SM212P Test 4, 4-28-2006

Prof Joyner Name: ANSWERS

USNA math tables, TI92s okay. Closed books, closed notes. No discussion of this exam until after 6th period. Work individually. Show all work.

1. Solve

$$\begin{cases} x' = -4y, & x(0) = 100, \\ y' = -x, & y(0) = 25, \end{cases}$$

using the eigenvalue method. (No points for any other method.) Show all your work.

Soln: The char poly is $p(x) = \det(\begin{matrix} 0 & -4 \\ -1 & 0 \end{matrix}) = x^2 - 4$, so the eigenvalues ar $\lambda_1 = 2$, $\lambda_2 = -2$. The eigenvectors are

$$v_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$

The soln is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} \exp(2t) + c_2 \begin{pmatrix} -4 \\ -2 \end{pmatrix} \exp(-2t).$$

Using $x(0) = 100$, $y(0) = 25$ gives

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 25e^{2t} + 75e^{-2t} \\ -\frac{25}{2}e^{2t} + \frac{75}{2}e^{-2t} \end{pmatrix}$$

2. (a) Find A^{-1} by hand for the matrix

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(b) Use A^{-1} from part (a) to solve the system of equations

$$\begin{aligned} y - z &= 1, \\ x + z &= 2, \\ -x + y &= 3. \end{aligned}$$

(No points for other methods.)

soln:

$$B = \begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Soln: The row ops are: $R_2 + R_3 \to R_3$, $R_1 + R_3 \to R_1$, $-\frac{1}{2}R_1 + R_3 \to R_3$, $R_2 - R_3 \to R_3$, $-\frac{1}{2}R_1 \to R_1$. Now permute the rows:

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{pmatrix}$$

so

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}$$

so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

3. Let

$$f(x) = \begin{cases} 0, & 0 < x \le \pi/2, \\ 1, & \pi/2 < x < \pi. \end{cases}$$

(a) Find the sine series of $f(x)$ having period 2π.

(b) Find its value at $x = 0,\ x = \pi/2,\ x = 10\pi,\ x = 2009$.

Soln:

$$b_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)\ dx = -\frac{2}{n\pi}\cos(nx)|_{\pi/2}^{\pi},$$

so $f(x) \sim \sum_{n=1}^{\infty} -\frac{2}{n\pi}(\cos(n\pi) - \cos(n\pi/2))$.

$SS(f)(0) = 0,\ SS(f)(\pi/2) = 1/2,\ SS(f)(10\pi) = 0,\ SS(f)(2009) = SS(f)(2009 - 640\pi) = SS(f)(-1.619...) = SS(f)(-.515..\pi) = -1.$

4. (a) Find the Fourier series of

$$f(x) = \begin{cases} 1, & -3 \le x \le 0, \\ 2, & 0 < x \le 3/2 \\ 3, & 3/2 < x < 3, \end{cases}$$

extended to the real line with period 6, $f(x) = f(x+6)$,

(b) Find the value of the function to which the Fourier series in (a) converges to at $x = -6, 0, 1, 9$.

(c) Find the sum of the first 4 non-zero terms in the Fourier series in (a).

Soln: $a_n = \frac{1}{3}\int_{-3}^{0} \cos(n\pi x/3)\ dx + \frac{1}{3}\int_{0}^{3/2} 2\cos(n\pi x/3)\ dx + \frac{1}{3}\int_{3/2}^{3} 3\cos(n\pi x/3)\ dx = -\frac{1}{n\pi}\sin(n\pi/2)$ and $b_n = \frac{1}{3}\int_{-3}^{0}\sin(n\pi x/3)\ dx + \frac{1}{3}\int_{0}^{3/2} 2\sin(n\pi x/3)\ dx + \frac{1}{3}\int_{3/2}^{3} 3\sin(n\pi x/3)\ dx = -\frac{1}{n\pi}\cos(n\pi x/3)|_{-3}^{0} - \frac{2}{n\pi}\cos(n\pi x/3)|_{0}^{3/2} - \frac{3}{n\pi}\cos(n\pi x/3)|_{3/2}^{3} = \frac{1}{n\pi}(1 - \cos(n\pi)) - \frac{2}{n\pi}(\cos(n\pi/2) - 1) - \frac{3}{n\pi}(\cos(n\pi) - \cos(n\pi/2)) = (\cos(n\pi/2) - 2\cos(n\pi) + 1)/(n\pi)$.

$$\begin{aligned} f(x) \sim\ & 7/4 + (-1/\pi)\cos(\pi x/3) + (3/\pi)\sin(\pi x/3) + 0\cos(2\pi x/3) + \\ & (-1/\pi)\sin(2\pi x/3) + (1/3\pi)\cos(3\pi x/3) + (1/\pi)\sin(3\pi x/3) + \\ & 0\cos(4\pi x/3) + 0\sin(4\pi x/3) + ... \end{aligned}$$

$FS(f)(-6) = FS(f)(0) = 3/2,\ FS(f)(0) = 3/2,\ FS(f)(1) = 2,\ FS(f)(9) = FS(f)(3) = 2.$

5. (a) Using separation of variables, solve

$$\begin{cases} \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \\ u(x,0) = 3\cos(\pi x), \quad 0 < x < 3, \\ u_x(0,t) = u_x(3,t) = 0. \end{cases}$$

Show all steps of the separation of variables process.

(b) Find the temperature of the wire at the point $x = \pi/4$ of the wire after 1 second (either an exact answer or an approximation to 2 decimal places is fine).

Soln: $u(x,t) = X(x)T(t)$, so $X''(x)/X(x) = \pi^2 T'(t)/T(t) = \lambda$, so $T(t) = c_1 e^{\lambda t/\pi^2}$. By physics, $\lambda = -r^2 \leq 0$, so $X(x) = a\cos(rx) + b\sin(rx)$, so $u(x,t) = X(x)T(t) = (a\cos(rx) + b\sin(rx))e^{-r^2 t/\pi^2}$. The BCs imply $b = 0$ and $r = n\pi/3$. The IC imply

$$u(x,t) = 3\cos(\pi x)e^{-(\frac{1}{\pi^2})(\pi^2)t} = 3\cos(\pi x)e^{-t}$$

so $u(x,1) = 3\cos(\pi^2/4)e^{-1} = -0.862...$

BONUS: Find the first 4 non-zero terms in the power series or Taylor series solution $y = y(x)$ to $y'' + xy = 0$, $y(0) = 1$, $y'(0) = 2$. More bonus points if you find the recursion relation for the coefficients c_n of the power series expansion of $y(x)$.

Soln: $y'' + xy = 0$ implies $a_2 = 0$, $y''' + y + xy' = 0$ implies $y'''(0) = -1$, $y'''' + 2y' + xy'' = 0$ implies $y''''(0) = -2$, $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = 1 + 2x + 0x^2 - (1/3!)x^3 - (2/4!)x^4....$