SM212, Final Examination, 9 December 2003

Name:

In accordance with the relevant USNA instruction, this exam is subject to USNA and federal regulations regarding sensitive and classified material.

There are 10 problems.

Show all work. You can use math tables and TI-92's but no text books, notes, cell phones, or PDAs.

1. The current, $i(t)$, in an LR series circuit can be found by solving the differential equation:

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0.$$

Use Laplace transforms to find the current, $i(t)$, in an LR series circuit when $i(0) = 0$, $L = 1h$, $R = 10\Omega$, and $E(t)$ is

$$E(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases}$$

2. Consider the DE $\frac{dy}{dx} - xy^2 = x$.

   (a) Find an explicit solution of this DE using the separation of variables method. Show all steps.

   (b) Determine the solution $y = \phi(x)$ to the initial value problem $\frac{dy}{dx} - xy^2 = x$, $y(0) = 1$.

   (c) Compute $\phi(1)$ to 2 decimal places of accuracy.

3. Given the system of differential equations:

$$\frac{dx}{dt} = 2x - 3y$$
$$\frac{dy}{dt} = x - 2y$$

   (a) Write in the matrix form $\vec{X}' = A\vec{X}$.

   (b) Find the eigenvectors and eigenvalues of A. Show all your work.

   (c) Use the eigenvectors and eigenvalues to find the general solution to the system.

4. Given the system of differential equations (note that in the above you solved for the homogeneous or complementary part of the solution):

$$\frac{dx}{dt} = 2x - 3y + e^{2t},$$
$$\frac{dy}{dt} = x - 2y + 1.$$

   (a) Write in the matrix form,

   (b) Use variation of parameters to find $\vec{X}_p$.

   (c) What is the general solution to the system?

5. A 2 lb "weight" is attached to a spring and stretches the spring 1/5 in. An additional 2 lb weight is attached to the spring and after this the spring is released from rest at a point 2 ft above its equilibrium position. The damping force is equal to the instantaneous velocity in value. No external driving force is applied to this spring-weight system.

   (a) Write the differential equation that models the spring motion in terms of the displacement $x(t)$ and time t.

   (b) Solve this DE.

   (c) State whether the spring is overdamped, underdamped, or critically damped (circle one).

6. (Do *not* use Laplace transforms to solve this problem.)

   (a) Find the general solution to $y'' + 4y' + 4y = 0$.

   (b) Solve the IVP: $y'' - 5y' + 6y = -e^{2x} + 6x - 5$.

7. Compute the following:

   (a) $\mathcal{L}[f(t)]$, where

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1, \\ e^t, & \text{if } 1 < t. \end{cases}$$

   (b) $\mathcal{L}[\delta(t-3) + te^{-t}\sin(t)]$.

   (c) $1 * t * t^2$.

8. Let $f(x) = x(\pi - x)$, $-\pi < x < \pi$.

   Find the cosine series of $f(x)$, $0 < x < \pi$, having period 2π, *and* graph the function to which the series converges to, $-2\pi < x < 2\pi$.

9. Solve the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{10}\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < \pi$$

   with insulated end points and $u(x,0) = x(\pi - x)$, $0 < x < \pi$. Show all steps of the separation of variables process. (Hint: Use the results of # 8 above.)

10. This is a two part problem.

   (a) Use three steps of Euler's method to approximate $y(1)$, where

$$y'' - 3y' + 2y = 1, \quad y(0) = 1, \ y'(0) = 0.$$

   (b) Use three steps of Euler's method to approximate $y(1)$, where

$$y' - xy = 1, \quad y(0) = 1.$$

Solutions:

1. Take Laplace transforms of both sides of the DE. The Laplace transform of $i(t)$ satisfies $I(s) = \frac{i(0)+LE(s)}{s+10}$, where $LE(s)$ is the Laplace transform of $E(t) = u(t) = u(t-\pi)$. The Math Tables give $I(t) = \frac{1}{10}(1-e^{-10t}) + \frac{1}{10}(1-e^{-10(t-\pi)})u(t-\pi)$.

2. Separation of variables gives: $\arctan(y) = \frac{1}{2}x + C$. Using the IC gives $C = \pi/4$, so $y = \phi(x) = \tan(\frac{1}{2}x + \frac{\pi}{4})$. Therefore, $\phi(1) = \tan(1/2 + \pi/4) = 3.408223450....$

3. Let $A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$, so the system is $\vec{X}' = A\vec{X}$. Eigenvalues and eigenvectors: $\lambda_1 = -1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\lambda_2 = 1$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. The solution is $x(t) = C_1 e^{-t} + C_2 e^{t}$, $y(t) = C_1 e^{-t} + \frac{C_2}{3} e^{t}$.

