SM212, Solutions
Final, Fall 1991
written by Mr Holcomb

1a) $y' - 4y = e^{4x}$: $y = \frac{\int e^{\int 4\,dx} e^{4x}\,dx + C}{e^{\int 4\,dx}} = \frac{\int e^{8x}\,dx + C}{e^{4x}} = \frac{e^{4x}}{8} + Ce^{-4x}$
1b) $x^2 y' = e^y$: $\frac{dx}{x^2} = \frac{dy}{e^y}$, so $-1/x = -e^{-y} + C$, so $x = 1/(C - e^{-y})$.
2a) $y' = y(y-2)$.
Sketch curves through $(0,-1)$, $(0,1/2)$, $(0,3/2)$, $(0,3)$ in the graph below.

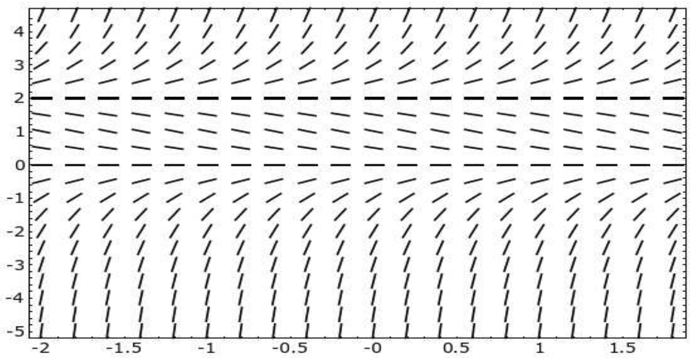

Figure 1: Direction field of $y' = y * (y-2)$

Equiv solns: $y(x) = 0$ (stable) and $y(x) = 2$ (unstable).
2b)

x	y	$hf(x,y) = (x - y^2)/10$
0	1	-1/10
1/10	9/10	-71/1000
1/5	829/1000	-487241/10000000

3ai) $D^2 - 4D - 5 = 0$. Roots: $r_1 = -1$, $r_2 = 5$, so $y = c_1 e^{-x} + c_2 e^{5x}$.

3aii) $D^2 + 4D + 13 = 0$. Roots: $r = -2 \pm i3$, so $y = c_1 e^{-2x}\cos(3x) + c_2 e^{-2x}\sin(3x)$.

3aiii) $D^2 - 6D - 9 = 0$. Roots: $r_1 = r_2 = 3$, so $y = c_1 e^{3x} + c_2 x e^{3x}$.

3bi) equilibrium position (with mass attached), $y = 0$

3bii) critically damped - $b^2 = 4mk$, over-damped - $b^2 > 4mk$, under-damped - $b^2 < 4mk$.

3biii) A - critically damped, B - over-damped, C - under-damped

4) $D^2 - D = 0$ has roots $r_1 = 0$, $r_2 = 1$, so $y_h = c_1 + c_2 e^x$. Since $f(x) = 2 + 6x$, we guess $y_p = A + Bx$. Are they any terms in y_p which agree

with those of y_h? Yes, so we multiply by x: $y_p = Ax + Bx^2$. Plug into DE $y'' - y' = 2 + 6x$:

$$\begin{aligned} y'' &= 2B \\ -y' &= -A - 2Bx \\ 2 + 6x &= 2B - A - 2Bx \end{aligned}$$

so $2B - A = 2$, $-2B = 6$, so $A = -8$, $B = -3$. Plugging this in: $y = y_h + y_p = c_1 + c_2e^x - 8 - 3x$. ICs: $y(0) = 10 \implies c_1 + c_2 - 8 = 10$. $y'(0) = 0 \implies c_2 - 3 = 0$. This implies $c_2 = 3$ and $c_1 = 15$, so $y = 15 + 3e^x - 8 - 3x$.

