SM212 Final Spring 2003-2004 Solutions

Written portion. Each problem is worth 20 points (Chairman says everyone *must* use same point grading scheme)

1. Taking LTs of each DE gives $(s-3)X(s)+6Y(s)=1$, $X(s)=(s-3)Y(s)$, so $((s-3)^2+6)Y(s)=1$, ie, $Y(s)=1/((s-3)^2+6)$. Taking the inverse LT gives $\underline{y(t)=\frac{1}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)}$. Since $x(t)=y'(t)-3y(t)$, we get $\underline{x(t)=e^{3t}\cos(\sqrt{6}t)}$.

2. (a) $u(x,t)=X(x)T(t)$ substituting into the PDE and separating gives $x\frac{X'(x)}{X(x)}=\frac{T'(t)}{T(t)}+1=c$, for some constant c. Therefore $\ln T(t)=(c-1)t+c'$, $\ln(x)=c\ln x+c''$, so $T(t)=Ae^{(c-1)t}$, $X(x)=Bx^c$, so $\underline{u(x,t)=Cx^ce^{(c-1)t}}$.

(b) $x^2=u(x,0)=Cx^c$, so $C=1$, $c=2$, so $\underline{u(x,t)=x^2e^t}$.

3. (a) $k=F/s=8/4=2$, $b=\sqrt{2}$, $m=8/32=1/4$, so $\underline{\frac{1}{4}x''+\sqrt{2}x'+2x=0,\ x(0)=0,\ x'(0)=5}$.

(b) roots of $\frac{1}{4}D^2+\sqrt{2}D+2=0$ are $-2\sqrt{2},-2\sqrt{2}$ (real, repeated), so $x(t)=c_1e^{-2\sqrt{2}t}+c_2te^{-2\sqrt{2}t}$. The ICs imply $\underline{x(t)=5te^{-2\sqrt{2}t}}$.

(c) <u>critically damped</u>

(d) $0=x'(t)=5e^{-2\sqrt{2}t}(1-2\sqrt{2}t)$, so $\underline{t=\frac{1}{2\sqrt{2}}}$.

4. $y_1=\cos(4x)$, $y2=\sin(4x)$,

$$u_1'=\frac{\begin{vmatrix}0 & \sin 4x\\ 2/\cos 4x & 4\cos 4x\end{vmatrix}}{\begin{vmatrix}\cos 4x & \sin 4x\\ -4\sin 4x & 4\cos 4x\end{vmatrix}}=\frac{-2\tan 4x}{4}=-\frac{1}{2}\tan 4x,$$

so $u_1=\frac{1}{8}\ln\cos(4x)$, and

$$u_2'=\frac{\begin{vmatrix}\cos 4x & 0\\ -4\sin 4x & 2/\cos 4x\end{vmatrix}}{\begin{vmatrix}\cos 4x & \sin 4x\\ -4\sin 4x & 4\cos 4x\end{vmatrix}}=\frac{2}{4}=\frac{1}{2},$$

so $u_2=\frac{x}{2}$. $y=y_1u_1+y_2u_2+c_1y_1+c_2y_2=\frac{1}{8}\ln\cos(4x)\cdot\cos(4x)+\frac{x}{2}\sin(4x)+c_1\cos(4x)+c_2\sin 4x$. The ICs imply $c_1=c_2=0$, so $\underline{y=\frac{1}{8}\ln\cos(4x)\cdot\cos(4x)+\frac{x}{2}\sin(4x)}$.

5.

x	y	$hf(x,y)=\frac{1}{2}(y^2-x^2)$
0	1	1/2
1/2	3/2	1
1	5/2	$--$

so $\underline{y(1)\cong 2.5}$.

6. (a) $\underline{f(t)=u(t)-u(t-3)}$, (b) $\underline{F(s)=\frac{1}{s}-\frac{e^{-3s}}{s}}$.

(c) Taking LTs of the DE and solving for $Q(s)=\mathcal{L}[q(t)]$ gives

$$Q(s) = \frac{1}{s((s+2)^2+2^2}.$$

The PF decomp of this is $\frac{1}{8}\frac{1}{s} - \frac{1}{8}\frac{s+2}{(s+2)^2+2^2} - \frac{1}{8}\frac{2}{(s+2)^2+2^2}$, so

$$\underline{q(t) = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos(2t) - \frac{1}{8}e^{-2t}\sin(2t) + (\frac{1}{8} - \frac{1}{8}e^{-2(t-3)}\cos(2(t-3)) - \frac{1}{8}e^{-2(t-3)}\sin(2(t-3)))u(t-3).}$$

7. (a) $b_n = \frac{2}{\pi}\int_0^{\pi/2}\sin(nx)\,dx = \frac{2}{n\pi}(1-\cos(\frac{n\pi}{2})$, so

$$\underline{f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos(\frac{n\pi}{2}) \cdot \sin(nx)}$$

(b) (i) 0, (ii) 1/2, (iii) -1.

(c)

$$\underline{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos(\frac{n\pi}{2}) \cdot \sin(nx) \cdot e^{-2n^2t}}$$

Multiple choice portion Each problem is worth 4 points

1. (a), 2. (e), 3. (e), 4. (d), 5. (c), 6. (e), 7. (e), 8. (b), 9. (d)
10. (a), 11. (c), 12. (b), 13. (d), 14. (c), 15. (d),