SM212, Annihilator method

PROBLEM: Solve

$$ay'' + by' + cy = f(x). \tag{1}$$

We *assume* that $f(x)$ is of the form $c \cdot p(x) \cdot e^{ax} \cdot \cos(bx)$, or $c \cdot p(x) \cdot e^{ax} \cdot \sin(bx)$, where a, b, c are constants and $p(x)$ is a polynomial.

soln:

- Write the ODE in symbolic form $(aD^2 + bD + c)y = f(x)$.
- Find the "homogeneous solution" y_h to $ay'' + by' = cy = 0$, $y_h = c_1y_1 + c_2y_2$.
- Find the differential operator L which annihilates $f(x)$: $Lf(x) = 0$. The following table may help.

function	annihilator
x^k	D^{k+1}
$x^k e^{ax}$	$(D-a)^{k+1}$
$x^k e^{\alpha x}\cos(\beta x)$	$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^{k+1}$
$x^k e^{\alpha x}\sin(\beta x)$	$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^{k+1}$

- Find the general solution to the homogeneous ODE, $L \cdot (aD^2 + bD + c)y = 0$.
- Let y_p be the function you get by taking the solution you just found and subtracting from it any terms in y_h.
- Solve for the undetermined coefficients in y_p as in the method of undetermined coefficients.

Example

Example 1 *Solve*

$$y'' - y = \cos(2x).$$

- *The DE is* $(D^2 - 1)y = \cos(2x)$.
- *The characteristic polynomial is* $r^2 - 1 = 0$, *which has* ± 1 *for roots. The "homogeneous solution" is therefore* $y_h = c_1 e^x + c_2 e^{-x}$.
- *We find* $L = D^2 + 4$ *annihilates* $\cos(2x)$.
- *We solve* $(D^2 + 4)(D^2 - 1)y = 0$. *The roots of the characteristic polynomial* $(r^2 + 4)(r^2 - 1)$ *are* $\pm 2i, \pm 1$. *The solution is*

  $$y = A_1 \cos(2x) + A_2 \sin(2x) + A_3 e^x + A_4 e^{-x}.$$

- *This solution agrees with* y_h *in the last two terms, so we guess*

  $$y_p = A_1 \cos(2x) + A_2 \sin(2x).$$

- *Now solve for* A_1 *and* A_2 *as before: Compute both sides of* $y_p'' - y_p = \cos(2x)$,

  $$(-4A_1 \cos(2x) - 4A_2 \sin(2x)) - (A_1 \cos(2x) + A_2 \sin(2x)) = \cos(2x).$$

  Next, equate the coefficients of $\cos(2x)$, $\sin(2x)$ *on both sides to get 2 equations in 2 unknowns. Solving, we get* $A_1 = -1/5$ *and* $A_2 = 0$.
- *The general solution:* $y = y_h + y_p = c_1 e^x + c_2 e^{-x} - \frac{1}{5}\cos(2x)$.