5ai) $\frac{12}{s^4} + \frac{6}{s^2+4} + \frac{4}{s+5}$

5aii) $\frac{s+2}{(s+2)^2+9}$

5aiii) $u(t) + (t-1)u(t-3)$

5b) $y'' + y = e^{-t}$, $y(0) = -1$, $y'(0) = 2$

$(s^2+1)Y(s) + s - 2 = \frac{1}{s+1}$, so $Y(s) = \frac{2-s}{s^2+1} + \frac{1}{(s+1)(s^2+1)} = \frac{1}{2(s+1)} - \frac{s-1}{2(s^2+1)}$, so $y(t) = 5\sin(t)/2 - 3\cos(t)/2 + e^{-t}/2$.

6a) $\mathcal{L}^{-1}[1/s^2] * \mathcal{L}^{-1}[1/(s+2)] = t * e^{-2t} = \int_0^t z^2 e^{-2(t-z)}\,dz = e^{2-2t}/4 - e^{-2t}/4$.

6b) Off the syllabus.

7ai) $D^2 + D + 1 = 0$: roots, $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, so $y_h = c_1e^{-t/2}\cos(\sqrt{3}t/2) + c_2e^{-t/2}\sin(\sqrt{3}t/2)$, $y_p = A\cos(t) + B\sin(t)$. Plug into DE $y'' + y' + y = \sin(t)$:

$$\begin{aligned} y'' &= -A\cos(t) - B\sin(t) \\ +y' &= B\cos(t) - A\sin(t) \\ y &= A\cos(t) + B\sin(t) \\ \sin(t) &= B\cos(t) - A\sin(t) \end{aligned}$$

so $-A = 1$, $B = 0$, so $A = -1$, $B = 0$. Plugging this in: $y = y_h + y_p = c_1e^{-t/2}\cos(\sqrt{3}t/2) + c_2e^{-t/2}\sin(\sqrt{3}t/2) - \cos(t)$. Steady state: $y_p = -\cos(t)$.

7aii) $4\cos(t) + 3\sin(t) = A\sin(t+\phi) \implies A = \sqrt{c_1^2 + c_2^2} = \sqrt{16+9} = 5$, $\phi = 2\tan^{-1}(\frac{c_1}{c_2+A}) = 2\tan^{-1}(\frac{-4}{8}) = 2\tan^{-1}(-1/2) = -0.927295218001612$ (about -53^o, though you should use radians).

7b) Referring to the network diagram, Kirchoff's current law implies $i_1 = i_2 + i_3$, Kirchoff's voltage law implies $\frac{1}{C}q_2 + Rq_1' = E(t)$ (or $\frac{1}{C}q_2 + R(q_2' + i_3) = E(t)$), $Li_3' = \frac{1}{C}q_2$.

8) Take LTs: $(s+4)X(s) + 2Y(s) = -3$, $-3X(s) + (s-1)Y(s) = 4$, augmented matrix: $A = \begin{pmatrix} s+4 & 2 & -3 \\ -3 & s-1 & 4 \end{pmatrix}$

$$rref(A) = \begin{pmatrix} 1 & 0 & \frac{-2\left(4-\frac{9}{s+4}\right)}{(s+4)\left(\frac{6}{s+4}+s-1\right)} - \frac{3}{s+4} \\ 0 & 1 & \frac{4-\frac{9}{s+4}}{\frac{6}{s+4}+s-1} \end{pmatrix}$$

so $X(s) = -2(4-9/(s+4))/((s+4)(6/(s+4)+s-1)) - 3/(s+4)$, so $x(t) = -2e^{-t} - e^{-2t}$, and $Y(s) = (4-9/(s+4))/(6/(s+4)+s-1)$, so $y(t) = 3e^{-t} + e^{-2t}$.

9a) Not on syllabus

9b) Not on syllabus

9ci) $L = \pi$, $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]$, $a_0 = 2$, $a_n = 0$ $(n > 0)$, $b_n = 2(1-\cos(n\pi))/(n\pi)$,

9cii) $1, 2, 1$

10a) insulated ends, initial temp at pt x is x

10b) Cosine series of $f(x) = x$ is:

$$x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L),$$

$a_0 = \pi$, $a_n = 2(1-\cos(n\pi))/(n^2\pi)$ $(n > 0)$,

Soln ($k = 1$, $L = \pi$):

$$u(x,t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-kn^2 t},$$

10c) $\pi/2$