SM212/P Final Examination: Multiple Choice Section

04 May 2009 1930

The Multiple Choice part of the exam counts 50%. Fill in the letter of the best answer on the bubble sheet. There is no penalty for wrong answers.

1. Which of the following differential equations is separable?

   a) $\frac{dy}{dx} = e^{xy}$ b) $\frac{dy}{dx} = y - x$ c) $\frac{dy}{dx} = xy^2$ d) $\frac{dy}{dx} = \cos(xy)$

   Ans: c)

2. The differential equation $y' + y = 1$ is

   a) linear b) homogeneous c) second order d) all of these

   Ans: a)

3. An integrating factor for the linear differential equation $(\cos x)\frac{dy}{dx} = x + (\sin x)y$ is

   a) $\cot x$ b) $\sec x$ c) $\tan x$ d) $\cos x$

   Ans: d): Standard form: $y' - (\tan x)y = x \to \mu = e^{-\int \tan x dx} = e^{\ln \cos x} = \cos x$

4. A tank contains 50 gallons of water with 10 lbs of salt dissolved in it. Saltwater with a concentration of 2 lb/gal is pumped into the tank at a rate of 2 gal/min. The well-stirred mixture is pumped out at the rate of 4 gal/min. A differential equation for the amount A (in pounds) of salt in the tank is

   a) $\frac{dA}{dt} = 10 - \frac{2}{25}A$ b) $\frac{dA}{dt} = 4 - \frac{2}{25}A$

   c) $\frac{dA}{dt} = 10 - \frac{4}{50-2t}A$ d) $\frac{dA}{dt} = 4 - \frac{2}{25-t}A$

   Ans: d): $\frac{dA}{dt} = rate_{in} - rate_{out} = 4 - \frac{A}{50-2t}4 = 4 - \frac{2}{25-t}A$

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, which of the following products does not exist

   a) CB b) AB c) AC d) BA

   Ans: a)

6. The linear system $\begin{aligned} x+2y+3z&=5 \\ -2x+6y-z&=10 \\ -4x+22y+3z&=40 \end{aligned}$ has solutions which satisfy

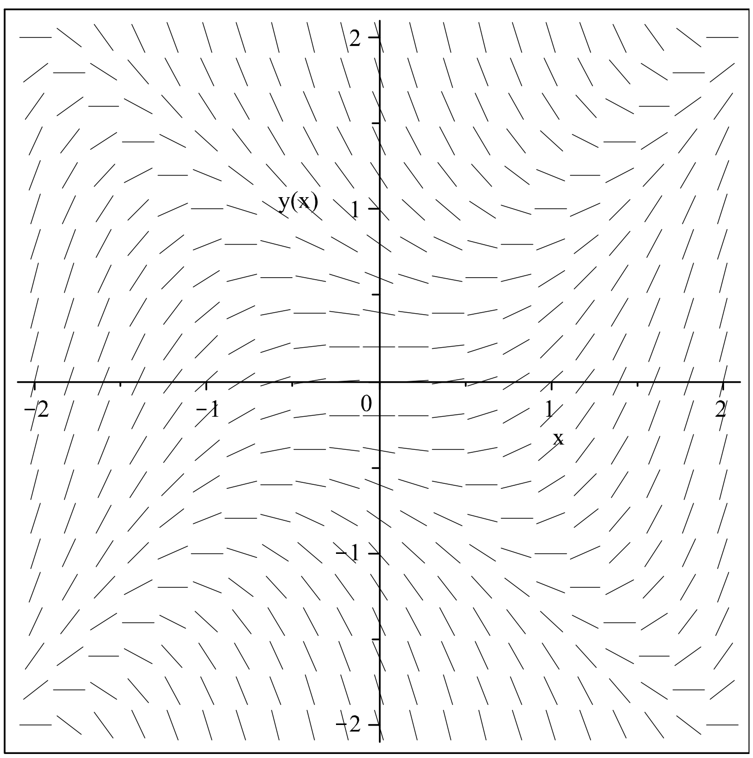
a) $\frac{1}{2}x+z=-\frac{1}{2}, y+\frac{1}{2}z=-2$ b) $x+2y=1, x+\frac{1}{2}z=2$
c) $x+2z=1, y+\frac{1}{2}z=1$ d) $x+2z=1, 2y+z=4$

