

Fall 1996 Final exam for SM 261
December 14, 1996

Calculators and USNA math tables may be used.

1. Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $C = (1, 3)$. Find those of the following expressions are defined.

(a) AB , (b) BA , (c) $4A + B$, (d) BC , (e) $B^t AB$.

2. a) Find a condition on g , h and k that makes the following system consistent.

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 &= g, \\ 4x_1 + 7x_2 - 4x_3 &= h, \\ -6x_1 - 3x_2 + x_3 &= k. \end{aligned}$$

b) Give an example of a system of two linear equations in three unknowns for which no solution exists.

3. a) State the definition of linear independence.

b) Is the following set of vectors linearly independent or linearly dependent?

$$\{(-2, 0, 0), (8, 0, -5), (-1, 0, 3)\}$$

c) Let A be an $n \times n$ matrix. Write five different statements, each equivalent to “ A is invertible”.

4. Let

Find (a) A^{-1} , (b) $\det(A)$, (c) $\det(A^3)$, (d) $\det(2A)$, (e) $\det(A^{-1})$.

5. Define a linear transformation T by

$$T(x_1, x_2) = (5x_2 - x_1, 0, 3x_1 + x_2, x_1).$$

a) Is T one-to-one? Why or why not?

b) Is T onto? Why or why not?

c) Let W be the set of all vectors in \mathbb{R}^3 of the form $(a, 2, -a)$, where a is a real number. Is W a subspace of \mathbb{R}^3 ? Why or why not?

d) Let $\mathcal{B} = \{(1+t)^2, (1-t)^2, 1\}$. Is \mathcal{B} a basis for the vector space P_2 of all polynomials of degree 2 or less? Why or why not?

6. Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

a) Find the eigenvalues and corresponding eigenvectors of A .

b) Diagonalize A , i.e., find a matrix P such that $P^{-1}AP$ is a diagonal matrix D .

c) Find PD^2P^{-1} .

7. The matrix

$$A = \begin{pmatrix} 1 & 1 & 5 & 0 & 3 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 2 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix}$$

row reduces to

$$B = \begin{pmatrix} 1 & 1 & 5 & 0 & 3 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find (a) rank A , (b) $\dim \ker(A)$, (c) rank A^t , (d) $\dim \ker(A^t)$,
(e) $\dim \text{row}(A)$, (f) a basis for $\text{col}(A)$, (g) a basis for $\text{row}(A)$,
(h) a basis for $\ker(A)$.

8. The matrix $M = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ can be interpreted geometrically as a rotation of ϕ radians counterclockwise about the origin in \mathbb{R}^2 . Let $L = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$.

- Give geometric interpretations of L and LM .
- Calculate $A = LM$ for $\phi = \pi/4$ and $r = 2$.
- Find the eigenvalues and corresponding eigenvectors of A .
- Find A^2 , $A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- Give a geometric interpretation of A^2 .

9. Let $b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The set $\mathcal{B} = \{b_1, b_2\}$ forms a basis of \mathbb{R}^2 .

- Find the \mathcal{B} -coordinates of $x = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.
- Let $A = \begin{pmatrix} .75 & -.25 \\ -.25 & .75 \end{pmatrix}$. Show that b_1 and b_2 are eigenvectors of A .
- Find the \mathcal{B} -coordinates of Ax . Hint: Use your answers from parts (a), (b).
- Find $A^4 b_1$ and $A^4 b_2$ without computing A^4 . Hint: Use your answers from part (b).
- Find $A^4 x$ using your answers from parts (a), (d).
- Find $\lim_{k \rightarrow \infty} (A^k x)$.

10. State whether the following is true or false. If true, give an argument why. If false, give a counterexample.

- If a system of linear equations has two different solutions then it must have infinitely many solutions.

- b) A 5×7 matrix cannot have a pivot position in every row.
- c) It is impossible for three vectors to span \mathbb{R}^4 .
- d) If a system $Ax = 0$ has a solution then so does the system $Ax = b$.
- e) If A is a 3×5 matrix, the smallest dimension of $\ker(A)$ is 2.
- f) If A is a 7×8 matrix such that $Ax = b$ has a solution for all $b \in \mathbb{R}^7$ then any two solutions of $Ax = 0$ must be multiples of each other.