SM212 Final Examination: Written Section

Practice Examination

In the following problems show your work.

1. Solve each of the initial value problems.

   (a) $\dfrac{dy}{dx} = (x+y+1)^2, \quad y(0) = 0$ *Ans*: $y = \tan(x + \frac{\pi}{4}) - x - 1$

   (b) $(\cos x)\frac{dy}{dx} + (\sin x)\, y = \cos^2 x, \quad y(0) = -1$ *Ans*: $y = (x-1)\cos x$

2. Torricelli's law for rate of change of the volume $V(t)$ of water in a tank with a hole in its bottom is $\frac{dV}{dt} = -a\sqrt{2gy}$, where a is the radius of the hole, $y(t)$ is the depth of the water at time t, and g is the acceleration due to gravity at the earth's surface. A cylindrical tank initially contains water to a depth of 9 ft. A bottom plug is removed at time $t = 0$ (hours). After one hour the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank? *Ans*: 3 hours

3. (a) Use undetermined coefficients to solve the initial value problem

   $$x'' + x = 4\cos t,\ x(0) = 0,\ x'(0) = 1.$$

   Ans: $x = \sin t + 2t\sin t$

   (b) The initial value problem in part (a) describes an undamped mass-spring system at resonance. What feature of the solution is characteristic of an undamped system at resonance? *Ans*: The presence of the $2t\sin t$ term causes the oscillations to grow without bound

4. (a) Find $\pounds^{-1}\left\{\frac{s}{s^2+2s+2}\right\}$. *Ans*: $e^{-t}(\cos t - \sin t)$

   (b) Use Laplace transforms to solve the initial value problem

$$x'' - x = e^t,\ x(0) = 1,\ x'(0) = -2$$

   Ans: $x = \frac{7}{4}e^{-t} - \frac{3}{4}e^t + \frac{1}{2}te^t$

5. Use eigenvalues/eigenvectors to find the general solution to the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= 3x + 2y \\ \frac{dy}{dt} &= -2x + 3y \\ x(0) &= 2, y(0) = 0 \end{aligned}$$

   Ans: $x = 2e^{3t}\cos 2t, y = -2e^{3t}\sin 2t$

6. A thin bar of length 4 and thermal diffusivity 1, which is initially has a temperature distribution given as a function of position by $5x$, has its ends insulated for times $t > 0$. This situation is described mathematically by

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \tfrac{\partial^2 u}{\partial x^2} \\ &\tfrac{\partial u}{\partial x}(0,t) = 0,\ \tfrac{\partial u}{\partial x}(4,t) = 0, t > 0 \\ &u(x,0) = 5x, 0 \le x \le 4 \end{aligned}$$

   (a) Find the temperature $u(x,t)$. Show all steps of the separation of variable process clearly. *Ans*: $u(x,t) = 10 + \sum_{n=1}^{\infty} \frac{40}{n^2\pi^2}\left((-1)^n - 1\right) e^{-\left(\frac{n\pi}{4}\right)^2 t} \cos\frac{n\pi x}{4}$

   (b) What is the final temperature distribution on the bar as $t \to \infty$? *Ans*: Uniformly $10°$.