4. Let A be as above and $\vec{F} = \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$, so the system is $\vec{X}' = A\vec{X} + \vec{F}$. Fundamental matrix is $\vec{v}_1 = \begin{pmatrix} 3\exp(t) & \exp(-t) \\ \exp(t) & \exp(-t) \end{pmatrix}$. The solution is $x(t) = C_1 e^{-t} + C_2 e^{t} + \frac{4e^{2t}}{3} + 3$, $y(t) = C_1 e^{-t} + \frac{C_2}{3} e^{t} + \frac{e^{2t}}{3} + 2$.

5. The DE is $\frac{1}{8}\frac{d^2}{dt^2}x(t) + \frac{d}{dt}x(t) + 10\,x(t) = 0$. The solution is $x(t) = -e^{-4t}\sin(8t) - 2e^{-4t}\cos(8t)$.

6. (a) $y(t) = C_1 e^{-2t} + C_2 e^{-2t}$.

   (b) $y(t) = C_2 e^{2t} + C_1 e^{3t} + te^{2t} + t + e^{2t}$.

7. (a) $s^{-1} + \frac{e^{-s+1}}{s-1} - \frac{e^{-s}}{s}$

   (b) $e^{-3s} + 2\frac{s+1}{\left((s+1)^2+1\right)^2}$

   (c) $\frac{1}{60}t^5$

8. The cosine series coefficients are $a_n = -2\frac{n\cos(\pi n)\pi + \pi n}{n^3\pi}$, $n > 0$, and $a_0 = \frac{\pi^2}{3}$. The drawing is:

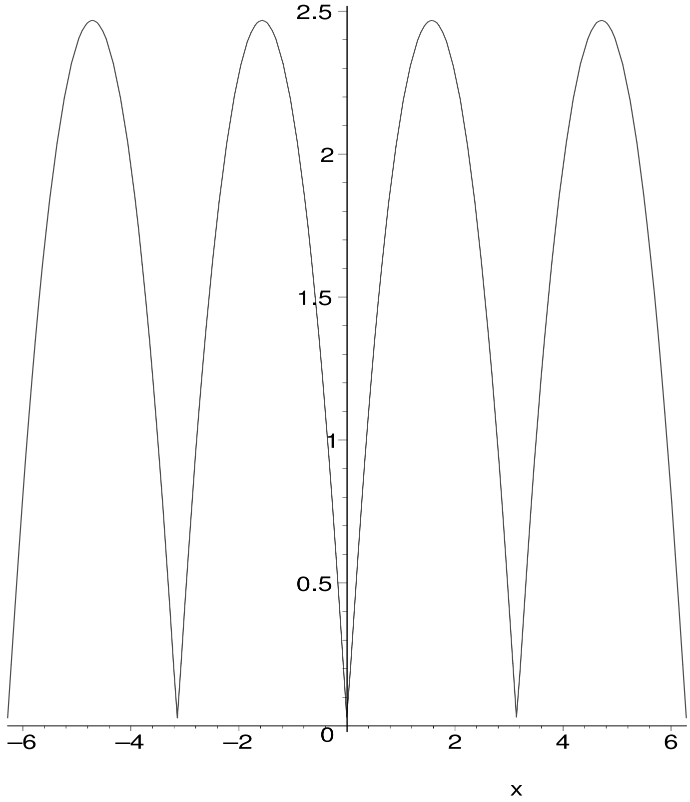

Figure 1: cosine series for $f(x) = x(\pi - x)$.

9. $\frac{X''(x)}{10X(x)} = \frac{T'(t)}{T(t)} = -\lambda^2$. $X(x) = c_1\cos(\lambda x)+c_2\sin(\lambda x)$, $T(t) = c_3 exp(-\lambda^2 t/10)$. BC's imply $u(x,t) = A\cos(nx)\exp(-n^2t/10)$. IC and superposition imply $u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} a_n\cos(nx)\exp(-n^2t/10)$, where a_n is as above.

10. (a)

x	y	$hf(x,y))$
0	1	1/3
1/3	4/3	$\frac{10}{27}$
2/3	$\frac{49}{27}$	$\frac{11}{27}$
1	$\frac{620}{243}$	4/9

(b)

x	y_1	$hf_1(x, y_1, y_2)$	y_2	$hf_2(x, y_1, y_2)$
0	1	0	0	$-1/3$
$1/3$	1	$-1/9$	$-1/3$	$-2/3$
$2/3$	$\frac{8}{9}$	$-1/3$	-1	$-\frac{34}{27}$
1	$5/9$	$-\frac{61}{81}$	$-\frac{61}{27}$	$-\frac{62}{27}$