SM212, Prof Joyner, 2009-4-17
Team Test 4

Name(s): Answer

Closed book, open notes. TI92 and USNA math tables okay. Work on your own. Show all work. No discussion of this exam except with your team mates until after 2nd period.

1. Consider the PDE

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + 3u.$$

(a) Using separation of variables, find the general solution to this PDE of the form $u(x,t) = X(x)T(t)$. Show all details of the separation of variables process.

$$X'(x)T(t) = 2X(x)T'(t) + 3X(x)T(t)$$

$$\frac{X'(x)}{X(x)} = 2\frac{T'(t)}{T(t)} + 3 = \lambda$$

$$X(x) = C_1 e^{\lambda x}, \quad T(t) = C_2 e^{(\lambda-3)t/2}$$

$$\boxed{u(x,t) = C e^{\lambda x + \frac{\lambda-3}{2}t}}$$

(b) Solve for that solution which also satisfies the condition $u(0,t) = e^{-t}$.

$$C e^{\frac{\lambda-3}{2}t} = e^{-t} \Rightarrow C = 1 \qquad \frac{\lambda-3}{2} = -1$$

$$\lambda - 3 = -2$$

$$\lambda = 1$$

$$\boxed{u(x,t) = e^{x-t}}$$

2. Let $f(x) = \begin{cases} -1, & \text{if } 0 < x < 2 \\ 2, & \text{if } 2 < x < 3. \end{cases}$

(a) Find the Fourier sine series for $f(x)$. Write the answer in summation notation and also find the first 3 non-zero terms.

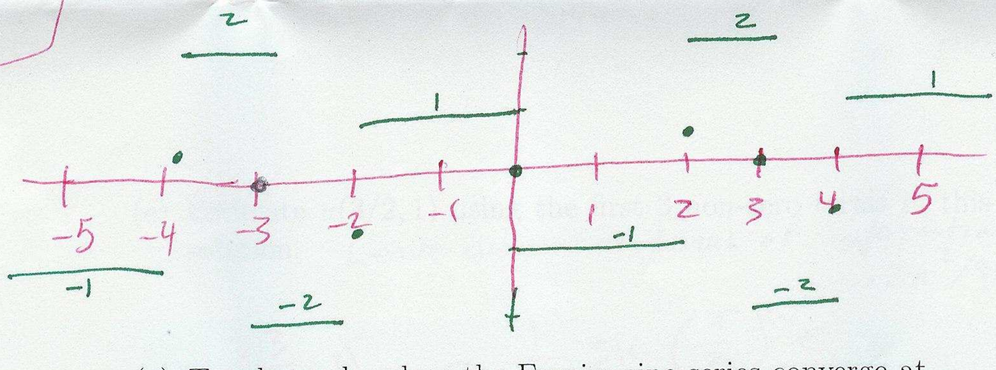
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$= -\frac{1}{\pi}\sin\left(\frac{\pi x}{3}\right) - \frac{9}{2\pi}\sin\left(\frac{2\pi x}{3}\right) + \frac{8}{3\pi}\sin(\pi x) + \cdots$$

$$\boxed{b_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{n\pi}\left(-1 - 2\cos(n\pi) + 3\cos\left(\frac{2n\pi}{3}\right)\right)}$$

$b_1 = -1/\pi$, $b_2 = -9/2\pi$, $b_3 = 8/3\pi$, $b_4 = -9/4\pi$, ...

(b) Graph the function to which its Fourier series (periodic with period 6) converges to, for $-5 < x < 5$.



(c) To what value does the Fourier sine series converge at

(i) $x = 2$? (ii) $x = -3$? (iii) $x = 2011$?

$1/2$

0

$FSS_f(2011)$
$= FSS_f(2010+1)$
$= FSS_f(1)$
$= -1$

(d) Use your answer in (a) to find the function $u(x,t)$ satisfying the heat equation

$$2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$
$$u(x,0) = f(x),$$
$$u(0,t) = u(3,t) = 0.$$

Write your answer in summation notation.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) e^{-2\left(\frac{n\pi}{3}\right)^2 t}$$

(e) Estimate $u(3/2, 1)$ using the first 3 non-zero terms of this series solution. Leave answer in terms of exponentials.

$$u\left(\frac{3}{2}, 1\right) = -\frac{1}{\pi}\sin\left(\frac{\pi}{2}\right)e^{-2\pi^2/9} - \frac{9}{2\pi}\sin(\pi)\, e^{---} + \frac{8}{3\pi}\sin\left(\frac{3\pi}{2}\right)e^{-2\pi^2}$$

$$= -\frac{1}{\pi}e^{-2\pi^2/9} \ \cancel{+} \ \frac{8}{3\pi}e^{-2\pi^2}$$

3. Consider the heat equation

$$\frac{\partial^2 u}{\partial x^2} = 3\frac{\partial u}{\partial t},$$
$$u(x,0) = -\tfrac{1}{2}\cos(\pi x),$$
$$u_x(0,t) = u_x(5,t) = 0.$$

(a) Describe the boundary conditions physically.

insulated

(b) Solve for $u(x,t)$.

(BONUS points if you show all steps of the separation of variables process clearly.)

$$u(x,t) = -\frac{1}{2}\cos(\pi x)\, e^{-\frac{1}{3}\pi^2 t}$$

(c) Find the approximate temperature at the middle of the bar at time $t = 0.1$.

$$u\left(\frac{5}{2}, \frac{1}{10}\right) = -\frac{1}{2}\cos\left(\frac{5\pi}{2}\right) e^{-\pi^2/30} = 0$$

BONUS: Use the value the cosine series of $f(x) = x^2$ at $x = 0$ to compute $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2 = 1 - 1/4 + 1/9 - 1/16 + -... =$ exactly.

$a_0 = \frac{2}{L}\int_0^L x^2 dx = 2L^2/3$, $\quad a_n = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx = -4L^2(-1)^{n+1}/(n\pi)^2$

$\therefore\ x^2 \sim \frac{a_0}{2} + \sum_1^\infty a_n\cos\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\sum_1^\infty (-1)^{n+1}\frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right) = FCS_f(x)$

$x = 0 \Rightarrow \quad 0 = \frac{L^2}{3} - \frac{4}{\pi^2}L^2\sum_1^\infty (-1)^{n+1}\frac{1}{n^2}$

$\Rightarrow \quad \boxed{\sum_1^\infty (-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{12}}$