Ans: d): Row reduce $\begin{bmatrix} 1 & 2 & 3 & 5 \\ -2 & 6 & -1 & 10 \\ -4 & 22 & 3 & 40 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to x+$
$2z=1, y+\frac{1}{2}z=2$

7. The matrix $A=\begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix}$ has eigenvalues $\lambda=1,5$ if $a=$

a) 0 b) 1 c) 2 d) 3

Ans: d): $\begin{vmatrix} a-1 & 2 \\ 2 & a-1 \end{vmatrix} = (a-1)^2-4=0 \to a-1=\pm 2 \to a=3,-1.$
$\begin{vmatrix} a-5 & 2 \\ 2 & a-5 \end{vmatrix} = (a-5)^2-4 \to a=5\pm 2=7,3$



8. The direction field above corresponds to the differential equation

a) $y' = x - y$ b) $y' = y - 1$ c) $y' = x^2 - y^2$ d) $y' = y^2$

Ans: c)

9. According to the direction field above the Euler method approximation to $y(1)$ with a step size of $h = 1$ and $y(0) = 1$ is

a) -2 b) -1 c) 0 d) 1

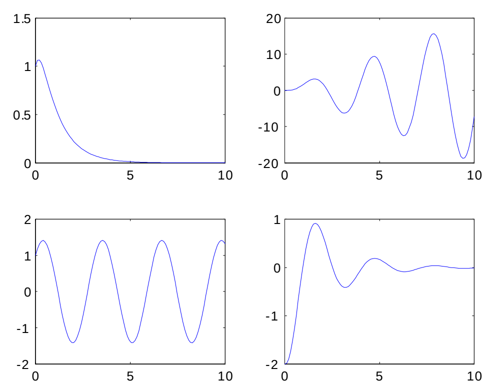
Ans: c)

10. The appropriate form for the particular solution to the differential equation $y'' - y = x^2 + e^x$ is $y_p =$

a) $Ax + Bx^2 + Ce^x$ b) $Ax^2 + Be^x$

c) $A + Bx + Cx^2 + Dxe^x$ d) $Ax + Bx^2 + Cxe^x$

Ans: c): Roots of complemetary: $m = -1, 1$. Roots of annihilator $A(D) = D^3(D-1)$: $m = 0, 0, 0, 1 \to y_{ap} = c_1 e^{-x} + c_2 e^x + c_3 x e^x + c_4 + c_5 x + c_6 x^2$



11. In the figure above the graph of a solution to the differential equation $x'' + 4x = 0$ is

a) upper-left b) upper-right c) lower-left d) lower-right

Ans: c)

12. In the figure above the graph of a solution to the differential equation $4x'' + 4x' + 17x = 0$ is

a) upper-left b) upper-right c) lower-left d) lower-right

Ans: d): $4m^2 + 4m + 17 = 0 \to m = \frac{-4 \pm \sqrt{4^2 - 4(4)17}}{8} = -\frac{1}{2} \pm 2i \to$underdamped

13. If the Laplace transform of a function $f(t)$ is $F(s) = \frac{3}{s^3+1}$, then $L\{tf(t)\} =$

a) $-\frac{3}{s^3+1}$ b) $\frac{3}{s^2(s^3+1)}$ c) $\frac{9s^2}{(s^3+1)^2}$ d) $-f'(t)$

Ans: c): $L\{tf(t)\} = (-1)\frac{d}{ds}\left(\frac{3}{s^3+1}\right) = \frac{9s^2}{(s^3+1)^2}$

14. Let $f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{-t} & t > 1 \end{cases}$. When written in terms of the unit step function $U(t)$, $f(t) =$

a) $e^{-t}U(t-1)$ b) $e^{-(t-1)}U(t-1)$

c) $e^{-t}(1-U(t-1))$ d) $e^{-(t+1)}U(t-1)$

Ans: a)

