugs, $b = 2$, $r'' + 8r' + 16r = 0$, $r = -4, -4$, $x = c_1e^{-4t} + c_2te^{-4t}$, $x(0) = -1, x'(0) = 0$, $x = -e^{-4t} - 4te^{-4t}$,

5.a.2. critically damped.

5.b.1. Amplitude: $\sqrt{c_1^2 + c_2^2} = \sqrt{36 + 16} = \sqrt{52}$, Period $= \pi$,

5.b.2. $\phi = arctan(c_2/c_1)$, $x(t) = (\sqrt{2})(e^{-t})\sin(3t + -\pi/4)$,

5.b.3. $\sqrt{2}e^{-t} < (1/10)\sqrt{6^2 + 4^2}$, $e^{-t} < \sqrt{26}/10$, $t > \log(10/\sqrt{26}$.

6.a.1. $24/s^4 - 8/(((s + 1)^2 + 4)$,

6.a.2 $(e^{-s})\mathcal{L}\{f(t - 1)\}$, $e^{-s}(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3})$

6.a.3 $\mathcal{L}\{t\cos t\} = (-1 + s^2)/(s^2 + 1)^2$

6.b.1 $u(t - a)- > e^{-as}/s - 1u(t - 2)$, $F(s) = (-2^{-3s})/s$

7.a.1 $u_c(t)f(t-c) = \mathcal{L}^-1\{e^{-cs}F(s)\}$, $1/3\sin 3t - \mathcal{L}^-1\{e^{-3s}(1/(s-2)^4)\}$, $f(t) = t^3e^{2t}$, $f(t - c) = (t - 3)^3e^{2t-6} = 1/3\sin 3t - 1/6u3(t)(t - 3)^3e^{2t-6}$,

7.a.2 $\mathcal{L}^{-1}\{(s+4)/(s^2+6s+25)\}$, $\mathcal{L}^{-1}\{(s+3)/((s+3)^2+16)+(1/((s+3)^2 + 16))\}$, $e^{-3t}\cos(4t) + 1/4e^{-3t}\sin(4t)$

7.b. Use the convolution theorem to find $\mathcal{L}^{-1}\{1/((s - 1)^2(s - 3)\}$, $\mathcal{L}^{-1}\{(1/(s - 1)^2)(1/(s - 3))\} = e^{-3t}/4 + (-t/2 - 1/4)e^t$.

8.a. $Li'_3 - (1/c)q = 0$, $i_1R + (1/c)q = 0$

8.b. $Lq'' + Rq' + (1/C)q = E(t)$, $q'' + (2/3)q' + 2q = 0$, $s^2Q(s) - sq(0) - q'(0) + 2/3Q(s) - q(0) + 2Q(s) = 0$, $Q(s)(s^2 + (2/3)s + 2) = 5s + 5$, $Q(s) = (5s+5)/(s^2+(2/3)s+2)$, $Q(s) = (5s+5)/(s^2)+(5s+5)/(2/3)s+(5s+5)/2$,

9. $x(t) = \cos(3t) + 2\sin(3t)$,$y(t) = -\sin(3t) + 2\cos(3t) - 2$

10.a. $b_n = (2/L)\int_0^L f(x)\sin(nx\pi/L)\ dx = \frac{1}{2}\int_2^4 50x\sin(nx\pi/4)\ dx = 200\frac{-2\sin(n\pi/2)+n\pi\cos(n\pi/2)-2n\pi\cos(n\pi)}{n^2\pi^2}$, $f(x) = \sum_n b_n \sin(n\pi x/4)$

b. (i) $X(x)T'(t) = X'(x)T(t)$, $\frac{1}{3}T'(t)/T(t) = X'(x)/X(x) = -K$, $T(t) = Ae^{-3Kt}$, with $K > 0$ and $X(x) = c_1\cos(\sqrt{K}x) + c_2\sin(\sqrt{K}x)$, $u(x,t) = \sum_n b_n \sin(\frac{xn\pi}{4})e^{-3n^2\pi^2 t/16}$. (ii) $u(1, 1.2) \cong 21.97$.