SM212 Examination

Time: 180 minutes Fall 1988

1. (a) Show that $y = x^{-2}\ln(x)$ is a solution for $x^2y'' + 5y' + 4y = 0$.
(b) Show that $(\cos y + 2x)dx + (3\exp(3y) - x\sin y)dy = 0$ is exact and find its general solution.

2. Consider the first order differential equation $y' = 1 + x - y$; $y(0) = 1$.

(a) Draw the direction field for the above equation by first plotting the three isoclines where $y' = 0$, $y' = -1$, and $y' = 1$. Also sketch the solution such that $y(0) = 1$ which "fits" your direction field.

(b) Solve the above differential equation and initial condition for its exact solution $y(x)$ and find $y(1)$. (Use $\exp(-1) = .3678794$.)

(c) Approximate $y(1)$ by using two steps of Euler's (constant step) method on the above problem.

3. (a) Solve $y'' - 6y' + 9y = 2\exp(3x)$; $y(0) = 1$, $y'(0) = 4$.
(b) Give an example of a fourth order linear homogeneous ordinary differential equation with constant coefficients and find its general solution.

4. (a) Use Laplace transforms and partial fractions to solve $y'' + 4y' + 8y = 8$; $y(0) = 2$, $y'(0) = -2$.

(b) Sketch $f(t) = t + (\cos(t) - t)u(t - \pi)$ and find $L\{f(t)\}$.

(c) Use convolution to evaluate $L^{-1}\{F(s)G(s)\}$ where you choose $F(s)$ and $G(s)$.

5. (a) A 4 lb weight is suspended from a spring with spring constant $k = 7/8$ lb/ft and damping constant $\beta = 1/2$ lb-sec/ft. If the weight is pulled 1 ft below equilibrium and given a velocity of 5 ft/sec upward, find its displacement in the form $x(t) = \exp(\alpha t)A\sin(wt + \phi)$.

(b) What value of β would give critical damping?

(d) If $\beta = 0$ and there is an external force of $10\cos(bt)$, what value of b would give resonance?

6. Consider the second order equation $y'' - x(y')^2 - y = 0, \;\; y(0) = 1, \, y'(0) = 2.$

(a)
- Write an equivalent system of two first order differential equations and initial conditions.
- Use 2 steps of Euler's method on the first order system in (i) to approximate $y(2)$ in the above differential equation.

(b) Write a second order homogeneous Cauchy-Euler differential equation and find its general solution.

7. (a) Write the following system of differential equation in operator notation and solve for $x(t)$ only $2x' + x + y' + y = 0, \, y' + x' - 2x - y = 0.$

(b) Write a system of 2 differential equations in operator notation involving only $I_2(t)$ and $q_2(t)$ for the electric circuit network on the right. Do not solve!

8. (a) Use the Taylor series method to find the first 4 non-zero terms of a series solution to the differential equation $y'' - (y')^2 = 0; \;\; y(0) = 1, \, y'(0) = 1.$

(b) Find the radius of convergence of the Taylor series.

9. (a) Find the Fourier series for the function of period 2π given by $f(x) = x, \, -\pi < x < \pi$. Write the series in summation notation and write out its first 4 non-zero terms.

(b) What does the Fourier series converge to if $x = \pi/2$?

(c) Use your results from (a) and (b) to find the sum of $1 - 1/3 + 1/5 - 1/7 + 1/9 \ldots.$

10. (a) Set up and solve a partial differential equation and conditions satisfied by the temperature $u(x,t)$ of a thin insulated wire of length 3 and diffusivity constant $\beta = 1$, whose left and right ends are kept at 0^o, and whose initial temperature is given by $f(x) = 5\sin(2\pi x/3)$ throughout the bar. Use separation of variables and show every step clearly.

(b) Find the temperature in the middle of the bar in part (a) $30\pi/2$ seconds later. Leave your answer in terms of exponentials.