15. If $f(t) = L^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}$, then $f(\frac{3\pi}{2}) + f(\frac{5\pi}{2}) =$

a) 2 b) 1 c) 0 d) $e^{-3\pi} + e^{-5\pi}$

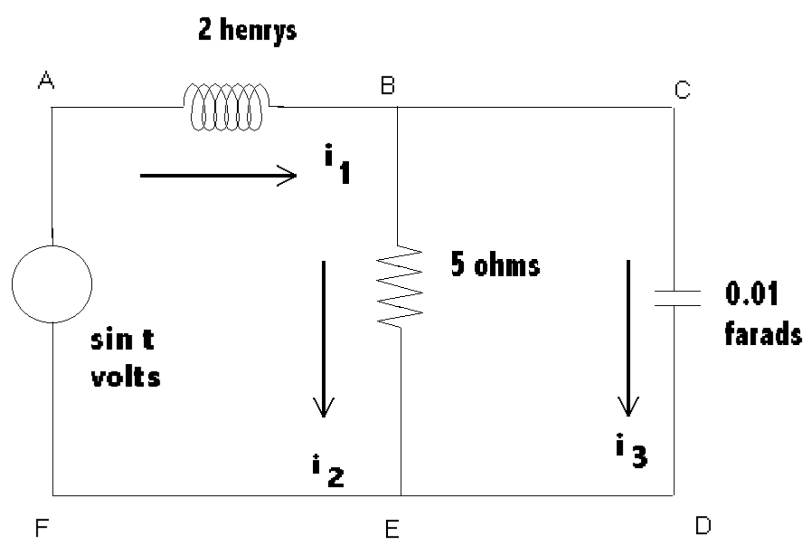
Ans: b): $f(t) = \sin(t-2\pi)U(t-2\pi) \to f(\frac{3\pi}{2}) + f(\frac{5\pi}{2}) = \sin(-\frac{\pi}{2})U(-\frac{\pi}{2}) + \sin(\frac{\pi}{2})U(\frac{\pi}{2}) = 1$

16. Let $Y(s)$ denote the Laplace transform of $y(t)$. For the initial value problem $y' - 2y = t^2, y(0) = -1$, $Y(s) =$

a) $\frac{2}{s^3(s-2)} - \frac{1}{s-2}$ b) $-\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^4}$

c) $\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^4}$ d) $-\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^4}$

Ans: a): $sY(s) + 1 - 2Y(s) = \frac{2}{s^3} \to (s-2)Y(s) = -1 + \frac{2}{s^3}$



17. For the network shown above let $q_3(t)$ denote the charge on the capacitor. A differential equation implied by Kirchoff's 2nd law (loop rule) for the the righthand loop (BCDEB) of this network and by Kirchoff's 1st law (junction rule) is

a) $2i_2' + 5i_2 = \sin t$ b) $q_3' = i_1 - 20q_3$

c) $2i_1' + 0.01q_3 = 0$ d) $5i_2' - 100q_3 = 0$

Ans: b): Loop rule:$100q_3 - 5i_2 = 0$. Junction rule: $i_1 = i_2 + i_3 = i_2 + q_3' \to 100q_3 - 5(i_1 - q_3') = 0 \to 20q_3 - i_1 + q_3' = 0$

18. For the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$ of $f(x) = 1 + x, -3 \le x \le 3$, $a_2 =$

a) 1 b) 0 c) $\frac{3}{\pi}$ d) $-\frac{3}{\pi}$

Ans: b): $a_2 = \frac{1}{3} \int_{-3}^{3} (1+x) \cos\left(\frac{2\pi}{3} x\right) dx = 0$

19. Let $FSS(x)$ denote the Fourier sine series of $f(x) = \left\{ \begin{array}{c} -5,\ 0 \le x \le 1 \\ 10x,\ 1 < x \le 2 \end{array} \right.$. Then $FSS(-1) =$

a) -10 b) 0 c) $\frac{\pi^2}{6}$ d) -2.5.

