FALL 2000 FINAL EXAM FOR SM261
1330 DECEMBER 11, 2000

**SHOW ALL WORK**

**Notations**
$M_{m\times n}$ is the vector space of all $m \times n$ matrices
$P_n$ is the vector space of all polynomials of degree $\leq n$.

1. Consider the linear system of equations

$$\begin{aligned} 3x + 5y - 4z &= 7 \\ -3x - 2y + 4z &= -1 \\ 6x + y + -8z &= -4 \end{aligned}$$

**a.** Write the system in the matrix form $Av = b$.

**b.** Put the augmented matrix in reduced echelon form.

**c.** Solve the system.

**d.** Is the solution set a subspace of $\mathbb{R}^3$? Explain.

**e.** Is the solution set of the corresponding homogenous system $Av = 0$ a subspace of $\mathbb{R}^3$ Explain.

2. Find the inverse of the matrix

$$\begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$$

3. For each of the following elementary row operations on a $2 \times 2$ matrix $A$, find a matrix $E$ (called an elementary matrix) such that multiplying $A$ on the left by $E$ has the same result as the elementary row operation.

**a.** Multiply row 2 by 4.

**b.** Switch row 1 and row 2.

**c.** Add 3 times row 2 to row 1.

4. Complete the definitions. $V$ is a vector space.

**a.** A subset $S$ of $V$ is a subspace of $V$ if ....

**b.** A set of vectors $\{v_1, \ldots, v_n\}$ in $V$ is linearly independent if ....

**c.** A set of vectors $\{v_1, \ldots, v_n\}$ in $V$ spans $V$ if ....

**d.** A set of vectors $\{v_1, \ldots, v_n\}$ in $V$ is a basis for $V$ if ....

5. Let $V = \mathbb{R}^4$, $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in V | x + w = y + z \right\}$. Determine if $S$ is a subspace of $V$. Justify your answer.

6. Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ a+c \\ b+c \\ a+b+c \end{pmatrix}.$$

Recall that the range of $T$ is defined to be $\{T(v) | v \in \mathbb{R}^3\}$. The range of $T$ is a subspace of $\mathbb{R}^4$. Find a basis $\mathcal{B}$ for the range of $T$.

7. **a.** Let

$$S = \{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 | x - y + 2z - 3w = 0 \text{ and } y - z + w = 0\}.$$

$S$ is a subspace of $\mathbb{R}^4$. Find a basis for $S$.

8. Let $V = \{A \in M_{4\times4} | A \text{ is upper triangular}\}$. $V$ is a vector space. Find the dimension of $V$.

9. True or false? If the statement is false, give a counterexample.

**a.** If a set of three vectors is linearly dependent, then one of the three vectors is a multiple of one of the other two vectors.

**b.** If $v$ and $w$ are eigenvectors for different eigenvalues, then $v$ and $w$ are linearly independent.

**c.** If $v$ and $w$ are eigenvectors for the same eigenvalue, then $v$ and $w$ are linearly dependent.

**d.** An $n \times n$ matrix is diagonalizable if it has at least $n$ eigenvectors.

**e.** An $n \times n$ matrix is diagonalizable if it has $n$ distinct eigenvalues.

10. Let $V = P_2$. Let $S$ be the subspace of $V$ with basis $\mathcal{B} = \{1 - t, 2t + t^2\}$.

**a.** Let $w = 2 + 8t + 5t^2$. Show $w \in S$.

**b.** Find $[w]_\mathcal{B}$, the coordinate vector of $w$ relative to $\mathcal{B}$.

11. Let $T : P_2 \to P_3$ be the linear transformation given by

$$T(p(t)) = (3 + 2t)p(t)$$

Find the matrix of $T$ relative to the ordered bases $\{1 + t, 1 + t^2, t + t^2\}$ for $P_2$ and $\{1, t, t^2, t^3\}$ for $P_3$.

12. Let

$$A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}.$$

**a.** Find the characteristic polynomial of $A$.

**b.** Find the eigenvalues of $A$.

**c.** Find a basis for each eigenspace of $A$.

**d.** Diagonalize $A$ by finding an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

**e.** Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ relative to which the matrix of the linear transformation $T(v) = Av$ is a diagonal matrix. Find the matrix of $T$ relative to this basis $\mathcal{B}$.

13. Let $A = \begin{pmatrix} .8 & .4 \\ -.9 & .8 \end{pmatrix}$.

**a.** Find a $2 \times 2$ matrix $C$ which is similar to $A$ and such that the linear transformation $T(v) = Cv$ is a rotation of $\mathbb{R}^2$.

**b.** Find the angle of rotation which is defined by $T$.

**c.** Give an example of a $2 \times 2$ matrix which has no real eigenvalues and which is not similar to a rotation.