SM212 Practice Test 3, Prof Joyner

1. Find the Laplace transform of $f(t)$, where

   (a) $f(t) = t^2 * \sin(t)$ (the convolution of t^2 and $\sin(t)$),

   (b) $f(t) = \sin(t)u(t - 2\pi)$.

2. Let

$$f(t) = \begin{cases} 1, & 0 < t < 1, \\ 0, & t \leq 0 \text{ or } t \geq 1. \end{cases}$$

   Solve $x'' + 4x = f(t)$, $x(0) = x'(0) = 0$.

3. Compute $\mathcal{L}^{-1}[\frac{s}{(s-1)(s-2)(s-3)}](t)$,

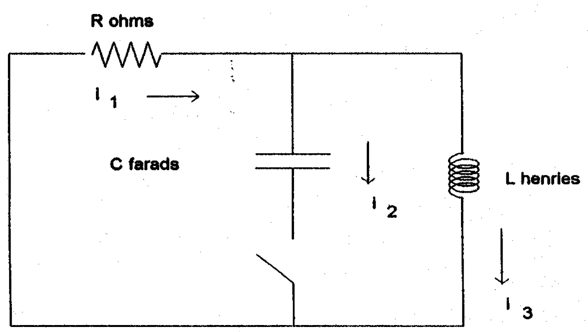
4. Compute

$$\mathcal{L}^{-1}[\frac{1}{s^2+2s+5} \cdot e^{-2\pi s}](t).$$

5. All 10 X-men (comic book characters) are fighting all 14 Y-men, the battle being modeled by

$$\begin{cases} x' = -4y, & x(0) = 10, \\ y' = -4x + 1, & y(0) = 14. \end{cases}$$

(Here $x(t)$ denotes the number of X-men still alive at time t and similarly for $y(t)$.) Solve this system using Laplace transforms. Who wins? Find out when they win to 2 decimal places.



6. (a) For the circuit in the circuit above show that the charge q on the capacitor and the current i_3 in the right branch satisfy the system of differential equations

$$\begin{aligned} q' + (1/RC)q + i_3 &= 0, \\ i_3' - (1/LC)q &= 0. \end{aligned}$$

(b) When the switch in the circuit is closed at time $t = 0$, the current i_3 is 0 amps and the charge on the capacitor is 5 coulombs. With $R = 2$, $L = 3$, $C = 1/6$ use Laplace transforms to find the charge $q(t)$ on the capacitor.

BONUS: (a) Solve

$$x + y + z = 1,$$
$$x - y + z = 1,$$
$$x + y - z = 1,$$

using Gauss elimination (elementary row operations on the augmented matrix). Show all details and label every step.

(b) Write the DE $y'' + y' - y = \sin(x)$ as a system of two first order equations in the new dependent variables $y_1 = y$ and $y_2 = y'$.

Answers

1 (a) $\frac{2}{s^3(s^2+1)}$, (b) $e^{-2\pi s}\mathcal{L}[\sin(t + 2\pi)](s) = e^{-2\pi s}\mathcal{L}[\sin(t)](s) = \frac{1}{s^2+1}e^{-2\pi s}$.

2. $(s^2 + 4)X(s) = \mathcal{L}[u(t) - u(t - 1)](s) = 1/s - e^{-s}/s$, so $X(s) = \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)}$, so $x(t) = \frac{1}{4}(1 - cos(2t) - (1 - \cos(2t - 2))u(t - 1)$.

3. partial fractions: $\frac{s}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$, so $s = A(s - 2)(s - 3) + B(s - 1)(s - 3) + C(s - 1)(s - 2)$. Plug in $s = 1$: $1 = 2A$, so $A = 1/2$. Plug in $s = 2$: $2 = -B$, so $B = -2$. Plug in $s = 3$: $3 = 2C$, so $C = 3/2$. Tables give: $\mathcal{L}^{-1}[\frac{s}{(s-1)(s-2)(s-3)}](t) = (1/2)e^t - 2e^{2t} + (3/2)e^{3t}$.

4. $f(t) = \mathcal{L}^{-1}[\frac{1}{s^2+2s+5}](t) = (1/2)\mathcal{L}^{-1}[\frac{2}{(s+1)^2+2^2}](t)$, which is $= (1/2)e^{-t}\sin(2t)$. By the translation theorem, $\mathcal{L}^{-1}[\frac{1}{s^2+2s+5} \cdot e^{-2\pi s}](t) = f(t - 2\pi)u(t - 2\pi)$.

5. $x(t) = \frac{95}{8}\,e^{-4\,t} - \frac{17}{8}\,e^{4\,t} + 1/4$, $y(t) = \frac{95}{8}\,e^{-4\,t} + \frac{17}{8}\,e^{4\,t}$. $x(t) = 0$ when $t = -(1/4)\ln(-1/95 + (4/95)101^{1/2}) = .2213032092$.

6.a. $Li_3' - (1/C)q = 0$, $i_1R + (1/C)q = 0$

6.b. Substituting one DE into the other, gives $Lq'' + Rq' + (1/C)q = E(t)$, so $q'' + 3q' + 2q = 0$. Taking LTs gives $s^2Q(s) - sq(0) - q'(0) + 3Q(s) - q(0) + 2Q(s) = 0$, $Q(s)(s^2 + 3s + 2) = 5s + 15$ (using the DE and ICs to get $q'(0)$), so $Q(s) = (5s + 15)/(s^2 + 3s + 2)$, $q(t) = \mathcal{L}^{-1}[(5s + 15)/(s^2 + 3s + 2)](t)$ $= 10e^{-2t} - 5e^{-t}$. Likewise, $i_3(t) = -10e^{-2t} + 10e^{-t}$.

Bonus: (a): You can check your answer by typing
```
rref([[1,1,1,1][1,-1,1,1][1,1,-1,1]])
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into your TI92.

(b): $y_1' = y_2$, $y_2' = -y_2 + y_1 + \sin(x)$.