Ans: d)

20. The partial differential equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$ has solutions of the form $u(x, y) = X(x)Y(y)$. For a constant μ, $X(x)$ and $Y(y)$ satisfy

a) $X'(x) = \mu X(x), Y'(y) = (1 - \mu)Y(y)$

b) $X'(x) = \mu X(x), Y'(y) = (1 + \mu)Y(y)$

c) $X'(x) = \mu X(x), Y'(y) = \mu Y(y)$ d) $X'(x) = \mu X(x), Y'(y) = -\mu Y(y)$

Ans: a): $X'(x)Y(y) + X(x)Y'Y(y) = X(x)Y(y) \to \frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = 1 \to \frac{X'(x)}{X(x)} = 1 - \frac{Y'(y)}{Y(y)} = \mu \to X'(x) = \mu X(x), Y'(y) = (1 - \mu)Y(y)$

SM212/P Final Examination: Written Section

04 May 2009, 1930

The Written part of the exam counts 50%. On the problems in the written section you are to show the steps of the solution process in detail.

1. (a) Find the explicit general solution to $x\frac{dy}{dx} + 3y = 1$.

   Ans: DE is first order linear. Standard form: $\frac{dy}{dx} + \frac{3}{x}y = \frac{1}{x}$. Integrating factor: $\mu = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3 \to \frac{d}{dx}\left(x^3 y\right) = x^2 \to x^3 y = \frac{x^3}{3} + C \to \boxed{y = \frac{1}{3} + \frac{C}{x^3}}$

   (b) Find the explicit solution to the initial value problem $\frac{dy}{dx} = 3x^2 e^y, y(0) = -1$.

   Ans: DE is separable. $\int e^{-y}dy = \int 3x^2 dx \to -e^{-y} = x^3 + C \to e^{-y} = -x^3 + C$. IC implies $e = C \to e^{-y} = -x^3 + e \to -y = \ln\left(-x^3 + e\right) \to \boxed{y = -\ln\left(-x^3 + e\right)}$

2. (a) A freshly poured cup of coffee is placed in a room whose temperature is 75 °F. After one minute the temperature of the coffee is 180 °F. After another minute its temperature is 160 °F. What was the temperature of the coffee when it was placed in the room?

   Ans: Newton'sLaw of Cooling: $\frac{dT}{dt} = k(T - 75)$. Also $T(1) = 180, T(2) = 160$. First order linear DE: $\frac{dT}{dt} - kT = -75k \to \mu = e^{-k\int dt} = e^{-kt} \to \left(e^{-kt}T\right)' = -75ke^{-kt} \to e^{-kt}T = 75e^{-kt} + C \to T = 75 + Ce^{kt}$. $T(1) = 180, T(2) = 160 \to 180 = 75 + Ce^k, 160 = 75 + Ce^{2k} \to 105 = Ce^k, 85 = Ce^{2k}$. Thus $C = 105e^{-k} \to 85 = 105e^{-k}e^{2k} = 105e^k \to k = \ln\left(\frac{85}{105}\right) = -0.211\,31 \to C = 105e^{0.211\,31} = 129.\,71 \to T(t) = 75 + 129.\,71e^{-0.211\,31t} \to T(0) = 75 + 129.\,71 = \boxed{204.\,71\,°\mathrm{F}}$

   (b) Use Euler's method with a step size of $h = 0.1$ to approximate $y(0.2)$ for the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = t,\ y(0) = 1, y'(0) = -1$$

   Ans: Write as first order system. Let $v = y' \to y' = v, v' = -2v - 3y + t$. Euler: $y_1 = 1 + 0.1 \times (-1) = 0.9, v_1 = -1 + 0.1 \times (-2 \times (-1) - 3 \times 1 + 0) = -1.\,1 \to y_2 = 0.9 + 0.1 \times (-1.1) = \boxed{0.79}$

3. Use undetermined coefficients, or annihilators, to solve the initial value problem
$$y'' + 5y' + 4y = 6e^{-x},\ y(0) = 1,\ y'(0) = -2.$$

   Ans: Step 1: Solve associated homogeneous $y'' + 5y' + 4y = 0 \to m^2 + 5m + 4 = 0 \to (m+1)(m+4) = 0 \to m = -1, -4 \to y_c = c_1e^{-x} + c_2e^{-4x}$.

   Step 2: Annihilator for the right hand side is $A(D) = D + 1$

   Step 3: Write DE in operator form, apply annihilator: $(D+1)(D+4)y = 6e^{-x} \to (D+1)(D+1)(D+4)y = 0$

   Step 4: Solve resulting homogeneous DE: $(m+1)(m+1)(m+4) = 0 \to m = -4, -1, -1 \to y_{ap} = c_1e^{-4x} + c_2e^{-x} + c_3xe^{-x}$

   Step 5: Cross out anything in y_{ap} that is in y_c: $y_p = c_3xe^{-x}$

   Step 6: Plug y_p into original DE and find y_p: $y_p' = c_3e^{-x} - c_3xe^{-x}, y_p'' = -2c_3e^{-x} + c_3xe^{-x} \to -2c_3e^{-x} + c_3xe^{-x} + 5(c_3e^{-x} - c_3xe^{-x}) + 4c_3xe^{-x} = 6e^{-x} \to 3c_3e^{-x} = 6e^{-x} \to c_3 = 2$

   Step 7:General solution is $y = y_c + y_p = c_1e^{-x} + c_2e^{-4x} + 2xe^{-x}$

   Step 8: Apply IC's: $y' = -c_1e^{-x} - 4c_2e^{-4x} + 2e^{-x} - 2xe^{-x}$. $y(0) = 1 \to c_1 + c_2 = 1, y'(0) = -2 \to -c_1 - 4c_2 + 2 = -2$. Add equations: $-3c_2 = -3 \to c_2 = 1 \to c_1 = 0 \to \boxed{y = e^{-4x} + 2xe^{-x}}$

4. Use Laplace transforms to find the solution to the initial value problem
$$y' + y = \begin{cases} \sin t,\ 0 \le t \le 2\pi \\ 0,\ t > 2\pi \end{cases}$$
$$y(0) = 1$$

   What is $y(3\pi)$ equal to?

   Ans: Right hand side $= \sin t - \sin tU(t-2\pi)$. $L\{\sin t - \sin tU(t-2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}L\{\sin(t+2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}L\{\sin t\} = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$.
   Take LT of DE: $sY(s) - 1 + Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} \to Y(s) = \frac{1}{s+1} + \frac{1}{(s+1)(s^2+1)} - \frac{e^{-2\pi s}}{(s+1)(s^2+1)} \to y(t) = L^{-1}\left\{\frac{1}{s+1} + \frac{1}{(s+1)(s^2+1)} - \frac{e^{-2\pi s}}{(s+1)(s^2+1)}\right\}$
   $= L^{-1}\left\{\frac{1}{s+1} + \frac{1}{2(s+1)} - \frac{s-1}{2(s^2+1)} - e^{-2\pi s}\left(\frac{1}{2(s+1)} - \frac{s-1}{2(s^2+1)}\right)\right\} = \frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t - \left(\frac{1}{2}e^{-(t-2\pi)} - \frac{1}{2}\cos(t-2\pi) + \frac{1}{2}\sin(t-2\pi)\right)U(t-2\pi) =$

   $\boxed{\frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t - \left(\frac{1}{2}e^{-(t-2\pi)} - \frac{1}{2}\cos t + \frac{1}{2}\sin t\right)U(t-2\pi)}$.

   $\boxed{y(3\pi) = \frac{3}{2}e^{-3\pi} - \frac{1}{2}e^{-\pi} = -2.1486 \times 10^{-2}}$

5. Use eigenvalues and eigenvectors to solve the initial value problem

$$\frac{dx}{dt} = x + 3y$$
$$\frac{dy}{dt} = x - y$$
$$x(0) = 4,\ y(0) = 8$$

Ans: Find eigenvalues of the coefficient matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \rightarrow |A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 4 = 0 \rightarrow \lambda = \pm 2$. Find eigenvalue for each eigenvector: $\lambda = -2 \rightarrow (A + 2I)\vec{v} = \vec{0} \rightarrow \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow v_1 + v_2 = 0 \rightarrow \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector. $\lambda = 2 \rightarrow (A - 2I)\vec{v} = \vec{0} \rightarrow \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -v_1 + 3v_2 = 0 \rightarrow \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector. General solution: $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{-2t} & 3e^{2t} \\ e^{-2t} & e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. IC's: $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}}$

6. The temperature $u(x,t)$ on a thin bar satisfies the following conditions

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = u(4,t) = 0,$$
$$u(x,0) = 10,\ 0 \le x \le 4$$

(a) Interpret the boundary conditions physically. *Ans*: Both ends are held at 0°

(b) Find $u(x,t)$. Show all steps of the separation of variables process clearly. Write your answer in summation notation.
Ans: 1. Find solutions to pde of the form $u(x,t) = X(x)T(t)$: $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \rightarrow X(x)T'(t) = 2X''(x)T(t) \rightarrow \frac{T'(t)}{T(t)} = 2\frac{X''(x)}{X(x)} = \mu$, a constant.

T-equation: $T'(t) = \mu T(t) \rightarrow T(t) = Ce^{2\mu t} \rightarrow \mu < 0$ so that the bar's temperature cools down to 0 degrees. Let $\mu = -\lambda^2 \rightarrow T(t) = Ce^{-\lambda^2 t}$.

X-equation: $X''(x) + 2\lambda^2 X(x) = 0 \rightarrow X(x) = A\cos\frac{\lambda}{\sqrt{2}}x + B\sin\frac{\lambda}{\sqrt{2}}x \rightarrow u(x,t) = X(x)T(t) = \left(A\cos\frac{\lambda}{\sqrt{2}}x + B\sin\frac{\lambda}{\sqrt{2}}x\right)Ce^{-\lambda^2 t} = \left(A\cos\frac{\lambda}{\sqrt{2}}x + B\sin\frac{\lambda}{\sqrt{2}}x\right)e^{-\lambda^2 t}$.

2. Fit separated solutions to boundary conditions:
Left BC: $u(0,t) = 0 \rightarrow Ae^{-\lambda^2 t} = 0 \rightarrow A = 0$.

Update: $u(x,t) = B \sin \frac{\lambda}{\sqrt{2}} x e^{-\lambda^2 t}$

Right BC: $u(4,t) = 0 \to B \sin \sin \left(\frac{\lambda}{\sqrt{2}} 4 \right) e^{-\lambda^2 t} = 0 \to \sin \left(\frac{4\lambda}{\sqrt{2}} \right) = 0 \to \frac{4\lambda}{\sqrt{2}} = n\pi \to \lambda = \sqrt{2} \frac{n\pi}{4}$.

Update: $u(x,t) = B \sin \frac{\sqrt{2} \frac{n\pi}{4}}{\sqrt{2}} e^{-\left(\sqrt{2} n\pi/4\right)^2 t} = B \sin \frac{n\pi}{4} x e^{-\frac{(n\pi)^2}{8} t}$

3. Superposition to get the general solution: $u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{4} x e^{-\frac{(n\pi)^2}{8} t}$.

4. Fit initial condition: $u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{4} = 10 \to B_n$'s are the Fourier sine coefficients of $f(x) = 10 \to B_n = \frac{2}{4} \int_0^4 10 \sin \frac{n\pi x}{4} dx = -\frac{20}{n\pi} \cos \frac{n\pi x}{4} \Big|_0^4 = \frac{20}{n\pi} (1 - \cos n\pi) = \frac{20}{n\pi} (1 - (-1)^n) \to \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{20}{n\pi} (1 - (-1)^n) \sin \frac{n\pi}{4} x e^{-\frac{(n\pi)^2}{8} t}}$

1. (a) Use the first two non-zero terms of your answer in part b to find the approximate temperature at the middle of the bar at time $t = 0.1$.

   Ans: $u(2, 0.1) \doteq \frac{20}{\pi} \left(1 - (-1)^1\right) \sin \frac{\pi}{2} e^{-\frac{(\pi)^2}{8} 0.1} + \frac{20}{2\pi} \left(1 - (-1)^2\right) \sin \pi e^{-\frac{(2\pi)^2}{8} 0.1} + \frac{20}{3\pi} \left(1 - (-1)^3\right) \sin \frac{3\pi}{2} e^{-\frac{(3\pi)^2}{8} 0.1} = 11.255 + 0 - 1.3982 = \boxed{9.8568}$